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EXPONENTIALLY GENERAL CONVEX FUNCTIONS

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Abstract. In this paper, we define and introduce some new classes of the exponentially convex functions involving an arbitrary function, which is called the exponentially general convex function. We investigate several properties of the exponentially general convex functions and discuss their relations with convex functions. Optimality conditions are characterized by a class of variational inequalities, which is called the exponentially general variational inequality. Several new results characterizing the exponentially general convex functions are obtained. Results obtained in this paper can be viewed as significant improvement of previously known results.

1. Introduction

Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. The development of convexity theory can be viewed as the simultaneous pursuit of two different lines of research. It have been shown that the minimum of the differentiable convex functions on the convex set can be characterized by the variational inequalities. Variational inequalities, the origin of which can be raced back to Bernoulli's brothers, Euler and Lagrange, provide us a power tool to discuss the behaviour of solutions (regarding its existence, uniqueness and regularity) to important classes of problems. Variational inequality theory also enables us to develop highly efficient and powerful new numerical methods to solve nonlinear problems, see [4, 12, 13, 14, 16, 17, 18, 19, 23, 27]. In recent years, various extensions and generalizations of convex functions and convex sets have been considered and studied using innovative ideas and techniques. It is known that more accurate and inequalities can be obtained using the logarithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, which have important applications in information theory, big data analysis, machine learning and statistic. Exponentially convex functions have appeared as significant generalization of the convex functions, the origin of which can be traced back to Bernstein[6]. Avriel[3, 4] introduced the concept of r-convex functions. Antczak [2] considered the (r, p) convex functions and discussed their applications in mathematical programming and optimization theory. Awan et al [3] also introduced a new class of exponentially convex functions. It worth mentioning that Noor and Noor [17, 18, 19, 20, 21] shown that the r-convex functions are equivalent to the exponentially convex functions. This alternative equivalent form has important applications in various fields of pure and applied sciences and has inspired much interest. It is worth mentioning that all these classes of exponentially convex functions have important applications in information sciences, data mining and statistics, see, for example, [1, 2, 3, 4, 5, 6, 7, 9, 17, 18, 19, 20, 21, 23, 26, 30] and the references therein.

It is known that a set may not be convex set. However, a set can be made convex set with respect to an arbitrary function. Motivated by this fact, Noor [24] introduced the

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concept of general convex sets involving an arbitrary function. It has been shown that the minimum of a differentiable general convex function on the general convex set can be characterized by the general variational inequalities, which were introduced by Noor [11] in 1988. The technique of the general variational inequalities can be used to consider the nonsymmetric, odd-order obstacle boundary values problems. Cristescu at al [8] have investigated algebraic and topological properties of the g-convex sets defined by Noor [14] in order to deduce their shape. They are a subclass of star-shaped sets, which have also a Youness [29] type convexity. A representation theorem based on extremal points is given for the class of bounded g-convex sets. Examples showing that this convexity is a frequent property in connection with a wild range of applications are given. For the formulation, applications, numerical methods,sensitivity analysis and other aspects of general variational inequalities, see [11, 12, 13, 14, 16, 22, 27] and the references es therein.

We would like to point out that the general convex functions and exponentially convex functions are two distinct generalizations of the convex functions, which have played a crucial and significant role in the development of various branches of pure and applied sciences. It is natural to unify these concepts. Motivated by these facts and observations, we now introduce a new class of convex functions, which is called exponentially general convex functions. We discuss the basic properties of the general exponentially convex functions. It is has been shown that the general exponentially convex(concave) have nice nice properties which convex functions enjoy. Several new concepts have been introduced and investigated. We show that the local minimum of the general exponentially convex functions is the global minimum. The optimal conditions of the differentiable exponentially convex functions are characterized by a class of variational inequalities, which is itself an interesting problem. The difference (sum) of the exponentially convex function and exponentially affine convex function is again a exponentially convex function. The ideas and techniques of this paper may be starting point for further research in these areas.

2. Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. We recall the well known facts and basic concepts.

Definition 1. [4]. *The set K in H is said to be a convex set, if*

$$u + t(v-u) \in K, \qquad \forall u, v \in K, t \in [0,1].$$

Definition 2. *A function F is said to be convex function, if*

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0,1]. \tag{1}$$

If the convex function F is differentiable, then $u \in K$ is the minimum of the F, if and only if, $u \in K$ satisfies the inequality

$$\langle F'(u), v-u \rangle \geq 0, \quad \forall v \in K. \tag{2}$$

The inequalities of the type (2) are called the variational inequalities, which were introduced and studied by Stampacchia [27] in 1964. For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, see [4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 22, 27] and the references therein.

We now define the exponentially convex functions, which are mainly due to Noor and

Noor [17, 18, 19, 20, 21, 23].

Definition 3. *A function F is said to be exponentially convex function, if*

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K, \quad t \in [0,1],$$

which can be written in the equivalent form

Definition 4. *A function F is said to be exponentially convex function, if*

$$F((1-t)a+tb) \leq \log[(1-t)e^{F(a)} + te^{F(b)}], \quad \forall a, b \in K, \quad t \in [0,1], \tag{3}$$

which is mainly due to Avriel [3, 4]. Antczak [2] discussed the applications of exponentially convex functions in the mathematical programming and optimization theory. A function is called the exponentially concave function f, if $-f$ is exponentially convex function. For the properties, generalizations and applications of the exponentially convex functions, see [1, 2, 3, 4, 5, 17, 25].
Noor [17] and Noor and Noor [18] proved that the minimum of a differentiable exponentially convex functions can be characterized by the inequality

$$\langle e^{F(u)} F'(u), v-u \rangle \geq 0, \quad \forall v \in K, \tag{4}$$

which is called exponentially variational inequality. For more details, see [18].

For the applications of the exponentially convex functions in the mathematical programming and information theory, see Antczak [2], Alirezaei and Mathar[1] and Pal and Wong [25].

Definition 5. [4].*The set K_g in H is said to be general convex set, if there exists an arbitrary function g, such that*

$$(1-t)u + tg(v) \in K_g, \quad \forall u, v \in H : u, g(v) \in K_g, t \in [0,1].$$

We now discuss some special cases of the general convex sets.

(I). If $g = I$, the identity operator, then general convex set reduces to the classical convex set. Clearly every convex set is a general convex set, but the converse is not true. Cristescu et al[8] discussed various applications of the general convex sets related to the necessity of adjusting investment or development projects out of environmental or social reasons. For example, the easiest manner of constructing this kind of convex sets comes from the problem of modernizing the railway transport system. Shape properties of the general convex sets with respect to a projection are investigated.

(II). If $g(v) = mv, \quad m \in [0.1]$, then the general convex set becomes the m-convex set, which is mainly due to Toader[28].

Definition 6. [25] *The set K_m is said to be m-convex set, if*

$$(1-t)u + tmv \in K_m, \qquad \forall u, v \in K_m, t \in [0,1].$$

For the sake of simplicity, we always assume that $\forall u, v \in H : u, g(v) \in K_g$, unless otherwise.

Definition 7. *A function F is said to be general convex(g-convex) function, if there exists an arbitrary non-negative function g, such that*

$$F((1-t)u + tg(v)) \leq (1-t)F(u) + tF(g(v)), \quad \forall u, g(v) \in K_g, \quad t \in [0,1]. \tag{5}$$

The general convex functions were introduced by Noor [14]. Noor [12, 13, 14, 23] proved that the minimum $u \in H : g(u) \in K_g$ of the differentiable general convex functions F can be characterized by the class of variational inequalities of the type:

$$\langle F'(u), g(v) - u\rangle \geq 0, \quad \forall v \in H : g(v) \in K_g, \tag{6}$$

which is known as general variational inequalities. For the applications of the general variational inequalities in various branches of pure and applied sciences, see [11, 12, 13, 14, 16, 17, 22, 29, 30] and the references therein.

We now introduce some new concepts of exponentially general convex functions, which is the main motivation of this paper.

Definition 8. *A function F is said to be exponentially general convex function with respect to an arbitrary non-negative function g, if*

$$e^{F((1-t)u+tg(v))} \leq (1-t)e^{F(u)} + te^{F(g(v))}, \quad \forall u, g(v) \in K_g, t \in [0,1]. \tag{7}$$

or equivalently

Definition 9. *A function F is said to be exponentially general convex function with respect to an arbitrary non-negative function g, if,*

$$F((1-t)u + tg(v)) \leq \log[(1-t)e^{F(u)} + te^{F(g(v))}], \quad \forall u, g(v) \in K_g, t \in [0,1]. \tag{8}$$

A function is called the exponentially general concave function f, if $-f$ is an exponentially general convex function.

Definition 10. *A function F is said to be exponentially general affine convex function with respect to an arbitrary non-negative function g, if*

$$e^{F((1-t)u+tg(v))} = (1-t)e^{F(u)} + te^{F(g(v))}, \quad ,\forall u, g(v) \in K_g, t \in [0,1]. \tag{9}$$

If $g = I$, the identity operator, then exponentially general convex functions reduce to: the exponentially convex functions. For the properties and applications of the exponentially convex functions, see [17, 18, 19, 20, 21].

If $g(v) = mv, \quad m \in [0,1]$, then Definition 10 reduces to:

Definition 11. *A function F is said to be exponentially m-convex function, if*

$$e^{F((1-t)u+tmv)} \leq (1-t)e^{F(u)} + te^{F(mv)}, \quad \forall u, v \in K, \quad t \in [0,1], \tag{10}$$

which can be rewritten in the following equivalent form.

Definition 12. *A function F is said to be an exponentially m-convex function, if*

$$F((1-t)u + tmv) \leq \log[(1-t)e^{F(u)} + te^{F(mv)}], \quad \forall u, v \in K, \quad t \in [0,1]. \tag{11}$$

For the properties of the exponentially m-convex functions, see Noor and Noor [17, 18, 19, 20, 21].

Definition 13. *The function F on the general convex set K_g is said to be exponentially general quasi convex, if*

$$e^{F(u+t(g(v)-u))} \leq \max\{e^{F(u)}, e^{F(g(v))}\}, \quad \forall u, g(v) \in K_g, t \in [0,1].$$

Definition 14. *The function F on the general convex set K_g is said to be exponentially general log-convex, if*

$$e^{F(u+t(g(v)-u))} \leq (e^{F(u)})^{1-t}(e^{F(g(v))})^{t}, \quad \forall u, g(v) \in K_g, t \in [0,1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} e^{F(u+t(g(v)-u)} &\leq (e^{F(u)})^{1-t}(e^{F(g(v))})^{t} \\ &\leq (1-t)e^{F(u)}+te^{F(g(v))}) \\ &\leq \max\{e^{F(u)},e^{F((v))}\}. \end{aligned}$$

This shows that every exponentially general log-convex function is a exponentially general convex function and every exponentially general convex function is a exponentially general quasi-convex function. However, the converse is not true.
We now define the exponentially general convex functions on the interval
$K_g = I_g = [a, g(b)]$.

Definition 15. *Let $I_g = [a, g(b)]$. Then F is exponentially general convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & g(b) \\ e^{F(a)} & e^{F(x)} & e^{F(g(b))} \end{vmatrix} \geq 0; \quad a \leq x \leq g(b).$$

One can easily show that the following are equivalent:

(1) F is exponentially convex function.
(2) $e^{F(x)} \leq e^{F(a)} + \frac{e^{F(g(b))}-e^{F(a)}}{g(b)-a}(x-a)$.
(3) $\frac{e^{F(x)}-e^{F(a)}}{x-a} \leq \frac{e^{F(g(b))}-e^{F(a)}}{g(b)-a}$.
(4) $(g(b)-x)e^{F(a)} + (a-g(b))e^{F(x)} + (x-a)e^{F(g(b))} \geq 0$.
(5) $\frac{e^{F(a)}}{(g(b)-a)(a-x)} + \frac{e^{F(x)}}{(x-g(b))(a-x)} + \frac{e^{F(g(b))}}{(g(b)-a)(x-g(b))} \leq 0$,

where $x = (1-t)a + tg(b) \in [a, g(b)]$.

3. Main Results

In this section, we consider some basic properties of exponentially general convex functions.

Theorem 1. *Let F be a strictly exponentially general convex function. Then any local minimum of F is a global minimum.*

Proof. Let the strictly exponentially convex function F have a local minimum at $u \in K_g$. Assume the contrary, that is, $F(g(v)) < F(u)$ for some $g(v) \in K_g$. Since F is strictly exponentially general convex function, so

$$e^{F(u+t(g(v)-u))} < te^{F(g(v))} + (1-t)e^{F(u)}, \quad \text{for} \quad 0 < t < 1.$$

Thus

$$e^{F(u+t(g(v)-u))} - e^{F(u)} < -t[e^{F(g(v))} - e^{F(u)}] < 0,$$

from which it follows that

$$e^{F(u+t(g(v)-u)} < e^{F(u)},$$

for arbitrary small $t > 0$, contradicting the local minimum. □

Theorem 2. *If the function F on the general convex set K_g is exponentially general convex, then the level set*

$$L_\alpha = \{u \in K_g : e^{F(u)} \leq \alpha, \quad \alpha \in R\}$$

is a general convex set.

Proof. Let $u, g(v) \in L_\alpha$. Then $e^{F(u)} \leq \alpha$ and $e^{F(g(v))} \leq \alpha$. Now, $\forall t \in (0,1)$, $g(w) = u + t(g(v) - u) \in K_g$, since K_g is a convex set. Thus, by the exponentially general convexity of F, we have

$$Fe^{(u+t(g(v)-u)} \leq (1-t)e^{F(u)} + te^{F(g(v))} \leq (1-t)\alpha + t\alpha = \alpha,$$

from which it follows that $u + t(g(v) - u) \in L_\alpha$ Hence L_α is a general convex set. □

Theorem 3. *The function F is exponentially general convex function, if and only if,*

$$epi(F) = \{u, \alpha) : u \in K_g : e^{F(u)} \leq \alpha, \alpha \in R\}$$

is a general convex set.

Proof. Assume that$F$ is exponentially general convex function. Let

$$(u, \alpha), \quad (g(v), \beta) \in epi(F).$$

Then it follows that $e^{F(u)} \leq \alpha$ and $e^{F(g(v))} \leq \beta$. Hence, we have

$$e^{F(u+t(g(v)-u))} \leq (1-t)e^{F(u)} + te^{F(g(v))} \leq (1-t)\alpha + t\beta,$$

which implies that

$$((1-t)u + tg(v)), (1-t)\alpha + t\beta) \in epi(F).$$

Thus $epi(F)$ is a general convex set. Conversely, let $epi(F)$ be a general convex set. Let $u, g(v) \in K_g$. Then $(u, e^{F(u)} \in epi(F)$ and $(g(v, e^{F(g(v))}) \in epi(F)$. Since $epi(F)$ is a general convex set, we must have

$$(u + t(g(v) - u), (1-t)e^{F(u)} + te^{F(g(v))} \in epi(F),$$

which implies that

$$e^{F((1-t)u+tg(v))} \leq (1-t)e^{F(u)} + te^{F(g(v))}.$$

This shows that F is an exponentially general convex function. □

Theorem 4. *The function F is exponentially general quasi convex, if and only if, the level set*

$$L_\alpha = \{u \in K_g, \alpha \in R : e^{F(u)} \leq \alpha\}$$

is a general convex set.

Proof. Let $u, g(v) \in L_\alpha$. Then $u, g(v) \in K_g$ and $\max(e^{F(u)}, e^{F(g(v))}) \leq \alpha$. Now for $t \in (0,1), g(w) = u + t(g(v) - u) \in K_g$. We have to prove that $u + t(g(v) - u) \in L_\alpha$. By the exponentially general convexity of F, we have

$$e^{F(u+t(g(v)-u))} \leq \max\left(e^{F(u)}, e^{F(g(v))}\right) \leq \alpha,$$

which implies that $u + t(g(v) - u) \in L_\alpha$, showing that the level set L_α is indeed a general convex set.

Conversely, assume that L_α is a general convex set. Then, $\forall \quad u, g(v) \in L_\alpha, t \in [0,1]$, $u + t(g(v) - u) \in L_\alpha$. Let $u, g(v) \in L_\alpha$ for

$$\alpha = max(e^{F(u)}, e^{F(g(v))}) \quad \text{and} \quad e^{F(g(v))} \leq e^{F(u)}.$$

Then from the definition of the level set L_α, it follows that

$$e^{F(u+t(g(v)-u)} \leq \max\left(e^{F(u)}, e^{F(g(v))}\right) \leq \alpha.$$

Thus F is an exponentially general quasi convex function. This completes the proof. □

Theorem 5. *Let F be an exponentially general convex function. Let $\mu = \inf_{u\in K} F(u)$. Then the set*

$$E = \{u \in K_g : e^{F(u)} = \mu\}$$

is a general convex set of K_g. If F is strictly exponentially general convex function , then E is a singleton.

Proof. Let $u, g(v) \in E$. For $0 < t < 1$, let $g(w) = u+t(g(v)-u)$. Since F is a exponentially general convex function, then

$$\begin{aligned} F(g(w)) &= e^{F(u+t(g(v)-u)} \\ &\leq (1-t)e^{F(u)} + te^{F(g(v))} = t\mu + (1-t)\mu = \mu, \end{aligned}$$

which implies $g(w) \in E$. and hence E is a general convex set. For the second part, assume to the contrary that $F(u) = F(g(v)) = \mu$. Since K is a general convex set, then for $0 < t < 1, u + t(g(v) - u) \in K_g$. Further, since F is strictly exponentially general convex function, so

$$e^{F(u+t(g(v)-u))} < (1-t)e^{F(u)} + te^{F(g(v))} = (1-t)\mu + t\mu = \mu.$$

This contradicts the fact that $\mu = \inf_{u\in K_g} F(u)$ and hence the result follows. □

Theorem 6. *If the function F is exponentially general convex such that*

$$e^{F(g(v))} < e^{F(u)}, \forall u, g(v) \in K_g,$$

then F is a strictly exponentially general quasi function.

Proof. By the exponentially general convexity of the function F, we have

$$\begin{aligned} e^{F(u+t(g(v)-u))} &\leq (1-t)e^{F(u)} + te^{F(g(v))}, \forall u, g(v) \in K_g, t \in [0,1] \\ &< e^F(u), \end{aligned}$$

since $e^{F(g(v))} < e^{F(u)}$, which shows that the function F is strictly exponentially general quasi convex. □

We now show that the difference of exponentially convex function and exponentially affine convex function is again an exponentially general convex function.

Theorem 7. *Let f be a exponentially general affine convex function. Then F is a exponentially general convex function, if and only if, $H = F-$ is a exponentially convex function.*

Proof. Let f be exponentially general affine convex function. Then

$$e^{f((1-t)u+tg(v))} = (1-t)e^{f(u)} + te^{f(g(v))}, \quad \forall u, g(v) \in K_g, t \in [0,1]. \tag{12}$$

From the exponentially general convexity of F, we have

$$e^{F((1-t)u+tg(v))} \leq (1-t)e^{F(u)} + te^{F(g(v))}, \quad \forall u, g(v) \in K_g, t \in [0,1]. \tag{13}$$

From (12) and (13), we have

$$\begin{aligned} &e^{F((1-t)u+tg(v))} - e^{f((1-t)u+tg(v))} \\ &\leq (1-t)(e^{F(u)} - e^{f(u)}) + t(e^{F(g(v))} - e^{f(g(v))}), \end{aligned} \tag{14}$$

from which it follows that

$$\begin{aligned} e^{H((1-t)u+tg(v))} &= e^{F((1-t)u+tg(v)))} - e^{f((1-t)f(u)+tf(g(v))} \\ &\leq (1-t)(e^{F(g(u))} - e^{f(g(u))}) + t(e^{F(g(v))} - e^{f(g(v))}), \end{aligned}$$

which show that $H = F - f$ is an exponentially general convex function.
The inverse implication is obvious. □

Definition 16. *A function F is said to be a exponentially general pseudo convex function, if there exists a strictly positive bifunction $b(.,.)$, such that*

$$\begin{aligned} e^{F(g(v))} &< e^{F(u)} \\ &\Rightarrow \\ e^{F(u+t(g(v)-u))} &< e^{F(g(u))} + t(t-1)b(g(v),u), \quad \forall u, g(v) \in K_g, t \in [0,1]. \end{aligned}$$

Theorem 8. *If the function F is exponentially general convex function such that*

$$e^{F(g(v))} < e^{F(u)},$$

then the function F is an exponentially general pseudo convex function.

Proof. Since $e^{F(g(v))} < e^{F(u)}$ and F is exponentially general convex function, then $\forall u, g(v) \in K_g, \quad t \in [0,1]$, we have

$$\begin{aligned} e^{F(u+(1-t)(g(v)-u))} &\leq e^{F(u)} + t(e^{F(g(v))} - e^{F(u)}) \\ &< e^{F(u)} + t(1-t)(e^{F(g(v))} - e^{F(u)}) \\ &= e^{F(u)} + t(t-1)(e^{F(u)} - e^{F(g(v))})) \\ &< e^{F(u)} + t(t-1)b(u,g(v)), \end{aligned}$$

where $b(u,g(v)) = e^{F(u)} - e^{F(g(v))} > 0$. This shows that the function F is exponentially general pseudo convex function. □

We now study some properties of the differentiable exponentially general convex functions.

Theorem 9. *Let F be a differentiable function on the general convex set K_g. Then the function F is exponentially general convex function, if and only if,*

$$e^{F(g(v))} - e^{F(u)} \geq \langle e^{F(u)} F'(g(u)), g(v) - u\rangle, \quad \forall g(v), u \in K_g. \tag{15}$$

Proof. Let F be a exponentially general convex function. Then

$$e^{F(u+t(g(v)-u))} \leq (1-t)e^{F(u)} + te^{F(g(v))}, \qquad \forall u, g(v) \in K_g,$$

which can be written as

$$e^{F(g(v))} - e^{F(u)} \geq \lim_{t \to 0} \{\frac{e^{F(u+t(g(v)-u))} - e^{F(u)}}{t}\} = \langle e^{F(u)} F'(u), g(v) - u)\rangle,$$

which is (15), the required result.

Conversely, let (15) hold. Then $\forall u, g(v) \in K_g, t \in [0,1]$,

$$g(v_t) = u + t(g(v) - u) \in K_g,$$

we have

$$\begin{aligned} e^{F(g(v))} - e^{F(g(v_t))} &\geq \langle e^{F(g(v_t))} F'(g(v_t)), g(v) - g(v_t))\rangle \\ &= (1-t)\langle e^{F(g(v_t))} F'(g(v_t)), g(v) - u\rangle. \end{aligned} \tag{16}$$

In a similar way, we have

$$\begin{aligned} e^{F(u)} - e^{F(g(v_t))} &\geq \langle e^{F(g(v_t))} F'(g(v_t)), u - g(v_t))\rangle \\ &= -t\langle e^{F(g(v_t))} F'(g(v_t)), g(v) - u\rangle. \end{aligned} \tag{17}$$

Multiplying (16) by t and (17) by $(1-t)$ and adding the resultant, we have

$$e^{F(u+t(g(v)-u))} \leq (1-t)e^{F(u)} + te^{F(g(v))},$$

showing that F is a exponentially general convex function. □

Remark 1. *From* (15)*, we have*

$$e^{F(g(v))-F(u)} - 1 \geq \langle F'(u), g(v) - u\rangle, \quad \forall g(v), u \in K_g,$$

which can be written as

$$F(g(v)) - F(u) \geq \log\{1 + \langle F'(u), g(v) - u\rangle, \} \quad \forall g(v), u \in K_g, \tag{18}$$

Changing the role of u and v in (18)*, we also*

$$F(u) - F(g(v)) \geq \log\{1 + \langle F'(g(v)), u - g(v)\rangle, \} \quad \forall g(v), u \in K_g, \tag{19}$$

Adding (18) *and* (19)*, we have*

$$\langle F'(u) - F'(g(v)), u - g(v)\rangle \geq (\langle F'(u), u - g(v)\rangle)(F'(g(v)), u - g(v)\rangle)$$

which express the monotonicity of the differential $F'(.)$ of the exponentially general convex function.

Theorem 9 enables us to introduce the concept of the exponentially monotone operators, which appears to be new ones.

Definition 17. *The differential $F'(.)$ is said to be exponentially general monotone, if*

$$\langle e^{F(u)}F'(u) - e^{F(g(v))}F'(g(v)), u - g(v)\rangle \geq 0, \quad \forall u, v \in H.$$

Definition 18. *The differential $F'(.)$ is said to be exponentially general pseudo-monotone, if*

$$\begin{aligned} &\langle e^{F(u)}F'(u), g(v) - u\rangle \geq 0, \\ &\quad \Rightarrow \\ &\langle e^{F(g(v))}F'(g(v)), g(v) - u\rangle \geq 0, \quad \forall u, v \in H. \end{aligned}$$

From these definitions, it follows that exponentially general monotonicity implies exponentially general pseudo-monotonicity, but the converse is not true.

Theorem 10. *Let F be differentiable exponentially general convex function. Then,* (15) *holds, if and only if, F' satisfies*

$$\langle e^{F(u)}F'(u) - e^{F(g(v))}F'(g(v)), u - g(v)\rangle \geq 0, \quad \forall u, g(v) \in K_g. \tag{20}$$

Proof. Let F be a exponentially general convex function. Then, from Theorem 9, we have

$$e^{F(g(v))} - e^{F(u)} \geq \langle e^{F(u)}F'(u), g(v) - u\rangle, \quad \forall u, g(v) \in K_g. \tag{21}$$

Changing the role of u and v in (21), we have

$$e^{F(u} - e^{F(g(v))} \geq \langle e^{F(g(v))}F'(g(v)), u - g(v))\rangle, \quad \forall u, g(v) \in K_g. \tag{22}$$

Adding (21) and (22), we have

$$\langle e^{F(u)}F'(u) - e^{F(g(v))}F'(g(v)), u - g(v)\rangle \geq 0,$$

which shows that F' is exponentially general monotone.

Conversely, from (20), we have

$$\langle e^{F(g(v)}F'(g(v)), u - g(v)\rangle \leq \langle e^{F(u)}F'(u), u - g(v))\rangle. \tag{23}$$

Since K_g is a general convex set, $\forall u, g(v) \in K_g, \quad t \in [0, 1]$,

$$g(v_t) = u + t(g(v) - u) \in K_g.$$

Taking $g(v) = g(v_t)$ in (23), we have

$$\begin{aligned}\langle e^{F(g(v_t))}F'(g(v_t)), u - g(v_t)\rangle &\leq \langle e^{F(u)}F'(u), u - g(v_t)\rangle \\ &= -t\langle e^{F(u)}F'(u), g(v) - u\rangle,\end{aligned}$$

which implies that

$$\langle e^{F(g(v_t))}F'(g(v_t)), g(v) - u\rangle \geq \langle e^{F(u)}F'(u), g(v) - u\rangle. \tag{24}$$

Consider the auxiliary function

$$g(t) = e^{F(u+t(g(v)-u)},$$

from which, we have

$$g(1) = e^{F(g(v))}, \quad g(0) = e^{F(u)}.$$

Then, from (24), we have

$$g'(t) = \langle e^{F(g(v_t))}F'(g(v_t)), g(v) - u\rangle \geq \langle e^{F(u)}F'(u), g(v) - u\rangle. \tag{25}$$

Integrating (25) between 0 and 1, we have

$$g(1) - g(0) = \int_0^1 g'(t)dt \geq \langle e^{F(u)}F'(u), g(v) - u\rangle.$$

Thus it follows that

$$e^{F(g(v))} - e^{F(u)} \geq \langle e^{F(u)}F'(u), g(v) - u\rangle,$$

which is the required (15). □

We now give a necessary condition for exponentially general pseudo-convex function.

Theorem 11. *Let F' be exponentially general pseudomonotone. Then F is a exponentially general pseudo-convex function.*

Proof. Let F' be a exponentially general pseudomonotone. Then, $\forall u, g(v) \in K_g$,

$$\langle e^{F(u)}F'(u), g(v) - u\rangle \geq 0.$$

implies that

$$\langle e^{F(g(v))}F'(g(v)), g(v) - u\rangle \geq 0. \tag{26}$$

Since K_g is a general convex set, $\forall u, g(v) \in K_g, \quad t \in [0, 1]$,

$$g(v_t) = u + t(g(v) - u) \in K_g.$$

Taking $g(v) = g(v_t)$ in (26), we have

$$\langle e^{F(g(v_t))}F'(g(v_t)), g(v) - u\rangle \geq 0. \tag{27}$$

Consider the auxiliary function

$$g(t) = e^{F(u+t(g(v)-u))} = e^{F(g(v_t))}, \quad \forall u, g(v) \in K_g, t \in [0, 1],$$

which is differentiable, since F is differentiable function. Then, using (27), we have

$$g'(t) = \langle e^{F(g(v_t))}F'(g(v_t)), g(v) - u)\rangle \geq 0.$$

Integrating the above relation between 0 to 1, we have

$$g(1) - g(0) = \int_0^1 g'(t)dt \geq 0,$$

that is,

$$e^{F(g(v))} - e^{F(u)} \geq 0,$$

showing that F is a exponentially general pseudo-convex function. $\square$

Definition 19. *The function F is said to be sharply exponentially general pseudo convex, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned}&\langle e^{F(u)}F'(u), g(v)-u\rangle \geq 0\\ &\Rightarrow\\ &F(g(v)) \geq e^{F(g(v)+t(u-g(v)))}, \quad \forall u, g(v) \in K_g, t \in [0,1].\end{aligned}$$

Theorem 12. *Let F be a sharply exponentially general pseudo convex function. Then*

$$\langle e^{F(g(v))}F'(g(v)), u-g(v)\rangle \geq 0, \quad \forall u, g(v) \in K_g.$$

Proof. Let F be a sharply exponentially general pseudo convex function. Then

$$e^{F(g(v))} \geq e^{F(g(v)+t(u-g(v)))}, \quad \forall u, g(v) \in K_g, t \in [0,1].$$

from which we have

$$0 \leq \lim_{t\to 0}\{\frac{e^{F(g(v)+t(u-g(v)))} - e^{F(g(v))}}{t}\} = \langle e^{F(g(v))}F'(g(v)), u-g(v)\rangle,$$

the required result. $\square$

We now discuss the optimality condition for the differentiable exponentially convex functions, which is the main motivation of our next result.

Theorem 13. *Let F be a differentiable general exponentially convex function. Then $u \in K_g$ is the minimum of the function F, if and only if, $u \in K_g$ satisfies the inequality*

$$\langle e^{F(u)}F'(u), g(v)-u\rangle \geq 0, \quad \forall u, g(v) \in K_g. \tag{28}$$

Proof. Let $u \in K_g$ be a minimum of the function F. Then

$$F(u) \leq F(g(v)), \forall v \in H : g(v) \in K_g.$$

from which, we have

$$e^{F(u)} \leq e^{F(g(v))}, \forall g(v) \in K_g. \tag{29}$$

Since K_g is a general convex set, so, $\forall u, g(v) \in K_g, \quad t \in [0,1]$,

$$g(v_t) = (1-t)u + tg(v) \in K_g.$$

Taking $g(v) = g(v_t)$ in (29), we have

$$0 \leq \lim_{t\to 0}\{\frac{e^{F(u+t(g(v)-u))} - e^{F(u)}}{t}\} = \langle e^{F(u)}F'(u), g(v)-u\rangle. \tag{30}$$

Since F is an exponentially general convex function, so

$$e^{F(u+t(g(v)-u))} \leq e^{F(u)} + t(e^{F(g(v))} - e^{F(u)}, u, g(v) \in K_g, t \in [0,1],$$

from which, using (30), we have

$$\begin{aligned}e^{F(g(v))} - e^{F(u)} &\geq \lim_{t\to 0}\{\frac{e^{F(u+t(g(v)-u))} - e^{F(u)}}{t}\}\\ &= \langle e^{F(u)}F'(u), g(v)-u\rangle \geq 0.\end{aligned}$$

This implies that

$$F(u) \leq F(g(v)).$$

This shows that $u \in K_g$ is the minimum of a differentiable exponentially general convex function, the required result. $\square$

Remark 2. *The inequality of the type* (28) *is called the general exponentially variational inequality, which has been introduced and studied by Noor* [16]*. It is an interesting problem to develop some numerical methods for solving general exponentially variational inequalities.*

Conclusion

In this paper, we have introduced and studied a new class of convex functions which is called the exponentially general convex function. It has been shown that the exponentially general convex functions contain m-convex functions as a special case. It has been shown that exponentially general convex functions enjoy several properties which convex functions have. We have shown that the minimum of an differentiable expedientially general convex functions can be characterized by a new class of variational inequalities, which is called the general exponential variational inequality. The ideas and techniques of this paper may stimulate further research.

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