

**A NOTE ON NUMERICAL COMPARISON OF SOME
MULTIPLICATIVE BOUNDS RELATED TO WEIGHTED
ARITHMETIC AND GEOMETRIC MEANS**

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ABSTRACT. In this note we provided some numerical comparison for the upper and lower bounds in some recent inequalities related to the famous Young's inequality for two positive numbers. We drew the conclusion that neither of the inequalities below is always best.

1. INTRODUCTION

We consider the weighted *Arithmetic and Geometric means* defined by

$$A_\nu(a, b) := (1 - \nu)a + \nu b, \quad G_\nu(a, b) := a^{1-\nu}b^\nu,$$

where $a, b > 0$ and $\nu \in [0, 1]$. If $\nu = \frac{1}{2}$ we denote $A(a, b) := \frac{a+b}{2}$ and $G(a, b) := \sqrt{ab}$, for simplicity.

The following inequality

$$G_\nu(a, b) \leq A_\nu(a, b) \tag{1}$$

is well known in literature as either *weighted Arithmetic mean-Geometric mean inequality* or as *Young's inequality*.

We recall that *Specht's ratio* is defined by

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0, 1) \cup (1, \infty), \\ 1 & \text{if } h = 1. \end{cases}$$

It is well known that $\lim_{h \rightarrow 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality (1)

$$S\left(\left(\frac{a}{b}\right)^r\right) \leq \frac{A_\nu(a, b)}{G_\nu(a, b)} \leq S\left(\frac{a}{b}\right), \tag{2}$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{1 - \nu, \nu\}$.

The second inequality in (2) is due to Tominaga [9] while the first one is due to Furuichi [6].

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \quad h > 0. \tag{3}$$

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The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$\left[K\left(\frac{a}{b}\right)\right]^r \leq \frac{A_\nu(a, b)}{G_\nu(a, b)} \leq \left[K\left(\frac{a}{b}\right)\right]^R \quad (4)$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{1 - \nu, \nu\}$ and $R = \max\{1 - \nu, \nu\}$.

The first inequality in (4) was obtained by Zou et al. in [10] while the second by Liao et al. [7].

In [10] the authors also showed that

$$S(h^r) \leq K^r(h) \text{ for } h > 0 \text{ and } r \in \left[0, \frac{1}{2}\right]$$

implying that the lower bound in (4) is better than the lower bound from (2).

In the recent paper [2] Dragomir obtained for any $a, b > 0$, $\nu \in [0, 1]$,

$$\frac{A_\nu(a, b)}{G_\nu(a, b)} \leq \exp\left[4\nu(1 - \nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \quad (5)$$

while in [3] the first author obtained for any $a, b > 0$, $\nu \in [0, 1]$,

$$\begin{aligned} \exp\left[\frac{1}{2}\nu(1 - \nu)\frac{(b - a)^2}{\max^2\{a, b\}}\right] &\leq \frac{A_\nu(a, b)}{G_\nu(a, b)} \\ &\leq \exp\left[\frac{1}{2}\nu(1 - \nu)\frac{(b - a)^2}{\min^2\{a, b\}}\right], \end{aligned} \quad (6)$$

see also the equivalent result from [1] that has been stated only for $a < b$.

In [5] the following asymmetric bound has been proven for any $a, b > 0$, $\nu \in [0, 1]$

$$\frac{A_\nu(a, b)}{G_\nu(a, b)} \leq \min\left\{\left(\frac{\exp\left(\frac{b}{a} - 1\right)}{\frac{b}{a}}\right)^\nu, \left(\frac{\frac{b}{a}}{\exp\left(1 - \frac{1}{\frac{b}{a}}\right)}\right)^{1 - \nu}\right\}. \quad (7)$$

It is therefore a natural question to compare the upper and lower bounds provided above for the quotient $\frac{A_\nu(a, b)}{G_\nu(a, b)}$.

In this note we provide some numerical comparison for the upper and lower bounds in the above inequalities related to the famous Young's inequality for two positive numbers.

2. NUMERICAL COMPARISON FOR WEIGHTED BOUNDS

In order to compare the upper bounds provided by the inequalities (4) and (5) we consider the difference

$$D_1(b, \nu) := (K(b))^{\max\{\nu, 1 - \nu\}} - \exp[4\nu(1 - \nu)(K(b) - 1)]$$

for $b > 0$ and $\nu \in [0, 1]$.

The plot of this difference in the $[0, 2] \times [0, 0.4]$ is depicted in Figure 1 showing that neither of the upper bounds in (4) and (5) is better always.

For the inequalities (4) and (6) we consider the difference

$$D_2(b, \nu) := (K(b))^{\max\{\nu, 1 - \nu\}} - \exp\left[\frac{1}{2}\nu(1 - \nu)\frac{(b - 1)^2}{\min^2\{1, b\}}\right]$$

for $b > 0$ and $\nu \in [0, 1]$.

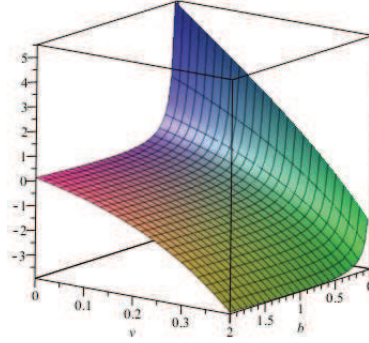


FIGURE 1

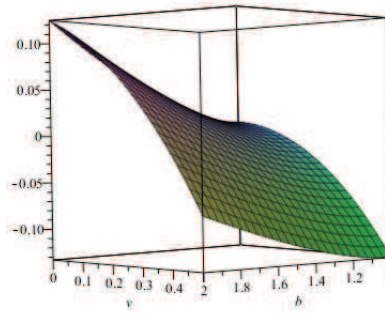


FIGURE 2

The plot of this difference in $[1, 2] \times [0, 0.5]$ is presented in Figure 2 showing that neither of the upper bounds in (4) and (6) is better always.

Now, if we want to compare the upper bounds provided by (4) and (7) we need to consider the difference

$$D_3(b, \nu) := (K(b))^{\max\{\nu, 1-\nu\}} - \min \left\{ \left(\frac{\exp(b-1)}{b} \right)^\nu, \left(\frac{b}{\exp(1-\frac{1}{b})} \right)^{1-\nu} \right\}$$

for $b > 0$ and $\nu \in [0, 1]$.

The graph of this difference in the box $[0, 2] \times [0, 1]$ is incorporated in Figure 3 proving that neither of the upper bounds in (4) and (7) is better always.

Now, for the inequalities (5) and (6) consider the difference

$$D_4(b, \nu) := \exp[4\nu(1-\nu)(K(b)-1)] - \exp \left[\frac{1}{2}\nu(1-\nu) \frac{(b-1)^2}{\min^2\{1, b\}} \right]$$

for $b > 0$ and $\nu \in [0, 1]$.

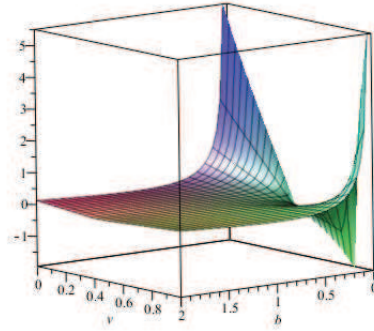


FIGURE 3

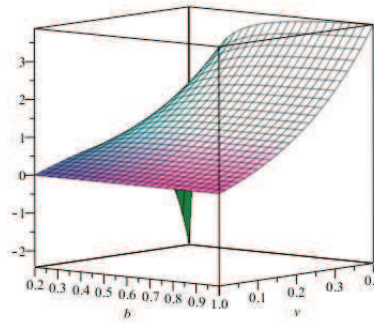


FIGURE 4

The graph of this difference in the box $[0, 0.2] \times [0, 0.4]$ is incorporated in Figure 4 and shows that neither of the upper bounds in (5) and (6) is always best.

Further, we consider for the inequalities (6) and (7) the difference

$$D_5(b, \nu) := \exp[4\nu(1-\nu)(K(b)-1)] - \min \left\{ \left(\frac{\exp(b-1)}{b} \right)^\nu, \left(\frac{b}{\exp(1-\frac{1}{b})} \right)^{1-\nu} \right\}$$

for $b > 0$ and $\nu \in [0, 1]$.

If we plot this difference on the box $[0, 10] \times [0, 0.1]$, see Figure 5, then we can conclude that neither of the upper bounds in (6) and (7) is always best.

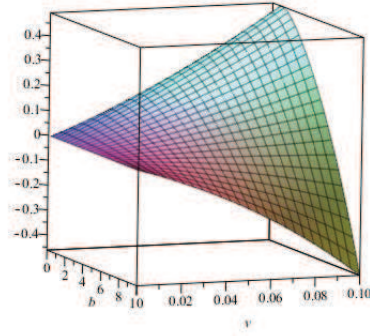


FIGURE 5

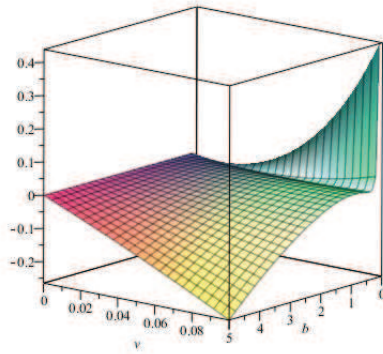


FIGURE 6

Moreover, consider the difference

$$D_6(b, \nu) := \exp \left[\frac{1}{2} \nu (1 - \nu) \frac{(b-1)^2}{\min^2 \{1, b\}} \right] - \min \left\{ \left(\frac{\exp(b-1)}{b} \right)^\nu, \left(\frac{b}{\exp(1 - \frac{1}{b})} \right)^{1-\nu} \right\}$$

for $b > 0$ and $\nu \in [0, 1]$.

The plot of this difference on the box $[0, 5] \times [0, 0.1]$ is incorporated in Figure 6 and shows that neither of the upper bounds in (5) and (7) is better always.

For the lower bounds, we need to study the differences

$$D_7(b, \nu) := S \left(b^{\min\{\nu, 1-\nu\}} \right) - \exp \left[\frac{1}{2} \nu (1 - \nu) \frac{(b-1)^2}{\max^2 \{1, b\}} \right]$$

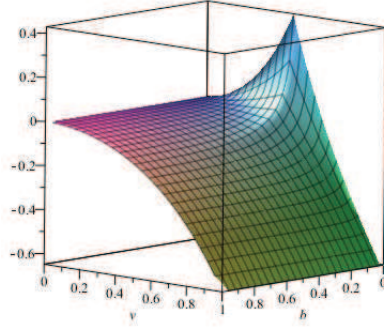


FIGURE 7

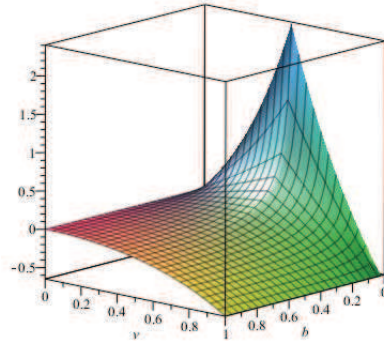


FIGURE 8

and

$$D_8(b, \nu) := (K(b))^{\min\{\nu, 1-\nu\}} - \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-1)^2}{\max^2\{1, b\}}\right]$$

for $b > 0$ and $\nu \in [0, 1]$.

The plot of the difference $D_7(b, \nu)$ in the box $[0, 1] \times [0, 1]$ is depicted in Figure 7 showing that neither of the lower bounds in (2) and (6) is always best.

Finally, if we plot the difference $D_8(b, \nu)$ on the box $[0, 1] \times [0, 1]$, see Figure 8, we conclude that neither of the lower bounds from (4) and (6) is always best.

3. NUMERICAL COMPARISON WITH TOMINAGA'S BOUND

In this section we provide a numerical comparison of the weighted upper bounds from (4), (5), (6) and (7) with Tominaga's upper bound (2) which does not depend on the weight $\nu \in [0, 1]$.

Consider the difference $D_9(b, \nu)$ defined by

$$D_9(b, \nu) := (K(b))^{\max\{\nu, 1-\nu\}} - S(b)$$

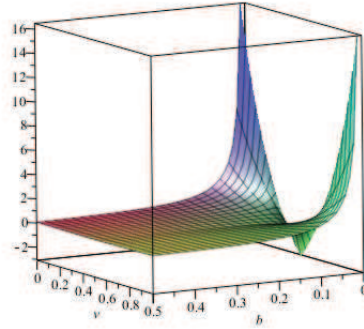


FIGURE 9

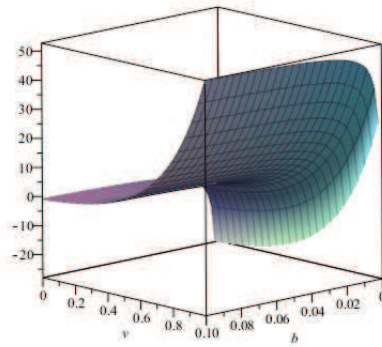


FIGURE 10

for $b > 0$ and $\nu \in [0, 1]$.

The plot of $D_9(b, \nu)$ in the box $[0, 0.5] \times [0, 1]$ is depicted in Figure 9 and shows that neither of the upper bounds (4) and (2) is always best.

Now, consider the difference $D_{10}(b, \nu)$ defined by

$$D_{10}(b, \nu) := \exp [4\nu(1 - \nu)(K(b) - 1)] - S(b)$$

for $b > 0$ and $\nu \in [0, 1]$.

The graph of the difference $D_{10}(b, \nu)$ in the box $[0, 0.1] \times [0, 1]$ is incorporated in Figure 10 showing that neither of the bounds (5) and (2) is always best.

Further, consider the difference

$$D_{11}(b, \nu) := \exp \left[\frac{1}{2} \nu(1 - \nu) \frac{(b - 1)^2}{\min^2 \{1, b\}} \right] - S(b)$$

for $b > 0$ and $\nu \in [0, 1]$.

The plot of $D_{11}(b, \nu)$ in the box $[1, 2] \times [0, 0.3]$ is depicted in Figure 11 and shows that neither of the upper bounds (6) and (2) is always best.

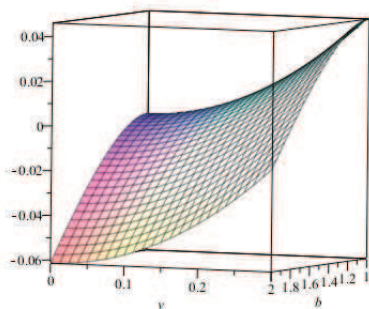


FIGURE 11

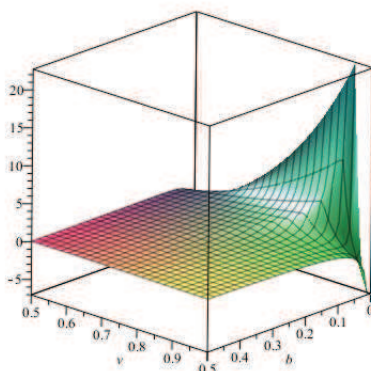


FIGURE 12

Finally, define the difference

$$D_{12}(b, \nu) := \min \left\{ \left(\frac{\exp(b-1)}{b} \right)^\nu, \left(\frac{b}{\exp(1-\frac{1}{b})} \right)^{1-\nu} \right\} - S(b)$$

for $b > 0$ and $\nu \in [0, 1]$.

The plot of this difference in the box $[0, 0.5] \times [0.5, 1]$ from Figure 12 proves that neither of the upper bounds (7) and (2) is always best.

We can draw the final conclusion that: Neither of the upper and lower bounds from the inequalities (2), (4), (5), (6) and (7) is always best.

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