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A FINITE DIFFERENCE METHOD FOR SOLUTION OF NONLINEAR TWO POINT BOUNDARY VALUE PROBLEM WITH A NEUMANN BOUNDARY CONDITIONS

PRAMOD KUMAR PANDEY, FAISAL AL-SHOWAIKH

Abstract. A finite difference scheme for the solution of two point boundary value problem in ordinary differential equations subject to the Neumann boundary conditions presented in this article. The propose scheme is tested on linear and non-linear problems. The solution of the discretized problems was solved by iterative methods, i.e. Gauss-Seidel and Newton-Raphson method. The computational results demonstrate reliability and efficiency of the developed finite difference method. Moreover, numerical results confirm that scheme has second order accuracy.

1. Introduction

Two point boundary value problems occur in all branches of engineering and science. In these problems the boundary conditions are specified at two points. In general the governing differential equation of such problems is nonlinear and often conveniently solved with finite difference methods using a uniform mesh spacing h. We consider the general second order nonlinear differential equation

$$y'' = f(x, y, y') \tag{1}$$

subject to the boundary conditions

$$y'(a) = \alpha, \quad y'(b) = \beta \tag{2}$$

Here α and β are finite constants, $-\infty < a \leq x \leq b < \infty$. We assume that, for $a \leq x \leq b$ and $-\infty < y, y' < \infty$.

In general with different boundary conditions other than (2) if
(i) $f(x, y, y')$ is continuous
(ii) $\partial f/\partial y$ and $\partial f/\partial y'$ exist and are continuous, and
(iii) $\partial f/\partial y > 0$ and $|\partial f/\partial y'| \leq W$, for some positive constant W
then the problem (1) posses a unique solution [1]. In this article it is assumed that the unique solution of equation (1) exists and the specific restrictions on f to ensure the existence and uniqueness will not be considered [2].

An $O(h^4)$ difference scheme for the general problem (1) with mixed boundary conditions proposed in [3]. However, in practice it is often required to consider well-suited schemes for different types of nonlinear problems. For example, the simple $O(h^2)$, $O(h^4)$ or higher order methods reported there in [4–7] and arbitrary accurate order difference schemes [8] for two point boundary value problem with Dirichlet boundary conditions. Recently a difference schemes of high order accuracy for the problem (1) with mixed

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boundary conditions reported in [9].

The objective of this article is to develop an economical $O(h^2)$ difference scheme that is well suited for particular types of nonlinear problems. The computational complexity of the scheme increases with the desired level of accuracy, so for one evaluation of $f(x, y, y')$ is required at each mesh point in the $O(h^2)$ scheme.

In making an evaluation of the performance of method, there is a balance between the level of accuracy achieved and computational efficiency of the method. An important practical consideration in the solution of nonlinear problems is the choice of an efficient method, but often the particular equation or test problem to be solved will restrict the choices among the methods. The Newton-Raphson iterative method, it converges quadratically applied to test problems to get its solution. In section 2 we describe the finite difference method. In section 3 we discuss the derivation of the method and also obtain the local truncation error. In section 4 consider test problems to illustrate the method and its convergence. A summary of the result and conclusion is given in section 5.

2. The finite difference method

Let N be a positive integer, mesh spacing $h = (b-a)/N$ and $x_i = a+ih, i = 0, 1, \cdots, N$. The values of exact solution $y(x)$ at mesh point x_i are denoted by y_i, similarly $y'_i = y'(x_i), y''_i = f(x_i, y_i, y'_i) = f_i$.

We also set $x_{i+\frac{1}{2}} = x_i + \frac{h}{2}, i = 0, 1, \cdots, N-1$, $y_{i+\frac{1}{2}} = y(x_i + \frac{h}{2})$, $y'_{i+\frac{1}{2}} = y'(x_i + \frac{h}{2})$ and $y''_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}, y'_{i+\frac{1}{2}}) = f_{i+\frac{1}{2}}$.

Let

$$\hat{y}_{\frac{1}{2}} = \frac{26y_{\frac{3}{2}} - y_{\frac{5}{2}} - 24hy'_0}{25} \tag{3}$$

$$\hat{y}'_{\frac{1}{2}} = \frac{y_{\frac{5}{2}} - y_{\frac{3}{2}} + 3hy'_0}{4h} \tag{4}$$

$$\hat{y}_{N-\frac{1}{2}} = \frac{26y_{N-\frac{3}{2}} - y_{N-\frac{5}{2}} + 24hy'_N}{25} \tag{5}$$

$$\hat{y}'_{N-\frac{1}{2}} = \frac{y_{N-\frac{3}{2}} - y_{N-\frac{5}{2}} + 3hy'_N}{4h} \tag{6}$$

set

$$\hat{f}_{\frac{1}{2}} = f(x_{\frac{1}{2}}, \hat{y}_{\frac{1}{2}}, \hat{y}'_{\frac{1}{2}}),$$
$$\hat{f}_{N-\frac{1}{2}} = f(x_{N-\frac{1}{2}}, \hat{y}_{N-\frac{1}{2}}, \hat{y}'_{N-\frac{1}{2}})$$

Let

$$\hat{\hat{y}}_{\frac{1}{2}} = \hat{y}_{\frac{1}{2}} - \frac{23h^2}{25}\hat{f}_{\frac{1}{2}} \tag{7}$$

$$\hat{\hat{y}}_{N-\frac{1}{2}} = \hat{y}_{N-\frac{1}{2}} - \frac{23h^2}{25}\hat{f}_{N-\frac{1}{2}} \tag{8}$$

$$\bar{y}'_{i+\frac{1}{2}} = \begin{cases} \frac{3(y_{i+\frac{3}{2}}-y_{i+\frac{1}{2}})+hy'_{i-1}}{4h}, & i=1 \\ \frac{y_{i+\frac{3}{2}}-y_{i-\frac{1}{2}}}{2h}, & 2 \le i \le N-3 \\ \frac{3(y_{i+\frac{1}{2}}-y_{i-\frac{1}{2}})+hy'_{i+2}}{4h}, & i=N-2 \end{cases} \tag{9}$$

and finally set

$$\bar{f}'_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}, \bar{y}'_{i+\frac{1}{2}})$$

Following the idea in [10], at each $x_{i+\frac{1}{2}}$ we propose the following discretization of the differential equation (1),

$$\begin{aligned} -2y_{i+\frac{1}{2}} + y_{i+\frac{3}{2}} &= -\hat{\hat{y}}_{i-\frac{1}{2}} + h^2\bar{f}_{i+\frac{1}{2}}, && \text{i=1} \\ y_{i-\frac{1}{2}} - 2y_{i+\frac{1}{2}} + y_{i+\frac{3}{2}} &= h^2\bar{f}_{i+\frac{1}{2}}, && 2 \le i \le N-3 \\ y_{i-\frac{1}{2}} - 2y_{i+\frac{1}{2}} &= -\hat{\hat{y}}_{i+\frac{3}{2}} + h^2\bar{f}_{i+\frac{1}{2}}, && i=N-2 \end{aligned} \tag{10}$$

The non-linear $(N-2)\times(N-2)$ system obtained from (10) in unknown $y_{\frac{3}{2}}, \dots y_{N-\frac{3}{2}}$, can be solved using the Newton-Raphson method. Let $u_{i+\frac{1}{2}}$ be approximate value of $y_{i+\frac{1}{2}}$, $i=1,\dots,N-2$. Using these approximate values, by second order interpolation, we have an approximate value u_{i-1} of y_{i-1}, $i=1,\dots,N+1$.

$$u_{i-1} = \begin{cases} u_{i-\frac{1}{2}} - \frac{h}{2}y'_{i-1}, & i=1 \\ \frac{1}{2}(u_{i-\frac{1}{2}} + u_{i-\frac{3}{2}}), & 2 \le i \le N \\ u_{i-\frac{3}{2}} + \frac{h}{2}y'_{i-1}, & i=N+1 \end{cases} \tag{11}$$

Note that in (11) $u_{\frac{1}{2}}$ and $u_{N-\frac{1}{2}}$ are approximate values of $\hat{y}_{\frac{1}{2}}$ and $\hat{y}_{N-\frac{1}{2}}$ respectively.

3. Derivation of the finite difference scheme

In this section we discuss the derivation of finite difference method and the local truncation error associated with it. We need $O(h^4)$-approximation for $y_{\frac{1}{2}}$.
Let

$$\hat{y}_{\frac{1}{2}} = a_0 \cdot y_{\frac{3}{2}} + a_1 \cdot y_{\frac{5}{2}} + b_0 \cdot h \cdot y'_0 \tag{12}$$

By Taylor series expansion about $y_{\frac{1}{2}}$ of terms in (12), comparing the coefficients of $h^p, p = 0, 1, 2$ both sides and solving the system of equations, we have

$$(a_0, a_1, b_0) = (\frac{26}{25}, -\frac{1}{25}, -\frac{24}{25}) \tag{13}$$

Thus (12) can be rewritten as

$$\hat{y}_{\frac{1}{2}} = y_{\frac{1}{2}} + \frac{23h^2}{25}y''_{\frac{1}{2}} + O(h^4) \tag{14}$$

Next we need $O(h^2)$-approximation for $y'_{\frac{1}{2}}$. Let

$$h\hat{y}'_{\frac{1}{2}} = a_2 \cdot y_{\frac{3}{2}} + a_3 \cdot y_{\frac{5}{2}} + b \cdot h \cdot y'_0 \tag{15}$$

proceed as above we obtained

$$(a_2, a_3, b_1) = (\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}) \tag{16}$$

Thus we can write (15) as

$$\hat{y}'_{\frac{1}{2}} = y'_{\frac{1}{2}} + O(h^2) \tag{17}$$

With the help (14) and (17) we find that

$$\hat{f}_{\frac{1}{2}} = f_{\frac{1}{2}} + O(h^2) \tag{18}$$

Let

$$\hat{\hat{y}}_{\frac{1}{2}} = \hat{y}_{\frac{1}{2}} + b_2 \cdot h^2 \cdot \hat{f}_{\frac{1}{2}} \tag{19}$$

With the help of (14) and (18), from (19) we find that if $b_2 = -\frac{23}{25}$ then

$$\hat{\hat{y}}_{\frac{1}{2}} = y_{\frac{1}{2}} + O(h^4) \tag{20}$$

Similarly we can find $O(h^4)$-approximation for $y_{N-\frac{1}{2}}$ i.e.

$$\hat{\hat{y}}_{N-\frac{1}{2}} = y_{N-\frac{1}{2}} + O(h^4) \tag{21}$$

Further let

$$h \cdot \bar{y}'_{\frac{3}{2}} = a_4 \cdot y_{\frac{3}{2}} + a_5 \cdot y_{\frac{5}{2}} + b_3 \cdot h \cdot y'_0 \tag{22}$$

By Taylor expansion, from (22) we fined that if

$$(a_4, a_5, b_3) = (-\frac{3}{4}, \frac{3}{4}, \frac{1}{4})$$

then

$$\bar{y}'_{\frac{3}{2}} = y'_{\frac{3}{2}} + O(h^2) \tag{23}$$

Finally let

$$A_0 \cdot \hat{\hat{y}}_{\frac{1}{2}} + A_1 \cdot y_{\frac{3}{2}} + A_2 \cdot y_{\frac{5}{2}} = B_0 \cdot h^2 \cdot \bar{f}_{\frac{3}{2}} \tag{24}$$

By Taylor expansion, using (20) and (23), from (24) we find that if

$$(A_0, A_1, A_2, B_0) = (1, -2, 1, 1)$$

then

$$-2y_{3/2} + y_{5/2} = -\hat{\hat{y}}_{1/2} + h^2 \cdot f_{3/2} + \theta_1(h) \tag{25}$$

Similarly we can discretize the differential equation (1) at each $x_{i+\frac{1}{2}}$ and obtain

$$y_{i-\frac{1}{2}} - 2y_{i+\frac{1}{2}} + y_{i+3/2} = h^2 f_{i+\frac{1}{2}} + \theta_i(h), \quad 2 \le i \le N-3 \tag{26}$$

$$y_{i-\frac{1}{2}} - 2y_{i+\frac{1}{2}} = -\hat{\hat{y}}_{i+\frac{3}{2}} + h^2 f_{i+\frac{1}{2}} + \theta_i(h), \quad i = N-2 \tag{27}$$

where $\theta_i(h) = O(h^4)$, $i = 1, 2, \ldots, N-2$.

Thus we define the discretization (10) for differential equation (1) and with the help of (25), (26) and (27) estimate local truncation error associated with (10). The emphasis here is on the actual formulae required for computation, the lengthy details concerning complete expansions for the local truncation errors $\theta_i(h)$ and formal bounds on errors are left out of this section in the interest of shortness.

4. Illustrations

In this section we have considered linear and nonlinear model problems to illustrate the method (10) presented in this article. We have solved each of the model problems for $N = 8, 16, \ldots, 256$. In computation of maximum absolute error MAE between the analytical solution $y(x_i)$ and computed numerical solution y_i of the problem, we have used the following formula,

$$\text{MAE} = \max_{1\leq i\leq N} |y(x_i) - y_i|$$

In the tables we have presented MAE using computer notation, i.e. $.164797e - 2$ for $.164797 \times 10^{-2}$ and computed order of the method. We have respectively applied Gauss-Seidel and Newton-Raphson method to solve the system of linear and nonlinear equations those arise from the method (10). The iteration is continued until either the maximum difference between two successive iterates is less than 10^{-8} or the number of iterations reached 10^5. All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC.

$$MAU = \max_i |y(x_i) - y_i| \qquad , \qquad i = 0, 1, \ldots, N$$

Problem 1. Consider the model linear boundary value problem

$$y'' = -2(1 - 2x^2)y, \quad 1 < x < 2$$

subject to boundary conditions

$$y'(0) = 0 \text{ and } y'(1) = \frac{-2}{\exp(1)}$$

The analytical solution is $y(x) = \exp(-x^2)$. The numerical results are given in table 1.

Problem 2. Consider the nonlinear diffusion equation in biology and reported in [11],

$$y'' = \frac{y}{1+y} + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$y'(0) = 0 \quad \text{and} \quad y'(1) = \frac{\exp(1) - 1}{\exp(1)}$$

and $f(x)$ calculated so that analytical solution is $y(x) = \exp(-x) + x - 1$. The numerical results are given in table 2.

Problem 3. Consider the model nonlinear boundary value problem

$$y'' = y^2y' + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$y'(0) = 1 \text{ and } y'(1) = \frac{\exp(1) + \exp(-1)}{2}$$

and $f(x)$ calculated so that analytical solution is $y(x) = \sinh(x)$. The numerical results are given in table 3.

Problem 4. Consider the model nonlinear boundary value problem

$$y'' = y' - \sin(y) - y^2 + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$y'(0) = 1 \text{ and } y'(1) = \frac{1}{4}$$

and $f(x)$ calculated so that analytical solution is $y(x) = \frac{-1}{1+x}$. The numerical results are given in table 4.

Table 1. Maximum absolute error and order (Problem 1).

	N					
	8	16	32	64	128	256
MAE	.164797e-2	.146400e-3	.458975e-4	.133228e-4	.371701e-5	.144364e-5
Order	–	3.4972	1.6734	1.7845	1.8416	1.3644

Table 2. Maximum absolute error and order (Problem 2).

	N					
	8	16	32	64	128	256
MAE	.224702e-1	.597457e-2	.154729e-2	.394165e-3	.994576e-4	.244385e-4
Order	–	1.9111	1.9490	1.9728	1.9866	2.0249

Table 3. Maximum absolute error and order (Problem 3).

	N					
	8	16	32	64	128	256
MAE	.416058e-2	.577071e-3	.851308e-4	.163014e-4	.550293e-5	.155248e-5
Order	–	2.8499	2.7609	2.3846	1.5667	1.8256

Table 4. Maximum absolute error and order (Problem 4).

	N					
	8	16	32	64	128	256
MAE	.193296e-1	.554621e-2	.146570e-2	.375294e-3	.948464e-4	.238187e-4
Order	–	1.8012	1.9199	1.9654	1.9843	1.9934

We have considered second order boundary value problems in ordinary differential equation to test the computational efficiency of the proposed finite difference method (10). We observed in numerical experiment for different values of N presented in tables, as N increases, i.e. h decreases, then maximum absolute error in computed solution decrease. It is evident from the tabulated results that method (10) is convergent and order of accuracy is at least quadratic in problems 2 and 4.

5. Conclusion

We introduced a second-order finite-difference method for a general two point non-linear second-order boundary value problems with Neumann boundary conditions. The proposed method (10) is economical as it requires only one function evaluation at each nodal point. The numerical results verify the second-order convergence of the method. The proposed method is useful for complicated non-linear equations and improvement in proposed finite difference method is possible. Work in this direction is in progress.

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Dyal Singh College (University of Delhi)
Department of Mathematics
Lodhi Road, New Delhi 110003 India
E-mail address: pramod_10p@hotmail.com

University of Bahrain
Department of Mathematics, College of Science
Sukhair, Bahrain
E-mail address: alshowaikh@gmail.com