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ON CERTAIN FAMILY OF MULTIVALENT HARMONIC FUNCTIONS ASSOCIATED WITH JUNG–KIM–SRIVASTAVA OPERATOR

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Abstract. New families of harmonic multivalent functions in terms Jung–Kim–Srivastava integral operator are introduced. Sufficient and necessary conditions for coefficients are obtained. Also extreme points, distortion bounds and convex combinations are investigated.

1. Introduction

Suppose $\mathcal{H}_{p,n}$ for fixed p and n $(p, n \in \mathbb{Z}^+)$ be the set of all harmonic p–valent and scene-preserving functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the type $f = h + \overline{g}$, where:

$$h(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=n+p+1}^{\infty} b_k z^k, \tag{1}$$

are analytic in $\mathbb{U}$ and $|b_{n+p-1}| < 1$. Also

$$\overline{\mathcal{H}}_{p,n} = \left\{ f = h + \overline{g} : h(z) = z^p - \sum_{k=n+p}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=n+p-1}^{\infty} |b_k| z^k, \quad |b_{n+p-1}| < 1 \right\}. \tag{2}$$

The Jung–Kim–Srivastava integral operator is defined by:

$$I^\sigma F(z) = \frac{(p+1)^\sigma}{2\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t} \right)^{\sigma+1} F(t) dt,$$

see [5].

If $F(z) = z^p + \sum\limits_{k=n+p}^{\infty} a_k z^k$, then:

$$I^\sigma F(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p+1}{n+1} \right)^\sigma a_k z^k, \tag{3}$$

and if

$$f(z) = h(z) + \overline{g(z)} = z^p + \sum_{k=n+p}^{\infty} |a_k| z^k + \sum_{k=n+p-1}^{\infty} |b_k| z^k, \qquad (|b_{n+p-1}| < 1),$$

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be in the class $\overline{\mathcal{H}}_{p,n}$, since I^σ is a linear operator, so:

$$\begin{aligned} I^\sigma f(z) &= I^\sigma h(z) + I^\sigma \overline{g(z)} \\ &= z^p - \sum_{k=n+p}^{\infty} \Big(\frac{p+1}{n+1}\Big)^\sigma |a_k| z^k + \sum_{k=n+p-1}^{\infty} \Big(\frac{p+1}{n+1}\Big)^\sigma |b_k| \overline{z}^k. \end{aligned} \tag{4}$$

A function $f \in \mathcal{H}_{p,n}$ is said to be member of $\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, if:

$$\operatorname{Re}\left\{(1-\alpha)(1-\beta)\frac{I^\sigma f(z)}{z^p} + (\alpha+\beta)\frac{\big(I^\sigma f(z)\big)'}{\big(z^p\big)'} - \lambda t \frac{\big(I^\sigma f(z)\big)''}{\big(z^p\big)''} + \lambda e^{it} - \alpha\beta\right\} \geqslant \frac{\eta}{p}, \tag{5}$$

where $0 \leqslant \eta < p$, $\alpha \geqslant 0$, $0 \leqslant \beta \leqslant 1$, $t \in \mathbb{R}$, $0 \leqslant \lambda \leqslant 1$ and:

$$\begin{aligned} \big(z^p\big)' &= \frac{\partial}{\partial\theta}\big(z^p\big) = ipz^p, \\ \big(z^p\big)'' &= \frac{\partial^2}{\partial\theta^2}\big(z^p\big) = -p^2 z^p, \\ \big(I^\sigma f(z)\big)' &= \frac{\partial}{\partial\theta}\big(I^\sigma f(re^{i\theta})\big) = iz\big(I^\sigma h(z)\big)' - iz\big(\overline{I^\sigma g(z)}\big)', \\ \big(I^\sigma f(z)\big)'' &= \frac{\partial^2}{\partial\theta^2}\big(I^\sigma f(re^{i\theta})\big) \\ &= -z\big(I^\sigma h(z)\big)' - z^2\big(I^\sigma h(z)\big)'' - z\big(\overline{I^\sigma g(z)}\big)' - z^2\big(\overline{I^\sigma g(z)}\big)''. \end{aligned} \tag{6}$$

Also

$$\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t) = \Big\{ f(z) \in \overline{\mathcal{H}}_{p,n} : \text{ (5) holds true } \Big\}.$$

The function $f = h + \overline{g}$ is scene-preserving in $\mathbb{U}$, if $|g'(z)| < |h'(z)|$. See [3] and [7].

Such as these families of harmonic multivalent functions were studied by many authors, for example see [1, 2] and [4, 6].

2. Main results

In this section, first we give the sufficient condition for $f(z) \in \mathcal{H}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$ and then we show that this sufficient bound is also necessary for $f(z) \in \overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$.

Theorem 1. *Suppose $f = h + \overline{g}$, h and g be given by (1) and:*

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \left| (\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p} \right| |a_k| \\ &+ \sum_{k=n+p-1}^{\infty} \left| (\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p} \right| |b_k| \\ &\leqslant \frac{(p-\eta)(n+1)^\sigma}{(p+1)^\sigma}, \end{aligned} \tag{7}$$

then $f(z) \in \mathcal{H}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$.

Proof. According to the fact that:

$$\operatorname{Re}\{W\} \geqslant \eta \Longleftrightarrow |W + 1 - \eta| \geqslant |W - 1 - \eta|,$$

or equivalently:

$$\operatorname{Re}\{W\} \geqslant \frac{\eta}{p} \Longleftrightarrow |Wp + p - \eta| \geqslant |Wp - p - \eta|,$$

and letting:

$$W=(1-\alpha)(1-\beta)\frac{I^\sigma f(z)}{z^p}+(\alpha+\beta)\frac{\big(I^\sigma f(z)\big)'}{\big(z^p\big)'}-\lambda e^{it}\frac{\big(I^\sigma f(z)\big)''}{\big(z^p\big)''}+\lambda e^{it}-\alpha\beta,$$

it is enough to show that:

$$|Wp+p-\eta|-|Wp-p-\eta|\geqslant 0.$$

But by using (4) and (6), we have:

$$\begin{aligned}|Wp+p-\eta|=\Bigg|&p(1-\alpha)(1-\beta)\Big(1+\sum_{k=n+p}^{\infty}\Big(\frac{p+1}{n+1}\Big)^{\sigma}a_k z^{k-p}+\sum_{k=n+p-1}^{\infty}\Big(\frac{p+1}{n+1}\Big)^{\sigma}b_k\overline{z}^{k-p}\Big)+\\ &+p(\alpha+\beta)\Big(1+\sum_{k=n+p}^{\infty}\Big(\frac{p+1}{n+1}\Big)^{\sigma}\frac{k}{p}a_k z^{k-p}-\sum_{k=n+p-1}^{\infty}\Big(\frac{p+1}{n+1}\Big)^{\sigma}\frac{k}{p}b_k\overline{z}^{k-p}\Big)\\ &-p\lambda e^{it}\left(\frac{1}{p}+\sum_{k=n+p}^{\infty}\Big(\frac{p+1}{n+1}\Big)^{\sigma}\frac{k}{p^2}a_k z^{k-p}+\frac{p-1}{p}+\sum_{k=n+p}^{\infty}\Big(\frac{p+1}{n+1}\Big)^{\sigma}\frac{k(k-1)}{p^2}a_k z^{k-p}\right.\\ &\left.+\sum_{k=n+p-1}^{\infty}\Big(\frac{p+1}{n+1}\Big)^{\sigma}\frac{k}{p^2}b_k\overline{z}^{k-p}+\sum_{k=n+p-1}^{\infty}\Big(\frac{p+1}{n+1}\Big)^{\sigma}\frac{k(k-1)}{p^2}b_k\overline{z}^{k-p}\right)\\ &+\lambda p e^{it}-\alpha\beta p+p-\eta\Bigg|\end{aligned}$$

and

$$\begin{aligned}|Wp-p-\eta|\leqslant\eta&+\sum_{k=n+p}^{\infty}\left|p+(\alpha+\beta)(k-p)-p\alpha\beta-\frac{\lambda k^2}{p}\right|\Big(\frac{p+1}{n+1}\Big)^{\sigma}|a_k|\left|\frac{z^k}{z^p}\right|\\ &+\sum_{k=n+p-1}^{\infty}\left|p-(\alpha+\beta)(k-p)+p\alpha\beta-\frac{\lambda k^2}{p}\right|\Big(\frac{p+1}{n+1}\Big)^{\sigma}|b_k|\left|\frac{z^k}{z^p}\right|.\end{aligned}$$

So by using (7), we have:

$$\begin{aligned}&|Wp+p-\eta|-|Wp-p-\eta|\geqslant\\ &2\left(\frac{(p-\eta)(n+1)^{\sigma}}{(p+1)^{\sigma}}-\sum_{k=n+p}^{\infty}\left|(\alpha+\beta)k+p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right||a_k|\right.\\ &\left.-\sum_{k=n+p-1}^{\infty}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right||b_k|\right)\geqslant 0.\end{aligned}$$

Hence the proof is complete. □

Remark 1. *The coefficient estimate* (7) *is sharp for the function:*

$$G(z) = z^p + \sum_{k=n+p}^{\infty} \frac{u_k}{\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right|} z^k + \sum_{k=n+p-1}^{\infty} \frac{w_k}{\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right|} \overline{z}^k,$$

where

$$\frac{(p+1)^\sigma}{(p-\eta)(n+1)^\sigma}\Big(\sum_{k=n+p}^{\infty}|u_k| + \sum_{k=n+p-1}^{\infty}|w_k|\Big) = 1.$$

Theorem 2. *Let* $f = h + \overline{g} \in \overline{\mathcal{H}}_{p,n}$. *Then* $f(z) \in \overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, *if and only if:*

$$\begin{aligned}&\sum_{k=n+p}^{\infty}\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right||a_k| \\ &+ \sum_{k=n+p-1}^{\infty}\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right||b_k| \leqslant \frac{(p-\eta)(n+1)^\sigma}{(p+1)^\sigma}.\end{aligned} \tag{8}$$

Proof. From Theorem 1, and since $\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t) \subseteq \mathcal{H}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, it is enough to prove the "only if" part.

Suppose $f(z) \in \overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, thus for $z = re^{i\theta} \in \mathbb{U}$, we have:

$$\begin{aligned}&\operatorname{Re}\left\{(1-\alpha)(1-\beta)\frac{I^\sigma f(z)}{z^p} + (\alpha+\beta)\frac{\big(I^\sigma f(z)\big)'}{\big(z^p\big)'} - \lambda e^{it}\frac{\big(I^\sigma f(z)\big)''}{\big(z^p\big)''} + \lambda e^{it} - \alpha\beta\right\} \\
&= \operatorname{Re}\left\{(1-\alpha)(1-\beta)\Big(1 + \sum_{k=n+p}^{\infty}\Big(\frac{p+1}{n+1}\Big)^\sigma a_k z^{k-p} + \sum_{k=n+p-1}^{\infty}\Big(\frac{p+1}{n+1}\Big)^\sigma b_k \overline{z}^{k-p}\Big)\right. \\
&+ \frac{\alpha+\beta}{p}\Big(p + \sum_{k=n+p}^{\infty}\Big(\frac{p+1}{n+1}\Big)^\sigma k a_k z^{k-p} - \sum_{k=n+p-1}^{\infty}\Big(\frac{p+1}{n+1}\Big)^\sigma k b_k \overline{z}^{k-p}\Big) \\
&- \frac{\lambda e^{it}}{p^2}\left(p + \sum_{k=n+p}^{\infty}\Big(\frac{p+1}{n+1}\Big)^\sigma a_z k^{k-p} + p(p-1) + \sum_{k=n+p}^{\infty}\Big(\frac{p+1}{n+1}\Big)^\sigma k(k-1) a_k z^{k-p}\right. \\
&\left.\left. + \sum_{k=n+p-1}^{\infty}\Big(\frac{p+1}{n+1}\Big)^\sigma k b_k \overline{z}^{k-p} + \sum_{k=n+p-1}^{\infty}\Big(\frac{p+1}{n+1}\Big)^\sigma k(k-1) b_k \overline{z}^{k-p}\right) + \lambda e^{it} - \alpha\beta\right\} \\
&= 1 - \frac{1}{p}\sum_{k=n+p}^{\infty}\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right||a_k|\Big(\frac{p+1}{n+1}\Big)^\sigma r^{k-p} \\
&- \frac{1}{p}\sum_{k=n+p-1}^{\infty}\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right||b_k|\Big(\frac{p+1}{n+1}\Big)^\sigma r^{k-p} \\
&\geqslant \frac{\eta}{p}.\end{aligned}$$

The above inequality holds for all $z \in \mathbb{U}$, so if $z = r \to 1$, we obtain the required result (8). Hence the proof is complete. □

3. Extreme points and distortion bounds

In this section, we first introduce extreme points of $\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$ and then we obtain distortion bounds for the same functions. In the end we show $\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$ is a convex set.

Theorem 3. *$f=h+\overline{g}\in\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$ if and only if it can be expressed by:*

$$f(z)=x_pz^p+\sum_{k=n+p}^{\infty}x_kh_k(z)+\sum_{k=n+p-1}^{\infty}y_kg_k(z),\qquad (z\in\mathbb{U}),\tag{9}$$

where for $k=n+p,n+p+1,\ldots,$

$$h_k(z)=z^p-\frac{(p-\eta)(n+1)^{\sigma}}{(p+1)^{\sigma}\Big((\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\Big)}z^k,$$

and for $k=n+p-1,n+p,\ldots,$ we have:

$$g_k(z)=z^p+\frac{(p-\eta)(n+1)^{\sigma}}{(p+1)^{\sigma}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|}\overline{z}^k,$$

$x_p\geqslant 0$, $y_{n+p-1}\geqslant 0$, $x_p+\sum_{k=n+p}^{\infty}x_k+\sum_{k=n+p-1}^{\infty}y_k=1$, $x_k\geqslant 0$, $y_k\geqslant 0$ and $k=n+p,n+p-1,\ldots.$

Proof. If f be given by (9), then:

$$f(z)=z^p-\sum_{k=n+p}^{\infty}\frac{(p-\eta)(n+1)^{\sigma}}{(p+1)^{\sigma}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|}x_kz^k$$
$$+\sum_{k=n+p-1}^{\infty}\frac{(p-\eta)(n+1)^{\sigma}}{(p+1)^{\sigma}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|}y_k\overline{z}^k.$$

Since by (8), we have:

$$\sum_{k=n+p}^{\infty}\left(\frac{\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|(p+1)^{\sigma}}{(p-\eta)(n+1)^{\sigma}}\right.$$
$$\left.\times\frac{(p-\eta)(n+1)^{\sigma}|x_k|}{(p+1)^{\sigma}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|}\right)$$
$$+\sum_{k=n+p-1}^{\infty}\left(\frac{\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|(p+1)^{\sigma}}{(p-\eta)(n+1)^{\sigma}}\right.$$
$$\left.\times\frac{(p-\eta)(n+1)^{\sigma}|y_k|}{(p+1)^{\sigma}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|}\right)$$
$$=\sum_{k=n+p}^{\infty}|x_k|+\sum_{k=n+p-1}^{\infty}|y_k|=1-x_p\leqslant 1.$$

So $f(z)\in\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$.

Conversely, suppose $f(z) \in \overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$. If we consider:

$$x_p = 1 - \Big(\sum_{k=n+p}^{\infty} x_k + \sum_{k=n+p-1}^{\infty} y_k \Big),$$

where

$$x_k = \frac{\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right| (p+1)^{\sigma}}{(p-\eta)(n+1)^{\sigma}} |a_k|,$$

and

$$y_k = \frac{\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right| (p+1)^{\sigma}}{(p-\eta)(n+1)^{\sigma}} |b_k|,$$

we get:

$$\begin{aligned}
f(z) &= z^p - \sum_{k=n+p}^{\infty} |a_k| z^k + \sum_{k=n+p-1}^{\infty} |b_k| \overline{z}^k \\
&= z^p - \sum_{k=n+p}^{\infty} \frac{(p-\eta)(n+1)^{\sigma}}{\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right| (p+1)^{\sigma}} x_k z^k \\
&+ \sum_{k=n+p-1}^{\infty} \frac{(p-\eta)(n+1)^{\sigma}}{\left|(\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda k^2}{p}\right| (p+1)^{\sigma}} y_k \overline{z}^k \\
&= z^p - \sum_{k=n+p}^{\infty} \big(z^p - h_k(z)\big) x_k - \sum_{k=n+p-1}^{\infty} \big(z^p - g_k(z)\big) y_k \\
&= \Big(1 - \Big(\sum_{k=n+p}^{\infty} x_k + \sum_{k=n+p-1}^{\infty} y_k \Big)\Big) z^p + \sum_{k=n+p}^{\infty} h_k(z) x_k + \sum_{k=n+p-1}^{\infty} g_k(z) y_k \\
&= x_p z^p + \sum_{k=n+p}^{\infty} x_k h_k(z) + \sum_{k=n+p-1}^{\infty} y_k g_k(z),
\end{aligned}$$

that is the required representation. □

Theorem 4. *If $f(z) \in \overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, $|z| = r < 1$, then:*

$$\begin{aligned}
|f(z)| &\geqslant \big(1 - |b_{n+p-1}| r^{n-1}\big) r^p \\
&- \left(\frac{(p-\eta)(n+1)^{\sigma}}{(p+1)^{\sigma}\Big((\alpha+\beta)k - p(1-\alpha-\beta+\alpha\beta) - \frac{\lambda(n+p)^2}{p}\Big)} \right. \\
&- \left. \frac{(\alpha+\beta)(n+2p-1) - p(1+\alpha\beta) - \frac{\lambda(n+p-1)^2}{p}}{(\alpha+\beta)n + p(1+\alpha\beta) - \frac{\lambda(n+p)^2}{p}} |b_{n+p-1}| \right) r^{n+p},
\end{aligned} \tag{10}$$

and

$$
\begin{aligned}
|f(z)| \leqslant & \big(1+|b_{n+p-1}r^{n-1}|r^p \\
& + \left(\frac{(p-\eta)(n+1)^\sigma}{(p+1)^\sigma\Big((\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}\Big)} \right. \\
& \left. - \frac{(\alpha+\beta)(n+2p-1)-p(1+\alpha\beta)-\frac{\lambda(n+p-1)^2}{p}}{(\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}}|b_{n+p-1}| \right) r^{n+p}.
\end{aligned} \tag{11}
$$

Proof. Suppose $f(z) \in \overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, then by (8), we have:

$$
\begin{aligned}
|f(z)| &= \left| z^p - \sum_{k=n+p}^{\infty} |a_k| z^k + \sum_{k=n+p-1}^{\infty} |b_k| \overline{z}^k \right| \\
&= \left| z^p + |b_{n+p-1}|\overline{z}^{n+p-1} - \sum_{k=n+p}^{\infty} \big(|a_k|z^k + |b_k|\overline{z}^k\big) \right| \\
&\geqslant r^p - |b_{n+p-1}|r^{n+p-1} \\
&- \frac{(p-\eta)(n+1)^\sigma}{(p+1)^\sigma\Big((\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}\Big)} \\
&\times \left(\sum_{k=n+p}^{\infty} \left[\frac{\Big((\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}\Big)(p+1)^\sigma}{(p-\eta)(n+1)^\sigma}|a_k| \right.\right. \\
&\qquad \left.\left. + \frac{\Big((\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}\Big)(p+1)^\sigma}{(p-\eta)(n+1)^\sigma}|b_k| \right] r^k \right) \\
&\geqslant r^p - |b_{n+p-1}r^{n+p-1}| \\
&- \frac{(p-\eta)(n+1)^\sigma}{(p+1)^\sigma\Big((\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}\Big)} \\
&\times \left(\sum_{k=n+p}^{\infty} \left[\frac{\Big((\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\Big)(p+1)^\sigma}{(p-\eta)(n+1)^\sigma}|a_k| \right.\right. \\
&\qquad \left.\left. + \frac{\Big((\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\Big)(p+1)^\sigma}{(p-\eta)(n+1)^\sigma}|b_k| \right] r^k \right) \\
&\geqslant r^p - |b_{n+p-1}|r^{n+p+1} \\
&- \frac{(p-\eta)(n+1)^\sigma}{(p+1)^\sigma\Big((\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}\Big)} \\
&\times \left[1 - \frac{\Big((\alpha+\beta)(n+2p-1)-p(1+\alpha\beta)-\frac{\lambda(n+p-1)^2}{p}\Big)(p+1)^\sigma}{(p-\eta)(n+1)^\sigma}|b_{n+p-1}| \right] r^{n+p}
\end{aligned}
$$

$$= r^p - |b_{n+p-1}|r^{n+p-1} - \left[\frac{(p-\eta)(n+1)^\sigma}{(p+1)^\sigma\Big((\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}\Big)}\right.$$
$$\left. - \frac{(\alpha+\beta)(n+2p-1)-p(1+\alpha\beta)-\frac{\lambda(n+p-1)^2}{p}}{(\alpha+\beta)n+p(1+\alpha\beta)-\frac{\lambda(n+p)^2}{p}}|b_{n+p-1}|\right] r^{n+p}.$$

Relation (11) can be proved by using the similar statements. So the proof is complete. □

Finally, we show $\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$ is a convex set.

Theorem 5. *If $f_j(z)$, $j=1,2,\ldots$, belongs to $\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, then the function:*

$$F(z)=\sum_{j=1}^{\infty}\delta_j f_j(z),$$

is also in $\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, where $f_j(z)$ for $j=1,2,\ldots$ and $\sum_{j=1}^{\infty}\delta_j=1$, defined by:

$$f_j(z)=z^p-\sum_{k=n+p}^{\infty}a_{k,j}z^k+\sum_{k=n+p-1}^{\infty}b_{k,j}\overline{z}^k.$$

In the other words, $\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, is closed under convex combination, so it is a convex set.

Proof. Since $f_J(z)\in\overline{\mathcal{H}}^{\eta}_{p,n}(\alpha,\beta,\lambda,t)$, so by (8) for $j=1,2,\ldots$, we have:

$$\sum_{k=n+p}^{\infty}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right||a_{k,j}|$$
$$+\sum_{k=n+p-1}^{\infty}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right||b_{k,j}|\leqslant\frac{(p-\eta)(n+1)^\sigma}{(p+1)^\sigma}.$$

Also

$$F(z)=\sum_{j=1}^{\infty}\delta_j f_j(z)=z^p-\sum_{k=n+p}^{\infty}\Big(\sum_{j=1}^{\infty}\delta_j a_{k,j}\Big)z^k+\sum_{k=n+p-1}^{\infty}\Big(\sum_{j=1}^{\infty}\delta_j b_{k,j}\Big)\overline{z}^k.$$

Now, according to Theorem 2, we get:

$$\sum_{k=n+p}^{\infty}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|\left|\sum_{j=1}^{\infty}\delta_j a_{k,j}\right|$$
$$+\sum_{k=n+p-1}^{\infty}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right|\left|\sum_{j=1}^{\infty}\delta_j b_{k,j}\right|$$
$$=\sum_{j=1}^{\infty}\left\{\sum_{k=n+p}^{\infty}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right||a_{k,j}|\right.$$
$$\left.+\sum_{k=n+p-1}^{\infty}\left|(\alpha+\beta)k-p(1-\alpha-\beta+\alpha\beta)-\frac{\lambda k^2}{p}\right||b_{k,j}|\right\}\delta_j$$

$$\leqslant \frac{(p-\eta)(n+1)^{\sigma}}{(p+1)^{\sigma}} \sum_{j=1}^{\infty} \delta_j$$
$$= \frac{(p-\eta)(n+1)^{\sigma}}{(p+1)^{\sigma}}.$$

Thus $F(z) \in \overline{\mathcal{H}}_{p,n}^{\eta}(\alpha, \beta, \lambda, t)$ and the proof is complete. □

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