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SOME DISCRETE INEQUALITIES FOR CONVEX FUNCTIONS DEFINED ON LINEAR SPACES

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Abstract. In this paper we provide some discrete inequalities related to the Hermite-Hadamard result for convex functions defined on convex subsets in a linear space. Applications for norms and univariate real functions with an example for the logarithm, are also given.

1. Introduction

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda)a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g(a,b) : [0,1] \to \mathbb{R}$, $g(a,b)(t) := f[(1-t)a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s) = (\triangledown_{\pm} f[(1-s)a + sb])(b-a)$, $s \in [0,1)$
(ii) $g'_{+}(a,b)(0) = (\triangledown_{+} f(a))(b-a)$
(iii) $g'_{-}(a,b)(1) = (\triangledown_{-} f(b))(b-a)$

where $(\triangledown_{\pm} f(x))(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$(\triangledown_{+} f(x))(y) := \lim_{h\to 0+}\left[\frac{f(x+hy)-f(x)}{h}\right],$$
$$(\triangledown_{-} f(x))(y) := \lim_{k\to 0-}\left[\frac{f(x+ky)-f(x)}{k}\right],\quad x,\ y \in X.$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b] \subset X$:

$$f\left(\frac{a+b}{2}\right) \le \int_0^1 f[(1-t)a+tb]\,dt \le \frac{f(a)+f(b)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b) : [0,1] \to \mathbb{R}$

$$g(a,b)\left(\frac{1}{2}\right) \le \int_0^1 g(a,b)(t)\,dt \le \frac{g(a,b)(0)+g(a,b)(1)}{2}.$$

For other related results see the monograph on line [4].

We have the following result [2] related to the first Hermite-Hadamard inequality in (HH):

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Theorem 1. *Let X be a linear space, $a,b\in X$, $a\neq b$ and $f:[a,b]\subset X\to\mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s\in(0,1)$ one has the inequality*

$$\begin{aligned}&\frac{1}{2}\left[(1-s)^2\left(\triangledown_+ f\left[(1-s)a+sb\right]\right)(b-a)-s^2\left(\triangledown_- f\left[(1-s)a+sb\right]\right)(b-a)\right] \\ &\leq\int_0^1 f\left[(1-t)a+tb\right]dt-f\left[(1-s)a+sb\right] \\ &\leq\frac{1}{2}\left[(1-s)^2\left(\triangledown_- f(b)\right)(b-a)-s^2\left(\triangledown_+ f(a)\right)(b-a)\right].\end{aligned}\tag{1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s=0$ or $s=1$.

If $f:[a,b]\to\mathbb{R}$ is as in Theorem 1 and Gâteaux differentiable in $c:=(1-\lambda)a+\lambda b$, $\lambda\in(0,1)$ along the direction $b-a$, then we have the inequality:

$$\left(\frac{1}{2}-\lambda\right)\left(\triangledown f(c)\right)(b-a)\leq\int_0^1 f\left[(1-t)a+tb\right]dt-f(c).\tag{2}$$

If f is as in Theorem 1, then

$$\begin{aligned}0&\leq\frac{1}{8}\left[\triangledown_+ f\left(\frac{a+b}{2}\right)(b-a)-\triangledown_- f\left(\frac{a+b}{2}\right)(b-a)\right] \\ &\leq\int_0^1 f\left[(1-t)a+tb\right]dt-f\left(\frac{a+b}{2}\right) \\ &\leq\frac{1}{8}\left[\left(\triangledown_- f(b)\right)(b-a)-\left(\triangledown_+ f(a)\right)(b-a)\right].\end{aligned}\tag{3}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Also we have the following result [3] related to the second Hermite-Hadamard inequality in (HH):

Theorem 2. *Let X be a linear space, $a,b\in X$, $a\neq b$ and $f:[a,b]\subset X\to\mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s\in(0,1)$ one has the inequality*

$$\begin{aligned}&\frac{1}{2}\left[(1-s)^2\left(\triangledown_+ f\left[(1-s)a+sb\right]\right)(b-a)-s^2\left(\triangledown_- f\left[(1-s)a+sb\right]\right)(b-a)\right] \\ &\leq(1-s)f(a)+sf(b)-\int_0^1 f\left[(1-t)a+tb\right]dt \\ &\leq\frac{1}{2}\left[(1-s)^2\left(\triangledown_- f(b)\right)(b-a)-s^2\left(\triangledown_+ f(a)\right)(b-a)\right].\end{aligned}\tag{4}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s=0$ or $s=1$.

If $f:[a,b]\to\mathbb{R}$ is as in Theorem 2 and Gâteaux differentiable in $c:=(1-\lambda)a+\lambda b$, $\lambda\in(0,1)$ along the direction $b-a$, then we have the inequality:

$$\left(\frac{1}{2}-\lambda\right)\left(\triangledown f(c)\right)(b-a)\leq(1-\lambda)f(a)+\lambda f(b)-\int_0^1 f\left[(1-t)a+tb\right]dt.\tag{5}$$

If f is as in Theorem 2, then

$$0 \leq \frac{1}{8}\left[\bigtriangledown_{+}f\left(\frac{a+b}{2}\right)(b-a)-\bigtriangledown_{-}f\left(\frac{a+b}{2}\right)(b-a)\right] \tag{6}$$
$$\leq \frac{f(a)+f(b)}{2}-\int_0^1 f\left[(1-t)a+tb\right]dt$$
$$\leq \frac{1}{8}\left[\left(\bigtriangledown_{-}f(b)\right)(b-a)-\left(\bigtriangledown_{+}f(a)\right)(b-a)\right].$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

2. The Results

Let $f : C \subset X \to \mathbb{R}$ be a convex function on C. We define the function $F_f : C \times C \to \mathbb{R}$ by

$$F_f(x,y) := \int_0^1 f((1-t)x+ty)\,dt. \tag{7}$$

Theorem 3. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on C. Then the function F_f is convex on $C \times C$ and if $x_i,\ y_i \in C$ and $p_i \geq 0$ for $i = 1,...,n$ with $\sum_{i=1}^{n} p_i = 1$, then we have the inequalities*

$$\sum_{i=1}^{n} p_i \int_0^1 f((1-t)x_i+ty_i)\,dt \geq \int_0^1 f\left((1-t)\sum_{i=1}^{n}p_ix_i+t\sum_{i=1}^{n}p_iy_i\right)dt \tag{8}$$
$$\geq f\left(\sum_{i=1}^{n}p_i\left(\frac{x_i+y_i}{2}\right)\right),$$

$$\sum_{i=1}^{n}p_i\left(\frac{f(x_i)+f(y_i)}{2}\right) \geq \sum_{i=1}^{n}p_i\int_0^1 f((1-t)x_i+ty_i)\,dt \tag{9}$$
$$\geq \sum_{i=1}^{n}p_if\left(\frac{x_i+y_i}{2}\right) \geq f\left(\sum_{i=1}^{n}p_i\left(\frac{x_i+y_i}{2}\right)\right)$$

and

$$\sum_{i=1}^{n}p_i\left(\frac{f(x_i)+f(y_i)}{2}\right) \geq \frac{1}{2}\left[f\left(\sum_{i=1}^{n}p_ix_i\right)+f\left(\sum_{i=1}^{n}p_iy_i\right)\right] \tag{10}$$
$$\geq \int_0^1 f\left((1-t)\sum_{i=1}^{n}p_ix_i+t\sum_{i=1}^{n}p_iy_i\right)dt.$$

Proof. Let $(x,y), (u,v) \in C \times C$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned}
F_f(\alpha(x,y) + \beta(u,v)) &= F_f(\alpha x + \beta u, \alpha y + \beta v) \\
&= \int_0^1 f((1-t)(\alpha x + \beta u) + t(\alpha y + \beta v))\, dt \\
&= \int_0^1 f(\alpha[(1-t)x + ty] + \beta[(1-t)u + tv])\, dt \\
&\leq \int_0^1 [\alpha f((1-t)x + ty) + \beta f((1-t)u + tv)]\, dt \\
&= \alpha \int_0^1 f((1-t)x + ty)\, dt + \beta \int_0^1 f((1-t)u + tv)\, dt \\
&= \alpha F_f(x,y) + \beta F_f(u,v),
\end{aligned}$$

which proves the joint convexity of the function F_f.

By Jensen's inequality for the convex function F_f we have

$$\sum_{i=1}^n p_i F_f(x_i, y_i) \geq F_f\left(\sum_{i=1}^n p_i(x_i, y_i)\right) = F_f\left(\sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i\right),$$

which is equivalent to the first inequality in (8).

By Hermite-Hadamard inequality (HH) we have

$$\begin{aligned}
\int_0^1 f\left((1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right) dt &\geq f\left(\frac{\sum_{i=1}^n p_i x_i + \sum_{i=1}^n p_i y_i}{2}\right) dt \\
&= f\left(\sum_{i=1}^n p_i\left(\frac{x_i + y_i}{2}\right)\right)
\end{aligned}$$

and the second part of (8) is proved.

From (HH) we also have for each $i \in \{1, ..., n\}$ that

$$\frac{f(x_i) + f(y_i)}{2} \geq \int_0^1 f[(1-t)x_i + ty_i]\, dt \geq f\left(\frac{x_i + y_i}{2}\right).$$

If we multiply this inequality by $p_i \geq 0$ and sum over i from 1 to n we get the first and second inequality in (9).

The last part in (9) follows by Jensen's inequality.

Let $u := \sum_{i=1}^n p_i x_i$ and $v := \sum_{i=1}^n p_i y_i$. By Hermite-Hadamard inequality (HH) we also have

$$\frac{f(u) + f(v)}{2} \geq \int_0^1 f[(1-t)u + tv]\, dt,$$

which produces the second inequality in (10).

By Jensen's inequality for f we have

$$\sum_{i=1}^n p_i f(x_i) \geq f\left(\sum_{i=1}^n p_i x_i\right)$$

and

$$\sum_{i=1}^n p_i f(y_i) \geq f\left(\sum_{i=1}^n p_i y_i\right).$$

If we sum these two inequalities and divide by 2 we get the first inequality in (10). □

The following result also holds:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i\left(\nabla_+ f\left(\frac{x_i+y_i}{2}\right)(y_i-x_i)\right)\right. \\
&\qquad\left. - \sum_{i=1}^{n} p_i\left(\nabla_- f\left(\frac{x_i+y_i}{2}\right)(y_i-x_i)\right)\right] \\
&\leq \sum_{i=1}^{n} p_i \int_0^1 f\left[(1-t)\,x_i + t y_i\right] dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\
&\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i\left(\nabla_- f\left(y_i\right)\right)(y_i-x_i) - \sum_{i=1}^{n} p_i\left(\nabla_+ f\left(x_i\right)\right)(y_i-x_i)\right], \quad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i\left(\nabla_+ f\left(\frac{x_i+y_i}{2}\right)(y_i-x_i)\right)\right. \\
&\qquad\left. - \sum_{i=1}^{n} p_i\left(\nabla_- f\left(\frac{x_i+y_i}{2}\right)(y_i-x_i)\right)\right] \\
&\leq \sum_{i=1}^{n} p_i\left(\frac{f(x_i)+f(y_i)}{2}\right) - \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)\,x_i + t y_i\right) dt \\
&\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i\left(\nabla_- f\left(y_i\right)\right)(y_i-x_i) - \sum_{i=1}^{n} p_i\left(\nabla_+ f\left(x_i\right)\right)(y_i-x_i)\right]. \quad (12)
\end{aligned}
$$

We also have

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[\nabla_+ f\left(\sum_{i=1}^{n} p_i\left(\frac{x_i+y_i}{2}\right)\right)\left(\sum_{i=1}^{n} p_i\left(y_i-x_i\right)\right)\right. \\
&\qquad\left. - \nabla_- f\left(\sum_{i=1}^{n} p_i\left(\frac{x_i+y_i}{2}\right)\right)\left(\sum_{i=1}^{n} p_i\left(y_i-x_i\right)\right)\right] \\
&\leq \int_0^1 f\left((1-t)\sum_{i=1}^{n} p_i x_i + t\sum_{i=1}^{n} p_i y_i\right) dt - f\left(\sum_{i=1}^{n} p_i\left(\frac{x_i+y_i}{2}\right)\right) \\
&\leq \frac{1}{8}\left[\left(\nabla_- f\left(\sum_{i=1}^{n} p_i y_i\right)\right)\left(\sum_{i=1}^{n} p_i\left(y_i-x_i\right)\right)\right. \\
&\qquad\left. - \left(\nabla_+ f\left(\sum_{i=1}^{n} p_i x_i\right)\right)\left(\sum_{i=1}^{n} p_i\left(y_i-x_i\right)\right)\right] \quad (13)
\end{aligned}
$$

and

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[\triangledown_{+} f\left(\sum_{i=1}^{n} p_{i}\left(\frac{x_{i}+y_{i}}{2}\right)\right)\left(\sum_{i=1}^{n} p_{i}\left(y_{i}-x_{i}\right)\right)\right. \\ &\qquad \left. -\triangledown_{-} f\left(\sum_{i=1}^{n} p_{i}\left(\frac{x_{i}+y_{i}}{2}\right)\right)\left(\sum_{i=1}^{n} p_{i}\left(y_{i}-x_{i}\right)\right)\right] \\ &\leq \frac{1}{2}\left[f\left(\sum_{i=1}^{n} p_{i} x_{i}\right)+f\left(\sum_{i=1}^{n} p_{i} y_{i}\right)\right]-\int_{0}^{1} f\left((1-t) \sum_{i=1}^{n} p_{i} x_{i}+t \sum_{i=1}^{n} p_{i} y_{i}\right) dt \\ &\leq \frac{1}{8}\left[\left(\triangledown_{-} f\left(\sum_{i=1}^{n} p_{i} y_{i}\right)\right)\left(\sum_{i=1}^{n} p_{i}\left(y_{i}-x_{i}\right)\right)\right. \\ &\qquad \left. -\left(\triangledown_{+} f\left(\sum_{i=1}^{n} p_{i} x_{i}\right)\right)\left(\sum_{i=1}^{n} p_{i}\left(y_{i}-x_{i}\right)\right)\right]. \end{aligned} \tag{14}$$

Proof. From the inequality (3) we have for $a=x_i$ and $b=y_i$, where $i\in\{1,...,n\}$ that

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[\triangledown_{+} f\left(\frac{x_{i}+y_{i}}{2}\right)\left(y_{i}-x_{i}\right)-\triangledown_{-} f\left(\frac{x_{i}+y_{i}}{2}\right)\left(y_{i}-x_{i}\right)\right] \\ &\leq \int_{0}^{1} f\left[(1-t) x_{i}+t y_{i}\right] dt-f\left(\frac{x_{i}+y_{i}}{2}\right) \\ &\leq \frac{1}{8}\left[\left(\triangledown_{-} f\left(y_{i}\right)\right)\left(y_{i}-x_{i}\right)-\left(\triangledown_{+} f\left(x_{i}\right)\right)\left(y_{i}-x_{i}\right)\right], \end{aligned}$$

for any $i\in\{1,...,n\}$.

If we multiply this inequality by $p_i\geq 0$ and sum over i from 1 to n, then we get

$$\begin{aligned} 0 &\leq \frac{1}{8} \sum_{i=1}^{n} p_{i}\left[\triangledown_{+} f\left(\frac{x_{i}+y_{i}}{2}\right)\left(y_{i}-x_{i}\right)-\triangledown_{-} f\left(\frac{x_{i}+y_{i}}{2}\right)\left(y_{i}-x_{i}\right)\right] \\ &\leq \sum_{i=1}^{n} p_{i} \int_{0}^{1} f\left[(1-t) x_{i}+t y_{i}\right] dt-\sum_{i=1}^{n} p_{i} f\left(\frac{x_{i}+y_{i}}{2}\right) \\ &\leq \frac{1}{8} \sum_{i=1}^{n} p_{i}\left[\left(\triangledown_{-} f\left(y_{i}\right)\right)\left(y_{i}-x_{i}\right)-\left(\triangledown_{+} f\left(x_{i}\right)\right)\left(y_{i}-x_{i}\right)\right], \end{aligned}$$

which is equivalent to (11).

The inequality (12) follows in a similar way by employing the inequality (6).

The inequalities (13) and (14) follow by taking $a=\sum_{i=1}^{n}p_ix_i$ and $b=\sum_{i=1}^{n}p_iy_i$ in the inequalities (3) and (6). □

3. Examples for Norms

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s := (\triangledown_{+} f_0(y))(x) = \lim_{t\to 0+}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

(v) $\langle x,y\rangle_i := (\triangledown_{-} f_0(y))(x) = \lim_{s\to 0-}\left[\frac{\|y+sx\|^2-\|y\|^2}{2s}\right]$;

for any x, $y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [1] or [5]), assuming that $p,\ q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p=\|x\|^2$ for all $x\in X$;
(aa) $\langle \alpha x,\beta y\rangle_p=\alpha\beta\langle x,y\rangle_p$ if $\alpha,\ \beta\geq 0$ and $x,\ y\in X$;
(aaa) $\left|\langle x,y\rangle_p\right|\leq\|x\|\,\|y\|$ for all $x,\ y\in X$;
(av) $\langle \alpha x+y,x\rangle_p=\alpha\langle x,x\rangle_p+\langle y,x\rangle_p$ if $x,\ y\in X$ and $\alpha\in\mathbb{R}$;
(v) $\langle -x,y\rangle_p=-\langle x,y\rangle_q$ for all $x,\ y\in X$;
(va) $\langle x+y,z\rangle_p\leq\|x\|\,\|z\|+\langle y,z\rangle_p$ for all $x,\ y,\ z\in X$;
(vaa) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);
(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0\in X\backslash\{0\}$ if and only if $\langle y,x_0\rangle_s=\langle y,x_0\rangle_i$ for all $y\in X$; in general $\langle y,x\rangle_i\leq\langle y,x\rangle_s$ for all $x,\ y\in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i=\langle y,x\rangle=\langle y,x\rangle_s$ for all $x,\ y\in X$.

Applying inequality (HH) for the convex function $f_r(x)=\|x\|^r$, $r\geq 1$ one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^r\leq\int_0^1\|(1-t)\,x+ty\|^r\,dt\leq\frac{\|x\|^r+\|y\|^r}{2}\tag{15}$$

for any $x,y\in X$.

Let $(X,\|\cdot\|)$ be a normed linear space and $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$ be n-tuples of vectors in X, then for the probability distribution $p=(p_1,...,p_n)$ and $r\geq 1$ we have by Theorem 3 for the convex function $f(x)=\|x\|^r$ that

$$\begin{aligned}\sum_{i=1}^n p_i\int_0^1\|(1-t)\,x_i+ty_i\|^r\,dt&\geq\int_0^1\left\|(1-t)\sum_{i=1}^n p_ix_i+t\sum_{i=1}^n p_iy_i\right\|^r dt\\&\geq\left\|\sum_{i=1}^n p_i\left(\frac{x_i+y_i}{2}\right)\right\|^r,\end{aligned}\tag{16}$$

$$\begin{aligned}\sum_{i=1}^n p_i\left(\frac{\|x_i\|^r+\|y_i\|^r}{2}\right)&\geq\sum_{i=1}^n\int_0^1 p_i\,\|(1-t)\,x_i+ty_i\|^r\,dt\\&\geq\sum_{i=1}^n p_i\left\|\frac{x_i+y_i}{2}\right\|^r\geq\left\|\sum_{i=1}^n p_i\left(\frac{x_i+y_i}{2}\right)\right\|^r\end{aligned}\tag{17}$$

and

$$\begin{aligned}\sum_{i=1}^n p_i\left(\frac{\|x_i\|^r+\|y_i\|^r}{2}\right)&\geq\frac{1}{2}\left[\left\|\sum_{i=1}^n p_ix_i\right\|^r+\left\|\sum_{i=1}^n p_iy_i\right\|^r\right]\\&\geq\int_0^1\left\|(1-t)\sum_{i=1}^n p_ix_i+t\sum_{i=1}^n p_iy_i\right\|^r dt.\end{aligned}\tag{18}$$

If we use Theorem 4 for the convex function $f(x)=\frac{1}{2}\|x\|^2$ then for $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$ n-tuples of vectors in X and for the probability distribution $p=(p_1,...,p_n)$

we have

$$0 \leq \frac{1}{4}\left[\sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_s - \sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_i\right]$$
$$\leq \sum_{k=1}^{n} p_k \int_0^1 \|(1-t)\, x_k + t y_k\|^2 \, dt - \sum_{k=1}^{n} p_k \left\|\frac{x_k + y_k}{2}\right\|^2$$
$$\leq \frac{1}{4}\left[\sum_{k=1}^{n} p_k \langle y_k - x_k, y_k\rangle_i - \sum_{k=1}^{n} p_k \langle y_k - x_k, x_k\rangle_s\right], \quad (19)$$

and

$$0 \leq \frac{1}{4}\left[\sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_s - \sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_i\right]$$
$$\leq \sum_{k=1}^{n} p_k \left(\frac{\|x_k\|^2 + \|y_k\|^2}{2}\right) - \sum_{k=1}^{n} p_k \int_0^1 \|(1-t)\, x_k + t y_k\|^2 \, dt$$
$$\leq \frac{1}{4}\left[\sum_{k=1}^{n} p_k \langle y_k - x_k, y_k\rangle_i - \sum_{k=1}^{n} p_k \langle y_k - x_k, x_k\rangle_s\right]. \quad (20)$$

We also have

$$0 \leq \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_s\right.$$
$$\left. - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_i\right]$$
$$\leq \int_0^1 \left\|(1-t)\sum_{k=1}^{n} p_k x_k + t \sum_{k=1}^{n} p_k y_k\right\|^2 dt - \left\|\sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\|^2$$
$$\leq \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k y_k\right\rangle_i - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k x_k\right\rangle_s\right] \quad (21)$$

and

$$0 \leq \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_s\right.$$
$$\left. - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_i\right]$$
$$\leq \frac{1}{2}\left[\left\|\sum_{k=1}^{n} p_k x_k\right\|^2 + \left\|\sum_{k=1}^{n} p_k y_k\right\|^2\right] - \int_0^1 \left\|(1-t)\sum_{k=1}^{n} p_k x_k + t \sum_{k=1}^{n} p_k y_k\right\|^2 dt$$
$$\leq \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k y_k\right\rangle_i - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k x_k\right\rangle_s\right]. \quad (22)$$

4. Examples for Functions of a Real Variable

If $f : I \to \mathbb{R}$ is convex on the interval I and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then

$$\sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)\, x_i + t y_i\right) dt \geq \int_0^1 f\left((1-t) \sum_{i=1}^{n} p_i x_i + t \sum_{i=1}^{n} p_i y_i\right) dt \tag{23}$$

$$\sum_{i=1}^{n} p_i \left(\frac{f(x_i) + f(y_i)}{2}\right) \geq \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)\, x_i + t y_i\right) dt. \tag{24}$$

If $f : I \to \mathbb{R}$ is convex and differentiable on the interior of $\mathring{I}$ then for all $x_i \in \mathring{I}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then by (11) and (12) we get

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i \int_0^1 f\left[(1-t)\, x_i + t y_i\right] dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i + y_i}{2}\right) \\ &\leq \frac{1}{8} \sum_{i=1}^{n} p_i \left[f'(y_i) - f'(x_i)\right] (y_i - x_i), \end{aligned} \tag{25}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i \left(\frac{f(x_i) + f(y_i)}{2}\right) - \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)\, x_i + t y_i\right) dt \\ &\leq \frac{1}{8} \sum_{i=1}^{n} p_i \left[f'(y_i) - f'(x_i)\right] (y_i - x_i). \end{aligned} \tag{26}$$

If $f(t) = \frac{1}{t}$ with $t > 0$, then for $y_i \neq x_i$, $i \in \{1, ..., n\}$ we have

$$\int_0^1 f\left((1-t)\, x_i + t y_i\right) dt = \int_0^1 \frac{1}{(1-t)\, x_i + t y_i} dt = \frac{\ln y_i - \ln x_i}{y_i - x_i}$$

and

$$\int_0^1 \left((1-t) \sum_{i=1}^{n} p_i x_i + t \sum_{i=1}^{n} p_i y_i\right)^{-1} dt = \frac{\ln\left(\sum_{i=1}^{n} p_i x_i\right) - \ln\left(\sum_{i=1}^{n} p_i y_i\right)}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i},$$

provided $\sum_{i=1}^{n} p_i x_i \neq \sum_{i=1}^{n} p_i y_i$.

From (23) we get

$$\sum_{i=1}^{n} p_i \frac{\ln y_i - \ln x_i}{y_i - x_i} \geq \frac{\ln\left(\sum_{i=1}^{n} p_i x_i\right) - \ln\left(\sum_{i=1}^{n} p_i y_i\right)}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}$$

that is equivalent to

$$\ln\left(\prod_{i=1}^{n} \left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}}\right) \geq \ln\left[\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i y_i}\right)^{\frac{1}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}}\right]$$

and to

$$\prod_{i=1}^{n} \left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}} \geq \left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i y_i}\right)^{\frac{1}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}}. \tag{27}$$

From (24) we get in a similar way that

$$\exp\left[\sum_{i=1}^{n} p_i \left(\frac{x_i + y_i}{2 x_i y_i}\right)\right] \geq \prod_{i=1}^{n} \left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}}, \tag{28}$$

from (25) we get

$$1 \leq \frac{\prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i-x_i}}}{\exp\left[\sum_{i=1}^{n} p_i\left(\frac{2}{x_i+y_i}\right)\right]} \leq \exp\left(\frac{1}{8}\sum_{i=1}^{n} p_i \frac{(x_i+y_i)(y_i-x_i)^2}{x_i^2 y_i^2}\right) \tag{29}$$

and from (26) we get

$$1 \leq \frac{\exp\left[\sum_{i=1}^{n} p_i\left(\frac{x_i+y_i}{2x_i y_i}\right)\right]}{\prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i-x_i}}} \leq \exp\left(\frac{1}{8}\sum_{i=1}^{n} p_i \frac{(x_i+y_i)(y_i-x_i)^2}{x_i^2 y_i^2}\right). \tag{30}$$

The interested reader may apply some of the above inequalities for other instances of convex functions such as $f(t) = -\ln t$, $t\ln t$, $\exp t$ etc... and we omit the details.

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