ON SOME CONNECTED GROUPS OF AUTOMORPHISMS OF WEIL ALGEBRAS

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ABSTRACT. The method of direct calculation of the group of $\mathbb{R}$-algebra automorphisms of a Weil algebra is presented in detail. The paper is focused on the case of a one-componental group and presents two cases of values of the determinant of its linear part.

INTRODUCTION

The paper is devoted to groups of $\mathbb{R}$-algebra automorphisms of Weil algebras, which are important examples of algebraic groups. We consider the usual Euclidean topology. The essential fact ([1]) is that every algebraic group has as irreducible components the cosets modulo a closed connected normal subgroup of finite index, called the *identity component*. (It contains the neutral element which is the identity automorphism.) Those irreducible components turn out to be also the connected components. We denote the group of $\mathbb{R}$-algebra automorphisms of a Weil algebra A by $\operatorname{Aut} A$ and its identity component by G_A.

In differential geometry, Weil algebras play a fundamental role in the theory of Weil bundles $T^A M$ over a smooth manifold M which generalize well-known higher order velocities bundles $T_n^r M$. Contact elements are defined as orbits of the group action. In the classical case, they form the fiber bundle denoted by $K_n^1 M = \operatorname{reg} T_n^1 M / G_n^1$ where G_n^1 denotes the first order jet group. It is the bundle of contact elements; moreover the bundle of oriented contact elements is obtained if we replace the whole group G_n^1 by its identity component which is the subgroup of matrices with positive determinant, see [2]. We refer to paper [4] for the case of higher order jet groups G_n^r, bundles $K_n^r M = \operatorname{reg} T_n^r M / G_n^r$ of higher order contact elements and the generalizations: groups of Weil algebra $\mathbb{R}$-algebra automorphisms $\operatorname{Aut} A$ and bundles $K^A M = \operatorname{reg} T^A M / \operatorname{Aut} A$ of Weil contact elements.

We investigate the case $\operatorname{Aut} A = G_A$, it means Weil algebras possessing only one connected component. They represent an important case where orientation reversing reparametrizations for corresponding velocities are impossible.

1. BASIC CONCEPTS, DENOTATIONS AND THE METHOD

This section will, among other things, show how to effectively calculate a group of $\mathbb{R}$-algebra automorphisms of a Weil algebra.

The Weil algebra $\mathbb{D}_2^2 / \langle X^2, Y^2 \rangle$ represents the Weil algebra of the iterated tangent functor. Let the residue classes of X and Y will be denoted by the same symbol X and Y, respectively. Then we choose the basis $\{1, X, Y, XY\}$ and write elements $a + bX +$

$cY + dXY$ of this Weil algebra as vectors (a, b, c, d). The endomorphisms have a form

$$(a, b, c, d) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & B & C \\ 0 & D & E & F \\ 0 & G & H & I \end{pmatrix} = (a, Ab + Dc + Gd, Bb + Ec + Hd, Cb + Fc + Id)$$

which follows from the fact $1 \mapsto 1$. It is therefore sufficient to investigate the third order submatrix $\mathcal{M} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$. In fact, we restrict to subspace formed by nilpotent elements (which is an ideal in the algebra and which is usually denoted by $\mathfrak{n}$). For automorphisms, the matrix $\mathcal{M}$ must be non-singular. However, we have to describe *algebra* endomorphisms and automorphisms. As

$$X \mapsto AX + BY + CXY \quad \text{and} \quad Y \mapsto DX + EY + FXY,$$

we compute

$$XY \mapsto ADX^2 + (AE + BD)XY + BEY^2$$

and it follows $AD = 0$ and $BE = 0$. So, we have four variants

(i) $B = 0$, $D = 0$
(ii) $A = 0$, $E = 0$
(iii) $D = 0$, $E = 0$
(iv) $A = 0$, $B = 0$

and, as $G = AD = 0$, $H = BE = 0$, $I = AE + BD$, they correspond with matrices

$$\begin{pmatrix} A & 0 & C \\ 0 & E & F \\ 0 & 0 & AE \end{pmatrix}, \begin{pmatrix} 0 & B & C \\ D & 0 & F \\ 0 & 0 & BD \end{pmatrix}, \begin{pmatrix} A & B & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & C \\ D & E & F \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that the third and fourth matrices will be killed in the case of automorphisms (they have zero row). Thus, $\mathbb{R}$-algebra automorphisms are represented by matrices

$$\begin{pmatrix} A & 0 & C \\ 0 & E & F \\ 0 & 0 & AE \end{pmatrix}, \begin{pmatrix} 0 & B & C \\ D & 0 & F \\ 0 & 0 & BD \end{pmatrix},$$

in which $A \neq 0$, $E \neq 0$ and $B \neq 0$, $D \neq 0$, respectively. As non-zero coefficients can be either positive or negative, we find that the group of $\mathbb{R}$-algebra automorphisms has eight connected components. For the determinant $\mathfrak{D} = \det \mathcal{M}$, we have

$$\mathfrak{D} = A^2 E^2 \quad \text{or} \quad \mathfrak{D} = -B^2 D^2.$$

If we make a deeper restriction, only for the supspace formed by elements of $\mathfrak{n}/\mathfrak{n}^2$, we consider the matrix $\mathcal{M}_1$ having two possible forms

$$\begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix}, \begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix},$$

$(A \neq 0, E \neq 0, B \neq 0, D \neq 0)$, and, for the determinant $\mathfrak{D}_1 = \det \mathcal{M}_1$, we have

$$\mathfrak{D}_1 = AE \quad \text{or} \quad \mathfrak{D}_1 = -BD.$$

2. Theorem on a connected group of $\mathbb{R}$-algebra automorphisms of a Weil algebra

In the introductory example, we saw the group of automorphisms non-connected, G_A was a proper subgroup of $\operatorname{Aut} A$. We now want to focus only on algebras where the group is connected ($G_A = \operatorname{Aut} A$). We recall that in [3] is proved that the Weil algebra $\mathbb{D}_2^6/\langle X^3 + Y^4, X^4 + Y^5\rangle$ possesses a connected (one-componental) group of $\mathbb{R}$-algebra automorphisms and

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(see Theorem 1 and its proof in [3] for more details). and

$$\mathfrak{D}_1 = 1.$$

The following assertion holds.

Theorem 1. *Let A is a Weil algebra with connected group of $\mathbb{R}$-algebra automorphisms. Then there may occur one of the following two cases for the determinant $\mathfrak{D}_1$:*

(i) $\mathfrak{D}_1$ is constant with the value 1
(ii) $\mathfrak{D}_1$ takes all real values from the interval $(0, \infty)$.

Proof. It is obvious that the map assigning the determinant $\mathfrak{D}_1$ to an $\mathbb{R}$-algebra automorphism is a continuous function and, moreover, a group homomorphism. It is also clear that its image is a subset of the interval $(0, \infty)$. So it is clear that only possibilities (i) and (ii) can occur because we have no other connected subgroup of the group $(0, \infty)$ (with the usual multiplication of reals) than these two. The case (i) occurs for the Weil algebra

$$\mathbb{D}_2^6/\langle X^3 + Y^4, X^4 + Y^5\rangle.$$

It remains to show that the case (ii) may occur, too.

Let us consider $A = \mathbb{D}_2^4/\langle X^3Y, X^2Y^2, Y^4, X^3 - Y^3\rangle$. We choose the basis

$$\left\{1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, X^4\right\}.$$

Then the endomorphism $\varphi\colon A \to A$ maps

$$\phi(X) = AX + BY + CX^2 + DXY + EY^2 + FX^3 + GX^2Y + HXY^2 + IX^4$$

and

$$\phi(Y) = JX + KY + LX^2 + MXY + NY^2 + PX^3 + QX^2Y + RXY^2 + SX^4.$$

For φ to be an $\mathbb{R}$-algebra automorphism, it is necessary

(i) $AK - BJ \neq 0$
(ii) $\varphi(X^3Y) = \varphi(0) = 0$, but it means that in the expression of $\varphi(X^3Y)$ the coefficient standing at $X^4 - XY^3$ is zero, which gives the equation

$$A^3J - B^3J - 3AB^2K = 0 \tag{1}$$

(iii) $\varphi(X^2Y^2) = \varphi(0) = 0$, but it means that in the expression of $\varphi(X^2Y^2)$ the coefficient standing at $X^4 - XY^3$ is zero, which gives the equation

$$A^2J^2 - 2B^2JK - 2ABK^2 = 0 \tag{2}$$

(iv) $\varphi(Y^4) = \varphi(0) = 0$, but it means that in the expression of $\varphi(Y^4)$ the coefficient standing at $X^4 - XY^3$ is zero, which gives the equation

$$J^4 - 4JK^3 = 0 \tag{3}$$

We start with the equation (3).

(α) If $J = 0$, then $A \neq 0$, $K \neq 0$ and then $B = 0$ due to (1) and (2).

(β) If $J \neq 0$, then $J = \sqrt[3]{4}K$, (1) transforms to

$$\left(\sqrt[3]{4}A^3 - \sqrt[3]{4}B^3 - 3AB^2 \right) K = 0 \tag{4}$$

and (2) transforms to

$$\left(\sqrt[3]{8}A^2 - 2\sqrt[3]{4}B^2 - 2AB \right) K^2 = 0, \tag{5}$$

but $K \neq 0$ as $J \neq 0$. So the expressions in brackets have to be zero but the obtained system in unknowns A and B has only the solution $A = B = 0$ which is impossible.

So we continue with $B = 0$, $J = 0$, $A \neq 0$, $K \neq 0$. Moreover, it is necessary

(v) $\varphi(X^3 - Y^3) = \varphi(0) = 0$, but it means that in the expression of $\varphi(X^3 - Y^3)$ the coefficient standing at $X^3 - Y^3$ is zero, which gives the equation

$$A^3 - K^3 = 0 \tag{6}$$

with the solution $K = A$ and also in the expression of $\varphi(X^3 - Y^3)$ the coefficient standing at $X^4 - XY^3$ is zero, which gives the equation

$$3A^2CD - 3A^2M = 0 \tag{7}$$

with the solution $M = C$.

So, we have

$$\begin{aligned}
\phi(X) &= AX + CX^2 + DXY + EY^2 + FX^3 + GX^2Y + HXY^2 + IX^4 \\
\phi(Y) &= AY + LX^2 + CXY + NY^2 + PX^3 + QX^2Y + RXY^2 + SX^4 \quad \text{where } A \neq 0.
\end{aligned}$$

That is why we obtain

$$\mathcal{M} = \begin{pmatrix}
A & 0 & C & D & E & F & G & H & I \\
0 & A & L & C & N & P & Q & R & S \\
0 & 0 & A^2 & 0 & 0 & 2AC & 2AD & 2AE & 2AF + C^2 - 2DE \\
0 & 0 & 0 & A^2 & 0 & AL - AE & 2AC & AD + AN & AP + CL - AH - CE - DN \\
0 & 0 & 0 & 0 & A^2 & -2AN & 2AL & 2AC & L^2 - 2AR - 2CN \\
0 & 0 & 0 & 0 & 0 & 2A^3 & 0 & 0 & 6A^2C \\
0 & 0 & 0 & 0 & 0 & 0 & A^3 & 0 & A^2L - 2A^2E \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A^3 & -A^2D - 2A^2N \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2A^4
\end{pmatrix},$$

$$\mathcal{M}_1 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

and

$$\mathfrak{D} = 4A^{21} \quad \text{and} \quad \mathfrak{D}_1 = A^2.$$

$\square$

Remark 1. We note that the result remains in force for local algebras over an arbitrary field of the characteristics 0.

References

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