TJMM
11 (2019), No. 1-2, 163-170

OPIAL TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS VIA CONVEXITY

MEHMET ZEKI SARIKAYA AND CANDAN CAN BILISIK

Abstract. The main target addressed in this article are presenting Opial type inequalities for Katugampola conformable fractional integral. In accordance with this purpose we try to use more general type of function in order to make a generalization. Thus our results cover the previous published studies for Opial type inequalities.

1. Introduction & Preliminaries

In the year 1960, Opial established the following interesting integral inequality[13]:

Theorem 1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|\, dt \leq \frac{h}{4}\int_0^h (x'(t))^2\, dt \tag{1}$$

The constant $h/4$ is best possible

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see [4]-[6], [14]-[22].

The purpose of this paper is to establish some generalizations of Opial type inequalities for conformable integral. The structure of this paper is as follows:. Firstly, we give the definitions of the conformable derivatives and conformable integral and introduce several useful notations conformable integral used our main results. Later, the main results are presented.

In light of recent developments in mathematics, fractional calculus is becoming extremely popular in a lot of application areas such as control theory, computational analysis and engineering [11], see also [12]. Together with these developments a number of new definitions have been introduced in academia to provide the best method for fractional calculus. For instance in more recent times a new local, limit-based definition of a conformable derivative has been introduced in [1], [8], [10], with several follow-up papers [2], [3], [7]-[9]. In this study, we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \quad D_\alpha(f)(0) = \lim_{t\to 0} D_\alpha(f)(t), \tag{2}$$

2010 *Mathematics Subject Classification.* 26D15, 26A51, 26A33, 26A42, 34A40.
Key words and phrases. Opial inequality, Hölder's inequality, confromable fractional integrals.

provided the limits exist (for detail see, [8]). If f is fully differentiable at t, then

$$D_{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}(t). \tag{3}$$

A function f is α-differentiable at a point $t \geq 0$ if the limit in (2) exists and is finite. This definition yields the following results;

Theorem 2. *Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then*
i. $D_{\alpha}(af + bg) = aD_{\alpha}(f) + bD_{\alpha}(g)$, *for all* $a, b \in \mathbb{R}$,
ii. $D_{\alpha}(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$,
iii. $D_{\alpha}(fg) = fD_{\alpha}(g) + gD_{\alpha}(f)$,
iv. $D_{\alpha}\left(\frac{f}{g}\right) = \frac{fD_{\alpha}(g) - gD_{\alpha}(f)}{g^2}$
v. $D_{\alpha}(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$
vi. $D_{\alpha}(f \circ g)(t) = f'(g(t))D_{\alpha}(g)(t)$ *for f is differentiable at* $g(t)$.

Definition 1 (Conformable fractional integral). *Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\,d_{\alpha}x := \int_a^b f(x)\,x^{\alpha-1}dx$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L^1_{\alpha}([a,b])$.

Remark 1.

$$I^a_{\alpha}(f)(t) = I^a_1\left(t^{\alpha-1}f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 1. *Let the conformable differential operator D^{α} be given as in (2), where $\alpha \in (0,1]$ and $t \geq 0$, and assume the functions f and g are α-differentiable as needed. Then*
i. $D_{\alpha}(\ln t) = t^{-\alpha}$ *for* $t > 0$
ii. $D_{\alpha}\left[\int_a^t f(t,s)\,d_{\alpha}s\right] = f(t,t) + \int_a^t D_{\alpha}[f(t,s)]\,d_{\alpha}s$
iii. $\int_a^b f(x)\,D_{\alpha}(g)(x)\,d_{\alpha}x = fg|_a^b - \int_a^b g(x)\,D_{\alpha}(f)(x)\,d_{\alpha}x$.

Theorem 3 (Jensen Inequality). [2] *Let $\alpha \in (0,1]$, $a, b, c, d \in [0,\infty)$. If Let $w : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to (c,d)$ are nonnegative, continuous functions with $\int_a^b p(t)d_{\alpha}t > 0$, and $F : (c,d) \to \mathbb{R}$ is continuous and convex function. Then, we have*

$$F\left(\frac{\int_a^b w(t)g(t)d_{\alpha}t}{\int_a^b w(t)d_{\alpha}t}\right) \leq \frac{\int_a^b w(t)F(g(t))\,d_{\alpha}t}{\int_a^b w(t)d_{\alpha}t}.$$

Theorem 4 (Taylor Formula). [2] *Let $\alpha \in (0,1]$ and $n \in \mathbb{N}$. Suppose f is $n+1$ times $\alpha-$ fractional differentiable on $[0,\infty)$, and $s, t \in [0,\infty)$. Then we have*

$$f(t) = \sum_{k=0}^{n}\frac{1}{k!}\left(\frac{t^{\alpha} - s^{\alpha}}{\alpha}\right)^k D^k_{\alpha}f(s) + \frac{1}{n!}\int_s^t \left(\frac{t^{\alpha} - \tau^{\alpha}}{\alpha}\right)^n D^{n+1}_{\alpha}f(\tau)d_{\alpha}\tau.$$

Using the Taylor's Theorem, we define tthe remainder function by

$$R_{-1,f}(.,s) := f(s),$$

and for $n > -1$,

$$\begin{aligned} R_{n,f}(t,s) \quad : \quad &= f(s) - \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)^{k} D_{\alpha}^{k} f(s) \\ &= \frac{1}{n!}\int_{s}^{t}\left(\frac{t^{\alpha}-\tau^{\alpha}}{\alpha}\right)^{n} D_{\alpha}^{n+1} f(\tau) d_{\alpha}\tau. \end{aligned}$$

Opial inequality can be represented for conformable fractional integral forms as follows [20]:

Theorem 5. *Let $\alpha \in (0,1]$, $u : [a,b] \to \mathbb{R}$ be an α- fractional differentiable function, and $u(a) = 0$. Then, we have the following inequality*

$$\int_{a}^{b} |u(x)|\,|D_{\alpha}u(x)|\, d_{\alpha}x \leq \frac{b^{\alpha}-a^{\alpha}}{2\alpha}\int_{a}^{b} (|D_{\alpha}u(x|)^{2} d_{\alpha}x. \tag{4}$$

Now, we present the main results:

2. Opial type inequalities for conformable fractional integral

Theorem 6. *Let $\alpha \in (0,1]$, p be a continuous and positive such that $D^{\alpha}(p(t)) > 0$ and g, F be convex and increasing functions on $[0,\infty)$. If $u : [a,b] \to \mathbb{R}$ be an α- fractional differentiable function, and $u(a) = 0$. Then, we have the following inequality*

$$\begin{aligned} &\int_{a}^{b} D_{\alpha}p(t) g\left(\frac{|D_{\alpha}u(t)|}{D_{\alpha}p(t)}\right) F'\left(p(t) g\left(\frac{|u(t)|}{p(t)}\right)\right) d_{\alpha}t \qquad (5) \\ \leq \; & F\left(\int_{a}^{b} D_{\alpha}p(t) g\left(\frac{|D_{\alpha}u(t)|}{D_{\alpha}p(t)}\right) d_{\alpha}t\right). \end{aligned}$$

If F is concave, then inequality (5) is reversed.

Proof. We consider $y(t) = \int_{a}^{t} |D_{\alpha}u(s)|\, d_{\alpha}s$ such that $D_{\alpha}y(t) = |D_{\alpha}u(t)|$ and $y(t) \geq |u(t)|$. Since g is increasing, by using the Jensen inequality for conformable fractional integral, we get

$$\begin{aligned} g\left(\frac{|u(t)|}{p(t)}\right) \quad &\leq \quad g\left(\frac{y(t)}{p(t)}\right) = g\left(\frac{\int_{a}^{t} D_{\alpha}p(s)\frac{|D^{\alpha}u(s)|}{D^{\alpha}p(s)} d_{\alpha}s}{\int_{a}^{t} D_{\alpha}p(s) d_{\alpha}s}\right) \\ &\leq \quad \frac{1}{p(t)}\int_{a}^{t} D_{\alpha}p(s) g\left(\frac{|D_{\alpha}u(s)|}{D_{\alpha}p(s)}\right) d_{\alpha}s \\ &= \quad \frac{1}{p(t)}\int_{a}^{t} D_{\alpha}p(s) g\left(\frac{D_{\alpha}y(s)}{D_{\alpha}p(s)}\right) d_{\alpha}s. \qquad (6) \end{aligned}$$

Thus, since F is a convex function, by using inequality (6) and with the help of the (vi) property in Theorem 2, we have

$$\begin{aligned}
& \int_a^b D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right) F'\left(p(t) g\left(\frac{|u(t)|}{p(t)}\right)\right) d_\alpha t \\
\leq & \int_a^b D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right) F'\left(\int_a^t D_\alpha p(s) g\left(\frac{D_\alpha y(s)}{D_\alpha p(s)}\right) d_\alpha s\right) d_\alpha t \\
\leq & \int_a^b D_\alpha F\left(\int_a^t D_\alpha p(s) g\left(\frac{D_\alpha y(s)}{D_\alpha p(s)}\right) d_\alpha s\right) d_\alpha t \\
= & F\left(\int_a^b D_\alpha p(s) g\left(\frac{D_\alpha y(s)}{D_\alpha p(s)}\right) d_\alpha s\right) \\
= & F\left(\int_a^b D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right) d_\alpha t\right)
\end{aligned}$$

which completes the proof. □

Corollary 1. *Under the hypotheses of Theorem 6, if we choose $g(s) = s$, we have*

$$\int_a^b |D_\alpha u(t)| F'(|u(t)|) d_\alpha t \leq F\left(\int_a^b |D_\alpha u(t)| d_\alpha t\right). \tag{7}$$

Remark 2. *If we take $F(s) = \frac{s^2}{2}$ in Corollary 1, the inequality (7) reduces to the inequality*

$$\int_a^b |u(t)| |D_\alpha u(t)| d_\alpha t \leq \frac{1}{2}\left(\int_a^b |D_\alpha u(t)| d_\alpha t\right)^2.$$

By using the Cauchy-Schwarz inequality for conformable integral, it follows that

$$\begin{aligned}
\int_a^b |u(t)| |D_\alpha u(t)| d_\alpha t & \leq \frac{1}{2}\left(\int_a^b |D_\alpha u(t)| d_\alpha t\right)^2 \\
& \leq \frac{b^\alpha - a^\alpha}{2\alpha} \int_a^b (|D_\alpha u(t)|)^2 d_\alpha x
\end{aligned}$$

which is proved by Sarikaya and Bilisik in [20].

Corollary 2. *Under the hypotheses of Theorem 6, if we choose $g(s) = s$ and $F(s) = \frac{s^{n+1}}{n+1}$ for $n \geq 0$, we have*

$$\int_a^b |D_\alpha u(t)| F'(|u(t)|) d_\alpha t \leq \frac{(b^\alpha - a^\alpha)^n}{\alpha^n (n+1)} \int_a^b |D_\alpha u(t)|^{n+1} d_\alpha t.$$

Proof. By applying Hölder's inequality with indices $n+1$, $\frac{n+1}{n}$ such that $\frac{1}{n+1} + \frac{n}{n+1} = 1$, we have

$$\begin{aligned}
\int_a^b |D_\alpha u(t)| F'(|u(t)|) d_\alpha t & \leq \frac{1}{n+1}\left(\int_a^b |D_\alpha u(t)| d_\alpha t\right)^{n+1} \\
& \leq \frac{1}{n+1}\left(\int_a^b d_\alpha t\right)^n \left(\int_a^b |D_\alpha u(t)|^{n+1} d_\alpha t\right)
\end{aligned}$$

which completes the proof. □

Theorem 7. *Let $\alpha \in (0,1]$, p be a continuous and positive with $p(0)=0$ and ϕ is convex and increasing function on $[0,\infty)$ such that $\phi(0)=0$. Also, suppose that g, F be convex and increasing functions on $[0,\infty)$ and define*

$$z(t)=\int_0^t D_\alpha p(s) g\left(\frac{|D_\alpha u(s)|}{D_\alpha p(s)}\right) d_\alpha s$$

such that

$$D_\alpha (F\circ z)(t)\phi\left(\frac{1}{D_\alpha z(t)}\right) \leq \frac{F(z(b))}{z(b)}\phi'\left(\frac{t}{z(b)}\right). \tag{8}$$

If $u:[a,b]\to\mathbb{R}$ be an α- fractional differentiable function, and $u(0)=0$. Then, we have the following inequality

$$\begin{aligned}&\int_0^b \psi\left(D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right)\right) F'\left(p(t) g\left(\frac{|u(t)|}{p(t)}\right)\right) d_\alpha t\\ \leq\ & \Phi\left(\int_0^b D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right) d_\alpha t\right)\end{aligned}$$

where $\psi(r)=rh\left(\phi\left(\frac{1}{r}\right)\right)$ and $\Phi(r)=F(r)h\left(\phi\left(\frac{b}{r}\right)\right)$ where h is a concav and increasing function on $[0,\infty)$.

Proof. Consider $y(t)=\int_a^t |D_\alpha u(s)|\, d_\alpha s$. Then $D_\alpha y(t)=|D_\alpha u(t)|$ and $y(t)\geq |u(t)|$. Since g is increasing, by using the Jensen inequality for conformable fractional integral, we get

$$\begin{aligned}p(t) g\left(\frac{|u(t)|}{p(t)}\right) &\leq p(t) g\left(\frac{y(t)}{p(t)}\right) = p(t) g\left(\frac{\int_0^t D_\alpha p(s)\frac{|D_\alpha u(s)|}{D_\alpha p(s)} d_\alpha s}{\int_0^t D_\alpha p(s) d_\alpha s}\right)\\ &\leq \int_0^t D_\alpha p(s) g\left(\frac{|D_\alpha u(s)|}{D_\alpha p(s)}\right) d_\alpha s = z(t).\end{aligned} \tag{9}$$

Then, since F is a convex function, by using inequality (9) and with the help of the (vi) property in Theorem 2, we have

$$\begin{aligned}&\int_0^b \psi\left(D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right)\right) F'\left(p(t) g\left(\frac{|u(t)|}{p(t)}\right)\right) d_\alpha t\\ \leq\ & \int_0^b \psi\left(D_\alpha z(t)\right) F'(z(t))\, d_\alpha t\\ =\ & \int_0^b D_\alpha z(t) h\left(\phi\left(\frac{1}{D_\alpha z(t)}\right)\right) F'(z(t))\, d_\alpha t\\ =\ & \frac{\int_0^b D_\alpha (F\circ z)(t) h\left(\phi\left(\frac{1}{D_\alpha z(t)}\right)\right) d_\alpha t}{\int_0^b D_\alpha (F\circ z)(t) d_\alpha t}\int_0^b D_\alpha (F\circ z)(t) d_\alpha t\\ \leq\ & h\left(\frac{\int_0^b D_\alpha (F\circ z)(t)\phi\left(\frac{1}{D_\alpha z(t)}\right) d_\alpha t}{\int_0^b D_\alpha (F\circ z)(t) d_\alpha t}\right) F(z(b))\\ \leq\ & h\left(\frac{\int_0^b \frac{F(z(b))}{z(b)}\phi'\left(\frac{t}{z(b)}\right) d_\alpha t}{F(z(b))}\right) F(z(b)).\end{aligned}$$

Since ϕ is convex, by applying (5) and by using increasing h function, we obtain that

$$\begin{aligned}
&\int_0^b \psi\left(D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right)\right) F'\left(p(t) g\left(\frac{|u(t)|}{p(t)}\right)\right) d_\alpha t \\
\leq\ & h\left(\frac{\int_0^b \frac{F(z(b))}{z(b)} \phi'\left(\frac{t}{z(b)}\right) d_\alpha t}{F(z(b))}\right) F(z(b)) \\
\leq\ & F(z(b)) h\left(\phi\left(\frac{b}{z(b)}\right)\right)=\Phi(z(b)) \\
=\ & \Phi\left(\int_0^b D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right) d_\alpha t\right).
\end{aligned}$$

This completes the proof. □

Theorem 8. *Let $\alpha \in (0,1]$, p be a continuous and positive such that $D_\alpha(p(t))>0$ and g, F be convex and increasing functions on $[0,\infty)$. If $u:[a,b] \rightarrow \mathbb{R}$ be an n-times α-fractional differentiable function, and $u(a)=D_\alpha u(a)=\ldots=D_\alpha^{n-1} u(a)=0$. Then, we have the inequality*

$$\begin{aligned}
&\int_a^b D_\alpha p(t) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1} \frac{|D_\alpha^n u(t)|}{D_\alpha p(t)}\right) F'\left(p(t) g\left(\frac{|u(t)|}{p(t)}\right)\right) d_\alpha t \quad (10) \\
\leq\ & \frac{1}{b-a} \int_a^b F\left[(b-a) D_\alpha p(t) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1} \frac{|D_\alpha^n u(t)|}{D_\alpha p(t)}\right)\right] d_\alpha t.
\end{aligned}$$

Proof. Assume that

$$y(t)=\int_a^t \int_a^{t_{n-1}} \ldots \int_a^{t_1} |D_\alpha^n u(x)|\, d_\alpha t_1 \ldots d_\alpha t_{n-1} d_\alpha x.$$

In this case, $D_\alpha y(t), \ldots, D_\alpha^{n-1} y(t) \geq 0$ and $D_\alpha^n y(t)=|D_\alpha^n u(t)| \geq 0,\ y(t) \geq |u(t)|$. From the Taylor's formula

$$\begin{aligned}
y(t) &\leq \frac{1}{(n-1)!} \int_a^t \left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^{n-1} D_\alpha^n y(s) d_\alpha s \\
&\leq \frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1} \int_a^t D_\alpha^n y(s) d_\alpha s.
\end{aligned}$$

Since g is increasing, by using the Jensen inequality for conformable fractional integral, we get

$$\begin{aligned}
g\left(\frac{|u(t)|}{p(t)}\right) &\leq g\left(\frac{y(t)}{p(t)}\right)=g\left(\frac{\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1} \int_a^t D_\alpha p(s) \frac{D_\alpha^n y(s)}{D_\alpha p(s)} d_\alpha s}{\int_a^t D_\alpha p(s) d_\alpha s}\right) \\
&\leq \frac{1}{p(t)} \int_a^t D_\alpha p(s) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1} \frac{D_\alpha^n y(s)}{D_\alpha p(s)}\right) d_\alpha s. \quad (11)
\end{aligned}$$

Thus, since F is a convex function, by using inequality (11) and with the help of the (vi) property in Theorem 2, we have

$$\begin{aligned}
&\int_a^b D_\alpha p(t) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1}\frac{|D_\alpha^n u(t)|}{D_\alpha p(s)}\right)F'\left(p(t)g\left(\frac{|u(t)|}{p(t)}\right)\right)d_\alpha t\\
\leq\ &\int_a^b D_\alpha p(t) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1}\frac{|D_\alpha^n u(t)|}{D_\alpha p(s)}\right)\\
&\times F'\left(\int_a^t D_\alpha p(s) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1}\frac{D_\alpha^n y(s)}{D_\alpha p(s)}\right)d_\alpha s\right)d_\alpha t.
\end{aligned}$$

By applying (5), and by using Jensen's inequality, we obtain that

$$\begin{aligned}
&\int_a^b D_\alpha p(t) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1}\frac{|D_\alpha^n u(t)|}{D_\alpha p(s)}\right)F'\left(p(t)g\left(\frac{|u(t)|}{p(t)}\right)\right)d_\alpha t\\
\leq\ &\int_a^b D_\alpha\left[F\circ\left(\int_a^t D_\alpha p(s) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1}\frac{D_\alpha^n y(s)}{D_\alpha p(s)}\right)d_\alpha s\right)\right]d_\alpha t\\
=\ &F\circ\left(\int_a^b D_\alpha p(t) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1}\frac{D_\alpha^n y(t)}{D_\alpha p(t)}\right)d_\alpha t\right)\\
=\ &F\circ\left(\frac{1}{b-a}\int_a^b (b-a)\, D_\alpha p(t) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1}\frac{D_\alpha^n y(t)}{D_\alpha p(t)}\right)d_\alpha t\right)\\
\leq\ &\frac{1}{b-a}\int_a^b F\left[(b-a)\, D_\alpha p(t) g\left(\frac{1}{(n-1)!}\left(\frac{t^\alpha-a^\alpha}{\alpha}\right)^{n-1}\frac{D_\alpha^n y(t)}{D_\alpha p(t)}\right)\right]d_\alpha t
\end{aligned}$$

This completes the proof of the inequality (10). □

Theorem 9. *Let $\alpha \in (0,1]$, p be a continuous and positive such that $D_\alpha(p(t)) > 0$ and g, F be convex and increasing functions on $[0,\infty)$. If $u : [a,b] \to \mathbb{R}$ be an α- fractional differentiable function, and $u(a) = 0$. Then, we have the following inequality*

$$\begin{aligned}
&\int_a^b D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right)F'\left(p(t)g\left(\frac{|u(t)|}{p(t)}\right)\right)d_\alpha t\\
\leq\ &\frac{1}{b-a}\int_a^b F\left((b-a)\, D_\alpha p(t) g\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right)\right)d_\alpha t.
\end{aligned}$$

Proof. The proof of the Theorem follows immediately from the inequality (5) by applying Jensen's inequality. □

References

[1] T. Abdeljawad, On conformable fractional calculus, Journal of Computational and Applied Mathematics 279 (2015) 57–66.

[2] D. R. Anderson and D. J. Ulness, Results for conformable differential equations, preprint, 2016.

[3] A. Atangana, D. Baleanu, and A. Alsaedi, New properties of conformable derivative, Open Math. 2015; 13: 889–898.

[4] W.S. Cheung, Some new Opial-type inequalities, Mathematika, 37 (1990), 136–142.

[5] W.S. Cheung, Some generalized Opial-type inequalities, J. Math. Anal. Appl., 162 (1991), 317– 321.

[6] X. G. He, A short of a generalization on Opial' s inequailty,Journal of Mathematical Analysis and Applications,182, (1994), 299-300.

[7] O.S. Iyiola and E.R.Nwaeze, Some new results on the new conformable fractional calculus with application using D'Alambert approach, Progr. Fract. Differ. Appl., 2(2), 115-122, 2016.
[8] U. Katugampola, A new fractional derivative with classical properties, ArXiv:1410.6535v2.
[9] A. Zheng, Y. Feng and W. Wang, The Hyers-Ulam stability of the conformable fractional differential equation, Mathematica Aeterna, Vol. 5, 2015, no. 3, 485-492.
[10] R. Khalil, M. Al horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, Journal of Computational Apllied Mathematics, 264 (2014), 65-70.
[11] A. A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier B.V., Amsterdam, Netherlands, 2006.
[12] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordonand Breach, Yverdon et alibi, 1993.
[13] Z. Opial, Sur une inegaliti, Ann. Polon. Math. 8 (1960), 29-32.
[14] B. G. Pachpatte, On Opial-type integral inequalities , J. Math. Anal. Appl. 120 (1986), 547–556.
[15] B. G. Pachpatte, Some inequalities similar to Opial's inequality , Demonstratio Math. 26 (1993), 643–647.
[16] B. G. Pachpatte, A note on some new Opial type integral inequalities, Octogon Math. Mag. 7 (1999), 80–84..
[17] S. H. Saker, M.D. Abdou and I. Kubiaczyk, Opial and Polya type inequalities via convexity, Fasciculi Mathematici, 60(1), 145–159, 2018.
[18] M. Z. Sarikaya and H. Budak, Opial type inequalities for conformable fractional integrals, Journal of Applied Analysis, in press.
[19] M. Z. Sarikaya and H. Budak, New inequalities of Opial type for conformable fractional integrals, Turkish Journal of Mathematics, (2017) 41: 1164 - 1173.
[20] M. Z. Sarikaya and C.C. Bilisik, Some Opial type inequalities for conformable fractional integrals, July 2018, AIP Conference Proceedings 1991(1):020013.
[21] F. H. Wong, W. C. Lian, S. L. Yu and C. C. Yeh, Some generalizations of Opial's inequalities on time scales, Taiwanese Journal of Mathematics, Vol. 12, Number 2, April 2008, Pp. 463–471.
[22] C.-J. Zhao and W.-S. Cheung, On Opial-type integral inequalities and applications. Math. Inequal. Appl. 17 (2014), no. 1, 223–232.

Düzce University
Faculty of Science and Arts
Department of Mathematics
Düzce, Turkey
E-mail address: sarikayamz@gmail.com

Düzce University
Faculty of Science and Arts
Department of Mathematics
Düzce, Turkey
E-mail address: candancan@duzce.edu.tr, candancanbilisik@gmail.com