TJMM
11 (2019), No. 1-2, 183-191

NEW REPRESENTATIONS OF INTEGRALS OF POLYLOGARITHMIC FUNCTIONS WITH QUADRATIC ARGUMENT

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Abstract. In this paper we explore the representation and many connections between integrals of products of polylogarithmic functions with a quadratic argument and Euler sums. The connection between polylogarithmic functions and Euler sums is well known. Some examples of integrals of products of polylogarithmic functions in terms of Riemann zeta values and Dirichlet values will be given.

1. Introduction and Preliminaries

In this paper we investigate the representations of integrals of the type

$$\int_0^1 \frac{\mathrm{Li}_q(\pm x)\ \mathrm{Li}_t(x^2)}{x} dx, \tag{1}$$

for integers q and t and two other variations of (1). It is well known that integrals of products of polylogarithmic functions can be associated with Euler sums, see [10]. We give explicit representations of the integral in terms of Euler sums. Some examples are highlighted, most of which are not amenable to a computer mathematical package. We also give three new representations of Euler sums in terms of Riemann zeta functions. This work extends the results given by [10] and [13] who examined a similar integral with positive, negative and mixed arguments of the polylogarithm. Devoto and Duke [8] also list many identities of lower order polylogarithmic integrals and their relations to Euler sums. For purely negative arguments a similar integral to (1) was investigated by [13]. Another important source of information on polylogarithm functions is the work of [11]. In [2] and [7] the authors explore the algorithmic and analytic properties of generalized harmonic Euler sums systematically, in order to compute the massive Feynman integrals which arise in quantum field theories and in certain combinatorial problems. We define the alternating zeta function (or Dirichlet eta function) $\eta(z)$ as

$$\eta(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = \left(1 - 2^{1-z}\right)\zeta(z) \tag{2}$$

where $\eta(1) = \ln 2$. Let $H_n^{(p)}$ denote harmonic numbers of order p, put

$$S(p,q) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ H_n^{(p)}}{n^q},$$

2010 *Mathematics Subject Classification.* 11M06, 11M32,33B15.
Key words and phrases. Polylogarithm function, Euler sums, Zeta functions, Dirichlet functions.

then in the case where p and q are both positive integers and $p+q$ is an odd integer, Flajolet and Salvy [9] gave the identity:

$$\begin{aligned} 2S(p,q) &= \left(1-(-1)^{p}\right)\zeta(p)\,\eta(q)+2(-1)^{p}\sum_{i+2k=q}\binom{p+i-1}{p-1}\zeta(p+i)\;\eta(2k) \\ &\quad +\eta(p+q)-2\sum_{j+2k=p}\binom{q+j-1}{q-1}(-1)^{j}\,\eta(q+j)\;\eta(2k), \end{aligned} \tag{3}$$

where $\eta(0)=\frac{1}{2},\ \eta(1)=\ln 2,\ \zeta(1)=0$, and $\zeta(0)=-\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. We also know, from the work of [6] that for odd weight $(p+q)$ we have

$$BW(p.q)=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q}=(-1)^{p}\sum_{j=1}^{\left[\frac{p}{2}\right]}\binom{p+q-2j-1}{p-1}\zeta(p+q-2j)\,\zeta(2j) \tag{4}$$

$$+\frac{1}{2}\left(1+(-1)^{p+1}\right)\zeta(p)\,\zeta(q)+(-1)^{p}\sum_{j=1}^{\left[\frac{p}{2}\right]}\binom{p+q-2j-1}{q-1}\zeta(p+q-2j)\,\zeta(2j)$$

$$+\frac{\zeta(p+q)}{2}\left(1+(-1)^{p+1}\binom{p+q-1}{p}+(-1)^{p+1}\binom{p+q-1}{q}\right),$$

where $[z]$ is the integer part of z. Eulers identity states, for $m\in\mathbb{N}\geq 2$,

$$EU(m)=\sum_{n=1}^{\infty}\frac{H_n}{n^m}=(m+2)\zeta(m+1)-\sum_{j=1}^{m-2}\zeta(m-j)\,\zeta(j+1), \tag{6}$$

and from [12], let $\Lambda(m)=\zeta(m)+\eta(m)$, then we have

$$HE(m)=\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^m}=\frac{m}{2}\Lambda(m+1)-\Lambda(m)\ln 2-\frac{1}{4}\sum_{j=1}^{m-2}\Lambda(j+1)\,\Lambda(m-j) \tag{7}$$

Other works including, [1], [3], [4], [5], [14], [15], [16] and [17] cite many identities of polylogarithmic integrals and Euler sums, but none of these examine the negative argument case. The following results where obtained by [10] and [13].

Lemma 1. *For q and t positive integers, Freitas,* [10], *proved,*

$$\begin{aligned} A(q,t) &= \int_0^1\frac{Li_q(x)\;Li_t(x)}{x}dx=\sum_{j=1}^{q-1}(-1)^{j+1}\zeta(t+j)\,\zeta(q-j+1) \\ &\quad +(-1)^{q+1}EU(t+q) \end{aligned} \tag{8}$$

where $EU(m)$ is Euler's identity given in the next lemma. Also, the following results where obtained in [13].

$$\begin{aligned} B(q,t) &= \int_0^1\frac{Li_q(x)\;Li_t(-x)}{x}dx=-\int_{-1}^{0}\frac{Li_q(-x)\;Li_t(x)}{x}dx \qquad (9) \\ &= \sum_{j=1}^{q-1}(-1)^{j}\,\eta(t+j)\,\zeta(q-j+1)+(-1)^{q}S(1,q+t), \end{aligned}$$

and

$$C(q,t) = \int_0^1 \frac{Li_q(-x)\,Li_t(x)}{x}dx = \sum_{j=1}^{t-1}(-1)^j\,\eta(q+j)\,\zeta(t-j+1) + (-1)^t S(1,q+t) \tag{10}$$

$$D(q,t) = \int_0^1 \frac{Li_q(-x)\;\;Li_t(-x)}{x}dx = (-1)^{q+1}\left(2^{-t-q}-1\right)EU(t+q) + \sum_{j=1}^{q-1}(-1)^{j+1}\,\eta(t+j)\,\eta(q-j+1) + (-1)^{q+1}\,\kappa(t+q)\ln 2 + (-1)^{q+1}\,HE(q+t)\,. \tag{11}$$

was proved in [13], *where* $\kappa(m) = \zeta(t+q) + \eta(t+q)$ *. The quantities* $S(1,q+t)$ *,* $EU(t+q)$ *and* $HE(q+t)$ *are given by (3), (4) and (7) respectively.*

2. Summation identity

Theorem 1. *For positive integers* q *and* t*, the integral of the product of two polylogarithmic functions,*

$$I_1(q,t) = \int_0^1 \frac{Li_q(x)\;\;Li_t\left(x^2\right)}{x}dx = 2^{t-1}\left(\sum_{j=1}^{q-1}(-1)^{j+1}\,\zeta(t+j)\,\zeta(q-j+1) + (-1)^{q+1}\,EU(t+q)\right) + 2^{t-1}\left(\sum_{j=1}^{q-1}(-1)^{j}\,\eta(t+j)\,\zeta(q-j+1) + (-1)^{q}\,S(1,q+t)\right) \tag{12}$$

where the terms $EU(\cdot,\cdot)$ *,* $S(\cdot,\cdot)$ *are obtained from (6) and (3) respectively.*

Proof. By the definition of the polylogarithmic function we have

$$I_1(q,t) = \int_0^1 \frac{\mathrm{Li}_q(x)\;\;\mathrm{Li}_t\left(x^2\right)}{x}dx = \sum_{n=1}^{\infty}\frac{1}{n^t}\int_0^1 x^{2n-1}\;\mathrm{Li}_q(x)\;\,dx.$$

Successively integrating by parts leads to

$$\begin{aligned} I_1(q,t) &= \sum_{j=1}^{q-1}(-1)^{j+1}\,\zeta(q-j+1)\sum_{n\geq 1}\frac{1}{n^t\,(2n)^j} + \sum_{n\geq 1}\frac{(-1)^{q+1}}{n^t\,(2n)^{q-1}}\int_0^1 x^{2n-1}\;\mathrm{Li}_1(x)\,dx \\ &= \sum_{j=1}^{q-1}(-1)^{j+1}\,\zeta(q-j+1)\sum_{n\geq 1}\frac{1}{n^t\,(2n)^j} + \sum_{n\geq 1}\frac{(-1)^{q+1}\,H_{2n}}{n^t\,(2n)^q} \\ &= \sum_{j=1}^{q-1}\frac{(-1)^{j+1}}{2^j}\zeta(q-j+1)\,\zeta(t+j) + \frac{(-1)^{q+1}}{2^q}\sum_{n\geq 1}\frac{H_{2n}}{n^{t+q}}. \end{aligned} \tag{13}$$

Alternatively we have the relation

$$\mathrm{Li}_t\left(x^2\right) = 2^{t-1}\left(\mathrm{Li}_t(x) + \mathrm{Li}_t(-x)\right), \tag{14}$$

from which we obtain

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_q(x)\ \mathrm{Li}_t(x^2)}{x}dx &= 2^{t-1}\int_0^1 \frac{\mathrm{Li}_q(x)\ (\mathrm{Li}_t(x)+\mathrm{Li}_t(-x))}{x}dx \\ &= 2^{t-1}(A(q,t)+B(q,t)) \qquad (15)\end{aligned}$$

$$\begin{aligned}&= 2^{t-1}\left(\sum_{j=1}^{q-1}(-1)^{j+1}\zeta(t+j)\zeta(q-j+1)+(-1)^{q+1}EU(t+q)\right) \\ &\quad +2^{t-1}\left(\sum_{j=1}^{q-1}(-1)^{j}\eta(t+j)\zeta(q-j+1)+(-1)^{q}S(1,q+t)\right),\end{aligned}$$

hence the identity (12) is achieved. An alternative expansion of the integral

$$I_1(q,t)=\int_0^1 \frac{\mathrm{Li}_q(x)\ \mathrm{Li}_t(x^2)}{x}dx = \sum_{n\geq 1}\frac{1}{n^q}\sum_{r\geq 1}\frac{1}{r^t(n+2r)}.$$

By partial fraction expansion,

$$\begin{aligned}I_1(q,t) &= \sum_{n\geq 1}\frac{1}{n^q}\sum_{r\geq 1}\frac{1}{r^t(n+2r)} = (-1)^{t-1}2^{t-1}\sum_{n\geq 1}\frac{H_{\frac{n}{2}}}{n^{t+q}} \\ &\quad +\sum_{j=1}^{t-1}(-1)^{j+1}2^{j-1}\zeta(q+j)\zeta(t-j+1). \qquad (16)\end{aligned}$$

□

Remark 1. *We note that from (12) and (13) we are able to isolate a new Euler identity of weight $q+t+1$, in the form:*

$$W(a,k)=\sum_{n\geq 1}\frac{H_{an}}{n^k}.$$

For $a=2, k=q+t$

$$\begin{aligned}W(2,q+t) &= \sum_{n\geq 1}\frac{H_{2n}}{n^{t+q}} = (-1)^{q+1}2^{q+t-1}(A(q,t)+B(q,t)) \\ &\quad -\sum_{j=1}^{q-1}(-1)^{j+q}2^{q-j}\zeta(q-j+1)\zeta(t+j). \qquad (17)\end{aligned}$$

Similarly from (15) and (16), we obtain, for $a=\frac{1}{2}, k=q+t$, another new Euler identity

$$\begin{aligned}W\left(\frac{1}{2},q+t\right) &= \sum_{n\geq 1}\frac{H_{\frac{n}{2}}}{n^{t+q}} = (-1)^{t+1}(A(q,t)+B(q,t)) \\ &\quad +(-1)^t\sum_{j=1}^{t-1}(-1)^{j+1}2^{j-1}\zeta(q+j)\zeta(t-j+1). \qquad (18)\end{aligned}$$

The next theorem investigates the integral of the product of polylogarithmic functions with one positive and one negative argument.

Theorem 2. *Let (q,t) be positive integers, then for $t+q$ an even integer,*

$$I_2(q,t) = \int_0^1 \frac{Li_q(-x)\ Li_t(x^2)}{x} dx = \int_{-1}^0 \frac{Li_q(x)\ Li_t(x^2)}{x} dx \tag{19}$$
$$= 2^{t-1}\left(\sum_{j=1}^{t-1} (-1)^j \eta(q+j)\zeta(t-j+1) + S(1,q+t)\right).$$
$$+2^{t-1}\left(\sum_{j=1}^{q-1} (-1)^{j+1}\eta(t+j)\eta(q-j+1) + (-1)^{q+1}\kappa(t+q)\ln 2\right)$$
$$+2^{t-1}(-1)^{q+1}\left(2^{-t-q}-1\right) EU(t+q) + (-1)^{q+1} HO(q+t).$$

Proof. From (14), we have

$$\begin{aligned} I_2(q,t) &= \int_0^1 \frac{\mathrm{Li}_q(-x)\ \mathrm{Li}_t(x^2)}{x} dx = 2^{t-1}\int_0^1 \frac{\mathrm{Li}_q(-x)\ (\mathrm{Li}_t(x)+\mathrm{Li}_t(-x))}{x} dx \\ &= 2^{t-1}(C(q,t)+D(q,t)), \end{aligned} \tag{20}$$

replacing $C(q,t)$, and $D(q,t)$, with (10) and (11) respectively leads to (19), as required. We can also consider,

$$I_2(q,t) = \int_0^1 \frac{\mathrm{Li}_q(-x)\ \mathrm{Li}_t(x^2)}{x} dx = \sum_{n\geq 1} \frac{(-1)^n}{n^q} \int_0^1 x^{n-1}\ \mathrm{Li}_t(x^2)\, dx,$$

and successively integrating by parts, as in theorem 1, leads to

$$I_2(q,t) = \sum_{j=1}^{t-1} (-1)^j \zeta(t-j+1)\eta(q+j) + (-1)^t \sum_{n\geq 1} \frac{(-1)^{n+1} H_{\frac{n}{2}}}{n^{q+t}}. \tag{21}$$

Another expansion of

$$\begin{aligned} I_2(q,t) &= \int_0^1 \frac{\mathrm{Li}_q(-x)\ \mathrm{Li}_t(x^2)}{x} dx = \int_0^1 \sum_{r\geq 1}\sum_{n\geq 1} \frac{(-1)^n x^{n+2r-1}}{n^q r^t} dx \\ &= \sum_{r\geq 1}\sum_{n\geq 1} \frac{(-1)^n}{n^q r^t (n+2r)}. \end{aligned}$$

Partial fraction expansion, and summing

$$\begin{aligned} I_2(q,t) &= (-1)^q \sum_{j=1}^{q-1} (-1)^j 2^{j-q}\eta(1+j)\zeta(q+t-j) \\ &\quad + \frac{(-1)^{q+1}}{2^q} \sum_{n\geq 1}\left(\frac{H_n}{n^{q+t}} - \frac{H_{2n}}{n^{q+t}}\right), \end{aligned}$$

$$= (-1)^q \sum_{j=1}^{q-1} (-1)^j 2^{j-q}\eta(1+j)\zeta(q+t-j) + \frac{(-1)^{q+1}}{2^q}(EU(q+t) + W(2,q+t)).$$

hence (19) follows. We note that from (20) and (21) we obtain another new Euler sum, namely

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_{\frac{n}{2}}}{n^{q+t}} = (-1)^t\left(C(q,t)+D(q,t)\right) + (-1)^t\sum_{j=1}^{t-1}(-1)^{j+1}2^{j-t}\zeta(t-j+1)\eta(q+j). \tag{22}$$

For $q=0$, $\int\limits_0^1 \frac{\mathrm{Li}_t(x^2)}{1+x}dx = \zeta(t)\ln 2 + EU(t) - W(2,t)$. □

The next two theorems are variations of (1), and consider integrals of the product of polylogarithmic functions which include the inverse power of the argument. First we need the following corollary.

Corollary 1. *For q and t positive integers we have*

$$X1(q,t) = \int\limits_0^1 \frac{Li_q\left(-\frac{1}{x}\right)\ Li_t(-x)}{x}dx$$

$$= \eta(q+t+1) + \kappa(q+t)\ln 2 + \sum_{j=1}^{q-1}\eta(t+j)\eta(q-j+1) + \left(\frac{1}{2^{q+t}}-1\right)EU(q+t) + HE(q+t).$$

$$X2(q,t) = \int\limits_0^1 \frac{Li_q\left(-\frac{1}{x}\right)\ Li_t(x)}{x}dx$$

$$-\zeta(q+t+1) - \kappa(q+t)\ln 2 - \sum_{j=1}^{q-1}\zeta(t+j)\eta(q-j+1) + S(1,q+t) + \frac{1}{2^{q+t}}BW(1,q+t) - HE(q+t).$$

where $BW(1,q+t)$ is given by (4) and $HE(\cdot)$ is given by (7).

$$X3(q,t) = \int\limits_0^1 \frac{Li_q\left(\frac{1}{x}\right)\ Li_t(x)}{x}dx = EU(q+t)$$

$$= +\sum_{j=1}^{q-1}\zeta(t+j)\zeta(q-j+1) - i\pi\zeta(t+q) - \zeta(t+q+1).$$

$$X4(q,t) = \int\limits_0^1 \frac{Li_q\left(\frac{1}{x}\right)\ Li_t(-x)}{x}dx = \eta(t+q+1) + i\pi\eta(t+q) - S(1,q+t) - \sum_{j=1}^{q-1}\eta(t+j)\zeta(q-j+1).$$

Proof. The proof of these results are available from the works [13]. □

Theorem 3. *For positive integers q and t, the integral of the product of two polylogarithmic functions,*

$$I_3(q,t)=\int_0^1 \frac{Li_q\left(-\frac{1}{x}\right)\ Li_t\left(x^2\right)}{x}dx$$

$$\begin{aligned} &= -\frac{1}{2^{q+1}}\zeta(q+t+1)+\frac{1}{2^q}\left(EU(q+t)-W(2,q+t)\right) \\ &\quad -\sum_{j=1}^{q-1}\frac{1}{2^j}\zeta(q+t)\,\eta(q-j+1) \end{aligned} \tag{23}$$

where the terms $EU(\cdot)$, $W(\cdot,\cdot)$ are obtained from (6) and (17) respectively.

Proof. Following the same process as in theorem 2, we have

$$I_3(q,t)=\int_0^1 \frac{\mathrm{Li}_q\left(-\frac{1}{x}\right)\ \mathrm{Li}_t\left(x^2\right)}{x}dx=\sum_{n\geq 1}\frac{1}{n^t}\int_0^1 x^{2n-1}\ \mathrm{Li}_q\left(-\frac{1}{x}\right)dx$$

Integrating by parts, as in theorem 2, we have,

$$I_3(q,t) = -\sum_{j=1}^{q-1}\frac{1}{2^j}\eta(q-j+1)\,\zeta(t+j)+\sum_{n\geq 1}\frac{1}{n^t(2n)^{q-1}}\int_0^1 x^{2n-1}\ \mathrm{Li}_1\left(-\frac{1}{x}\right)dx$$

$$=-\sum_{j=1}^{q-1}\frac{1}{2^j}\eta(q-j+1)\,\zeta(t+j)-\frac{1}{2^q}\zeta(t+q)+\frac{1}{2^q}\sum_{n\geq 1}\frac{1}{n^{t+q}}\left(\frac{1}{2}H_n-\frac{1}{2}H_{n-\frac{1}{2}}-\ln 2\right),$$

from the polygamma multiplication formula, we can simplify so that

$$I_3(q,t)=-\sum_{j=1}^{q-1}\frac{1}{2^j}\eta(q-j+1)\,\zeta(t+j)-\frac{1}{2^q}\zeta(t+q)-\frac{1}{2^q}\sum_{n\geq 1}\frac{1}{n^{t+q}}\left(H_{2n}-H_n\right)$$

and the proof of theorem 3 is finalized. Similarly, we are able to write, using (14),

$$\begin{aligned} I_3(q,t) &= \int_0^1 \frac{\mathrm{Li}_q\left(-\frac{1}{x}\right)\ \mathrm{Li}_t\left(x^2\right)}{x}dx=2^{t-1}\int_0^1\frac{\mathrm{Li}_q\left(-\frac{1}{x}\right)\ \left(\mathrm{Li}_t(x)+\mathrm{Li}_t(-x)\right)}{x}dx \\ &= 2^{t-1}\left(X1(q,t)+X2(q,t)\right). \end{aligned}$$

For $q=0$, $\int_0^1 \frac{\mathrm{Li}_t\left(x^2\right)}{x(1+x)}dx=\zeta(t)\ln 2+EU(t)-W(2,t)+\frac{1}{2}\zeta(t+1)$. □

Theorem 4. *For positive integers q and t, the integral of the product of two polylogarithmic functions,*

$$\begin{aligned} I_4(q,t)=\int_0^1 \frac{Li_q\left(\frac{1}{x^2}\right)\ Li_t(x)}{x}dx &= -2^q\zeta(q+t+1)+2^{q-1}W\left(\frac{1}{2},q+t\right) \\ &\quad +\sum_{j=1}^{q-1}2^{j-1}\zeta(j+t)\,\zeta(q-j+1)-2^{q-1}i\pi\zeta(q+t) \end{aligned} \tag{24}$$

where $W\left(\frac{1}{2},q+t\right)$ is obtained from (18), and $i=\sqrt{-1}$.

Proof. Integrating by parts, we have

$$\begin{aligned} I_4(q,t) &= \sum_{j=1}^{q-1} 2^{j-1}\zeta(j+t)\zeta(q-j+1) + \sum_{n\geq 1} \frac{2^{q-1}}{n^{q+t-1}} \int_0^1 x^{n-1}\mathrm{Li}_1\left(\frac{1}{x^2}\right) dx \\ &= \sum_{j=1}^{q-1} 2^{j-1}\zeta(j+t)\zeta(q-j+1) - 2^{q-1}i\pi\zeta(q+t) - 2^q \ln 2 \\ &\quad +2^{q-1}\sum_{n\geq 1} \frac{H_{\frac{n}{2}}}{(n+2)^{q+t}}. \end{aligned}$$

Making a change in the index summation n, we finally arrive at (24). An alternative representation of (24) is,

$$\begin{aligned} I_4(q,t) &= \int_0^1 \frac{\mathrm{Li}_q\left(\frac{1}{x^2}\right)\ \mathrm{Li}_t(x)}{x}dx = 2^{q-1}\int_0^1 \frac{\mathrm{Li}_t(x)\ \left(\mathrm{Li}_q\left(\frac{1}{x}\right) + \mathrm{Li}_q\left(-\frac{1}{x}\right)\right)}{x}dx \\ &= 2^{q-1}\left(X2(q,t) + X3(q,t)\right). \end{aligned}$$

□

Example 1.

$$I_1(3,5) = \int_0^1 \frac{Li_3(x)\ Li_5\left(x^2\right)}{x}dx = \frac{521}{32}\zeta(9) - \frac{33}{4}\zeta(2)\zeta(7) - 2\zeta(4)\zeta(5)$$

$$I_2(4,4) = \int_0^1 \frac{Li_q(-x)\ Li_t\left(x^2\right)}{x}dx = \frac{501}{64}\zeta(9) - 4\zeta(2)\zeta(7) - \frac{11}{8}\zeta(4)\zeta(5),$$

$$\sum_{n\geq 1}\frac{H_{\frac{n}{2}}}{n^8} = \frac{521}{512}\zeta(9) - \frac{1}{64}\zeta(2)\zeta(7) - \frac{1}{16}\zeta(4)\zeta(5) - \frac{1}{4}\zeta(3)\zeta(6)$$

$$\begin{aligned} X2(2,1) &= \int_0^1 \frac{Li_2\left(-\frac{1}{x}\right)\ Li_1(x)}{x}dx = \frac{1}{2}\zeta(2)\ln^2 2 - \frac{3}{4}\zeta(4) \\ &\quad -\frac{7}{4}\zeta(3)\ln 2 - \frac{1}{12}\ln^4 2 - 2Li_4\left(\frac{1}{2}\right) = L(3) - \frac{7}{2}\zeta(4), \end{aligned}$$

where $L(3) = \frac{11}{4}\zeta(4) - \frac{7}{4}\zeta(3)\ln 2 + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - 2Li_4\left(\frac{1}{2}\right)$.

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_{\frac{n}{2}}}{n^6} = \frac{119}{128}\zeta(7) - \frac{7}{32}\zeta(4)\zeta(3) - \frac{1}{32}\zeta(5)\zeta(2)$$

$$X3(5,3) = \int_0^1 \frac{Li_5\left(\frac{1}{x}\right)\ Li_3(x)}{x}dx = 4\zeta(9) + \zeta(4)\zeta(5) - i\pi\zeta(8).$$

$$\begin{aligned} X4(3,7) &= \int_0^1 \frac{Li_3\left(\frac{1}{x}\right)Li_7(-x)}{x}dx = i\pi\eta(10) + \frac{31}{32}\zeta(6)\zeta(5) \\ &\quad +\frac{7}{8}\zeta(4)\zeta(7) - \frac{127}{256}\zeta(2)\zeta(9) - \frac{8183}{2048}\zeta(11). \end{aligned}$$

Concluding Remarks: These results build on the work of [10], where they explored integrals of polylogarithmic functions with positive arguments only, The results in [10], [13], give many identities of integrals of polylogarithmic functions with positive, negative and mixed arguments.

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