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GENERALIZATIONS OF OPIAL TYPE INEQUALITIES IN TWO VARIABLES USING p-NORMS

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Abstract. In this paper, using the p-norms, we obtain some generalized Opial type inequalities in two variables for two functions. The results in this paper generalize several inequalities obtained in earlier works.

1. Introduction

In [11], Opial established the following interesting integral inequality :

Theorem 1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|\,dt \le \frac{h}{4}\int_0^h (x'(t))^2\,dt. \tag{1}$$

The constant $h/4$ is best possible

Over the years a large number of papers have been appeared in the literature which deal with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, for some of them please see [4], [5], [8], [9], [12]-[15], [25]-[27].

In [28], Yang proved the following Opial type inequalities in two variables:

Theorem 2. *If $f(t,s)$, $f_1(t,s)$ and $f_{12}(t,s)$ are continuous functions on $[a,b]\times[c,d]$ and if $f(a,s) = f(b,s) = f_1(t,c) = f_1(t,d) = 0$ for $a \le t \le b$, $c \le s \le d$, then*

$$\int_a^b\int_c^d |f(t,s)|\,|f_{12}(t,s)|\,dsdt \le \frac{(b-a)(d-c)}{8}\int_a^b\int_c^d |f_{12}(t,s)|^2\,dsdt \tag{2}$$

where

$$f_1(t,s) = \frac{\partial}{\partial t}f(t,s) \text{ and } f_{12}(s,t) = \frac{\partial^2}{\partial t\partial s}f(t,s).$$

On the other hand, B. G. Pachpatte published some papers which focus on the generalizations of the inequality (2). For the some of these generalizations, please see [16]-[21]. Moreover using two functions and their partial derivatives, W. S. Cheung established some generalizations of the inequality (2) in [6]. For the other Opial type inequalities in higher dimession, please see [1], [3], [7], [22]-[24].

Recently in [2], Budak and Sarikaya have proved the following refinement of the inequality (2).

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Theorem 3. *Let $f(t,s)$, $f_1(t,s)$ and $f_{12}(t,s)$ be continuous functions on $[a,b]\times[c,d]$ and let $f_{12}\in L_2([a,b]\times[c,d])$. If $f(a,s)=f(b,s)=f_1(t,c)=f_1(t,d)=0=0$ for $(t,s)\in[a,b]\times[c,d]$, then for all $(t,s)\in[a,b]\times[c,d]$ we have*

$$\int_a^b\int_c^d |f_{12}(t,s)f(t,s)|\,dsdt \tag{3}$$

$$\leq \frac{1}{4}\left((b-a)\int_a^b\int_c^d Q(s)\left|f_{12}(t,s)\right|^2 dsdt+(d-c)\int_a^b\int_c^d P(t)\left|f_{12}(t,s)\right|^p dsdt\right)^{\frac{1}{2}}$$

$$\times\left((b-a)\int_a^b\int_c^d \left|\frac{c+d}{2}-s\right|\left|f_{12}(t,s)\right|^2 dsdt+(d-c)\int_a^b\int_c^d \left|\frac{a+b}{2}-t\right|\left|f_{12}(t,s)\right|^2 dsdt\right)^{\frac{1}{2}}$$

$$\leq \frac{(b-a)(d-c)}{8}\int_a^b\int_c^d \left|f_{12}(t,s)\right|^2 dsdt.$$

where

$$P(t)=\begin{cases} t-a, & a\leq t\leq \frac{a+b}{2}\\ b-t, & \frac{a+b}{2}\leq t\leq b\end{cases}\quad \text{and}\quad Q(s)=\begin{cases} s-c, & c\leq s\leq \frac{c+d}{2}\\ d-s, & \frac{c+d}{2}\leq s\leq d.\end{cases}$$

The aim of this paper is to establish some generalized Opial type inequalities in two independent variables. We also obtain the refinement of the inequality (2).

2. p-Norm Generalizations of Opial type inequalities

In this section, we obtain p-norm generalizations of the inequalities (3).

Theorem 4. *Let $f(s,t)$, $f_1(s,t)$, $f_{12}(s,t)$, $g(s,t)$, $g_1(s,t)$ and $g_{12}(s,t)$ be continuous on $[a,b]\times[c,d]$ and let $f_{12}\in L_p([a,b]\times[c,d])$ and $g_{12}\in L_q([a,b]\times[c,d])$. If $g(a,s)=g(b,s)=g_1(t,c)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, then for all $(x,y)\in[a,b]\times[c,d]$ we have*

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \tag{4}$$

$$\leq \frac{1}{4}\left((b-a)\int_a^b\int_c^d Q(s,y)\left|f_{12}(t,s)\right|^p dsdt+(d-c)\int_a^b\int_c^d P(t,x)\left|f_{12}(t,s)\right|^p dsdt\right)^{\frac{1}{p}}$$

$$\times\left((b-a)\int_a^b\int_c^d |y-s|\left|g_{12}(t,s)\right|^q dsdt+(d-c)\int_a^b\int_c^d |x-t|\left|g_{12}(t,s)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{4p}\left[(b-a)\int_a^b\int_c^d Q(s,y)\left|f_{12}(t,s)\right|^p dsdt+(d-c)\int_a^b\int_c^d P(t,x)\left|f_{12}(t,s)\right|^p dsdt\right]$$

$$+\frac{1}{4q}\left[(b-a)\int_a^b\int_c^d |y-s|\left|g_{12}(t,s)\right|^q dsdt+(d-c)\int_a^b\int_c^d |x-t|\left|g_{12}(t,s)\right|^q dsdt\right]$$

where $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$P(t,x) = \begin{cases} t-a, & a \le t \le x \\ \\ b-t, & x \le t \le b \end{cases} \quad \textit{and} \quad Q(s,y) = \begin{cases} s-c, & c \le s \le y \\ \\ d-s, & y \le s \le d. \end{cases}$$

Proof. In order to prove Theorem 4, we consider the following four cases:

Case I: Let $g(a,s) = g_1(t,c) = 0$ for $(t,s) \in [a,b] \times [c,d]$.

Using the assumptions $g(a,s) = g_1(t,c) = 0$, we can write

$$g(t,s) = \int_a^t \int_c^s g_{12}(u,v)dudv$$

for $(t,s) \in [a,b] \times [c,d]$. Then, by applying Hölder's inequality, we have

$$\int_a^b \int_c^d |f_{12}(t,s)g(t,s)|\, dsdt \tag{5}$$

$$= \int_a^b \int_c^d (t-a)^{\frac{1}{p}} (s-c)^{\frac{1}{p}} |f_{12}(t,s)| (t-a)^{-\frac{1}{p}} (s-c)^{-\frac{1}{p}} |g(t,s)|\, dsdt$$

$$= \int_a^b \int_c^d (t-a)^{\frac{1}{p}} (s-c)^{\frac{1}{p}} |f_{12}(t,s)| (t-a)^{-\frac{1}{p}} (s-c)^{-\frac{1}{p}} \left| \int_a^t \int_c^s g_{12}(u,v)dudv \right| dsdt$$

$$\le \left(\int_a^b \int_c^d (t-a)(s-c) |f_{12}(t,s)|^p\, dsdt \right)^{\frac{1}{p}}$$

$$\times \left(\int_a^b \int_c^d \left[(t-a)^{-\frac{1}{p}} (s-c)^{-\frac{1}{p}} \left| \int_a^t \int_c^s g_{12}(u,v)dudv \right| \right]^q dsdt \right)^{\frac{1}{q}}.$$

By aplying again Hölder's inequality, we get

$$(t-a)^{-\frac{1}{p}} (s-c)^{-\frac{1}{p}} \left| \int_a^t \int_c^s g_{12}(u,v)dudv \right|$$

$$\le (t-a)^{-\frac{1}{p}} (s-c)^{-\frac{1}{p}} \left(\int_a^t \int_c^s dudv \right)^{\frac{1}{p}} \left(\int_a^t \int_c^s |g_{12}(u,v)|^q\, dudv \right)^{\frac{1}{q}}$$

$$= \left(\int_a^t \int_c^s |g_{12}(u,v)|^q\, dudv \right)^{\frac{1}{q}}.$$

That is,

$$\left[(t-a)^{-\frac{1}{p}} (s-c)^{-\frac{1}{p}} \left| \int_a^t \int_c^s g_{12}(u,v)dudv \right| \right]^q \le \int_a^t \int_c^s |g_{12}(u,v)|^q\, dudv. \tag{6}$$

Substituting the inequality (6) in (5), we obtain

$$\begin{aligned}
&\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \qquad (7)\\
&\le \left(\int_a^b\int_c^d (t-a)(s-c)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_c^d\left(\int_a^t\int_c^s |g_{12}(u,v)|^q\,dudv\right)dsdt\right)^{\frac{1}{q}}.
\end{aligned}$$

By integration by parts, one can show that

$$\int_a^b\int_c^d\left(\int_a^t\int_c^s |g_{12}(u,v)|^q\,dudv\right)dsdt=\int_a^b\int_c^d (b-t)(d-s)|g_{12}(u,v)|^q\,dsdt. \qquad (8)$$

By using the equality (8) in (7), we obtain the following inequality

$$\begin{aligned}
&\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \qquad (9)\\
&\le \left(\int_a^b\int_c^d (t-a)(s-c)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_c^d (b-t)(d-s)|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}.
\end{aligned}$$

Case II: Let $g(a,s)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$.
We get

$$g(t,s)=-\int_a^t\int_s^d g_{12}(u,v)dudv$$

for $(t,s)\in[a,b]\times[c,d]$. Then it follows that

$$\begin{aligned}
&\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt\\
&= \int_a^b\int_c^d (t-a)^{\frac{1}{p}}(d-s)^{\frac{1}{p}}|f_{12}(t,s)|(t-a)^{-\frac{1}{p}}(d-s)^{-\frac{1}{p}}\left|\int_a^t\int_s^d g_{12}(u,v)dudv\right|dsdt.
\end{aligned}$$

By using Hölder's inequality two times, we get

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt$$
$$\leq \left(\int_a^b\int_c^d (t-a)(d-s)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^b\int_c^d \left[(t-a)^{-\frac{1}{p}}(d-s)^{-\frac{1}{p}}\left|\int_a^t\int_s^d g_{12}(u,v)dudv\right|\right]^q dsdt\right)^{\frac{1}{q}}$$
$$\leq \left(\int_a^b\int_c^d (t-a)(d-s)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_c^d\left(\int_a^t\int_s^d |g_{12}(u,v)|^q\,dudv\right)dsdt\right)^{\frac{1}{q}}.$$

By integration by parts, we have

$$\int_a^b\int_c^d\left(\int_a^t\int_s^d |g_{12}(u,v)|^q\,dudv\right)dsdt = \int_a^b\int_c^d (b-t)(s-c)|g_{12}(t,s)|^q\,dsdt.$$

This gives

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \tag{10}$$
$$\leq \left(\int_a^b\int_c^d (t-a)(d-s)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_c^d (b-t)(s-c)|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}.$$

Case III: Let $g(b,s) = g_1(t,c) = 0$ for $(t,s)\in[a,b]\times[c,d]$. Then we have

$$g(t,s) = -\int_t^b\int_c^s g_{12}(u,v)dudv.$$

Case IV: Let $g(b,s) = g_1(t,d) = 0$ for $(t,s)\in[a,b]\times[c,d]$. In this case we can write

$$g(t,s) = \int_t^b\int_s^d g_{12}(u,v)dudv.$$

By following similar to those in proof of (9) and (10), but with suitable modifications, we establish the following inequalities in Case III and Case IV:

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \tag{11}$$
$$\leq \left(\int_a^b\int_c^d (t-a)(d-s)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_c^d (t-a)(d-s)|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}},$$

$$\int_a^b \int_c^d |f_{12}(t,s)g(t,s)|\, dsdt \tag{12}$$
$$\leq \left(\int_a^b \int_c^d (b-t)(d-s)|f_{12}(t,s)|^p\, dsdt \right)^{\frac{1}{p}} \left(\int_a^b \int_c^d (t-a)(s-c)|g_{12}(t,s)|^q\, dsdt \right)^{\frac{1}{q}},$$

respectively.

By the assumption $g(a,s) = g_1(t,c) = 0$ for $(t,s) \in [a,b] \times [c,d]$, if we write the inequality (9) for the rectangles $[a,b] \times [c,y]$ and $[a,x] \times [c,d]$ for $(x,y) \in [a,b] \times [c,d]$, then we have

$$\int_a^b \int_c^y |f_{12}(t,s)g(t,s)|\, dsdt \tag{13}$$
$$\leq \left(\int_a^b \int_c^y (t-a)(s-c)|f_{12}(t,s)|^p\, dsdt \right)^{\frac{1}{p}} \left(\int_a^b \int_c^y (b-t)(y-s)|g_{12}(t,s)|^q\, dsdt \right)^{\frac{1}{q}}$$

and

$$\int_a^x \int_c^d |f_{12}(t,s)g(t,s)|\, dsdt \tag{14}$$
$$\leq \left(\int_a^x \int_c^d (t-a)(s-c)|f_{12}(t,s)|^p\, dsdt \right)^{\frac{1}{p}} \left(\int_a^x \int_c^d (x-t)(d-s)|g_{12}(t,s)|^q\, dsdt \right)^{\frac{1}{q}},$$

respectively.

As $g(a,s) = g_1(t,d) = 0$ for $(t,s) \in [a,b] \times [c,d]$, if we apply the inequality (10) for the rectangles $[a,b] \times [y,d]$ and $[a,x] \times [c,d]$ for $(x,y) \in [a,b] \times [c,d]$, then we get

$$\int_a^b \int_y^d |f_{12}(t,s)g(t,s)|\, dsdt \tag{15}$$
$$\leq \left(\int_a^b \int_y^d (t-a)(d-s)|f_{12}(t,s)|^p\, dsdt \right)^{\frac{1}{p}} \left(\int_a^b \int_y^d (b-t)(s-y)|g_{12}(t,s)|^q\, dsdt \right)^{\frac{1}{q}}$$

and

$$\int_a^x \int_c^d |f_{12}(t,s)g(t,s)|\, dsdt \tag{16}$$
$$\leq \left(\int_a^x \int_c^d (t-a)(d-s)|f_{12}(t,s)|^p\, dsdt \right)^{\frac{1}{p}} \left(\int_a^x \int_c^d (x-t)(s-c)|g_{12}(t,s)|^q\, dsdt \right)^{\frac{1}{q}}.$$

Similarly, since $g(b,s)=g_1(t,c)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we write the inequality (11) for the rectangles $[a,b]\times[c,y]$ and $[x,b]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\int_a^b\int_c^y |f_{12}(t,s)g(t,s)|\,dsdt \tag{17}$$

$$\leq \left(\int_a^b\int_c^y (b-t)(s-c)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_c^y (t-a)(y-s)|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}$$

and

$$\int_x^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \tag{18}$$

$$\leq \left(\int_x^b\int_c^d (b-t)(s-c)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_x^b\int_c^d (t-x)(d-s)|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}.$$

Finally, as $g(b,s)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we apply the inequality (12) for the rectangles $[a,b]\times[y,d]$ and $[x,b]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we obtain

$$\int_a^b\int_y^d |f_{12}(t,s)g(t,s)|\,dsdt \tag{19}$$

$$\leq \left(\int_a^b\int_y^d (b-t)(d-s)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_y^d (t-a)(s-y)|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}$$

and

$$\int_x^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \tag{20}$$

$$\leq \left(\int_x^b\int_c^d (b-t)(d-s)|f_{12}(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_x^b\int_c^d (t-x)(s-c)|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}.$$

Let $a_1,a_2,...,a_n$ and $b_1,b_2,...,b_n$ be real numbers. Then we have the following Hölder inequality

$$a_1b_1+a_2b_2+...+a_nb_n\leq\left(a_1^p+a_2^p+....+a_n^p\right)^{\frac{1}{p}}\left(b_1^q+b_2^q+....+b_n^q\right)^{\frac{1}{q}} \tag{21}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If we add the inequalities (13)-(20),then by using Hölder inequality (21), we obtain

$$
\begin{aligned}
& 4\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \qquad (22)\\
\leq & \left[\int_a^b\int_c^y (b-a)(s-c)|f_{12}(t,s)|^p\,dsdt+\int_a^x\int_c^d (t-a)(d-c)|f_{12}(t,s)|^p\,dsdt\right.\\
& \left.+\int_a^b\int_y^d (b-a)(d-s)|f_{12}(t,s)|^p\,dsdt+\int_x^b\int_c^d (b-t)(d-c)|f_{12}(t,s)|^p\,dsdt\right]^{\frac{1}{p}}\\
& \times\left[\int_a^b\int_c^y (b-a)(y-s)|g_{12}(t,s)|^q\,dsdt+\int_a^x\int_c^d (x-t)(d-c)|g_{12}(t,s)|^q\,dsdt\right.\\
& \left.+\int_a^b\int_y^d (b-a)(s-y)|g_{12}(t,s)|^q\,dsdt+\int_x^b\int_c^d (t-x)(d-c)|g_{12}(t,s)|^q\,dsdt\right]^{\frac{1}{q}}\\
= & \left[(b-a)\int_a^b\int_c^d Q(s,y)|f_{12}(t,s)|^p\,dsdt+(d-c)\int_a^b\int_c^d P(t,x)|f_{12}(t,s)|^p\,dsdt\right]^{\frac{1}{p}}\\
& \times\left[(b-a)\int_a^b\int_c^d |y-s|\,|g_{12}(t,s)|^q\,dsdt+(d-c)\int_a^b\int_c^d |x-t|\,|g_{12}(t,s)|^q\,dsdt\right]^{\frac{1}{q}}.
\end{aligned}
$$

If we divide by 4 both sides of the inequality (22), then we obtain the first inequality in (4).

The proof of the second inequality in (4) is obvious from the Young inequality

$$a_1^{1/p}a_1^{1/q}\leq \frac{1}{p}a_1+\frac{1}{q}a_2,$$

for $a_1, a_2 > 0$. □

Corollary 1. *If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Theorem 4, then we have the following inequality*

$$
\begin{aligned}
&\int_a^b \int_c^d |f_{12}(t,s) g(t,s)|\, dsdt \\
\le\ & \frac{1}{4}\left((b-a) \int_a^b \int_c^d Q(s) |f_{12}(t,s)|^p \, dsdt + (d-c) \int_a^b \int_c^d P(t) |f_{12}(t,s)|^p \, dsdt \right)^{\frac{1}{p}} \\
& \times \left((b-a) \int_a^b \int_c^d \left| \frac{c+d}{2} - s \right| |g_{12}(t,s)|^q \, dsdt + (d-c) \int_a^b \int_c^d \left| \frac{a+b}{2} - t \right| |g_{12}(t,s)|^q \, dsdt \right)^{\frac{1}{q}} \\
\le\ & \frac{1}{4p}\left[(b-a) \int_a^b \int_c^d Q(s) |f_{12}(t,s)|^p \, dsdt + (d-c) \int_a^b \int_c^d P(t) |f_{12}(t,s)|^p \, dsdt \right] \\
& + \frac{1}{4q}\left[(b-a) \int_a^b \int_c^d \left| \frac{c+d}{2} - s \right| |g_{12}(t,s)|^q \, dsdt + (d-c) \int_a^b \int_c^d \left| \frac{a+b}{2} - t \right| |g_{12}(t,s)|^q \, dsdt \right]
\end{aligned}
$$

where

$$
P(t) = \begin{cases} t-a, & a \le t \le \frac{a+b}{2} \\ b-t, & \frac{a+b}{2} \le t \le b \end{cases} \quad \text{and} \quad Q(s) = \begin{cases} s-c, & c \le s \le \frac{c+d}{2} \\ d-s, & \frac{c+d}{2} \le s \le d. \end{cases}
$$

Corollary 2. *If we choose $f(t,s) = g(t,s)$ for $(t,s) \in [a,b] \times [c,d]$ in Corollary 1, then we have the following inequality*

$$
\begin{aligned}
&\int_a^b \int_c^d |f_{12}(t,s) f(t,s)|\, dsdt \qquad (23)\\
\le\ & \frac{1}{4}\left((b-a) \int_a^b \int_c^d Q(s) |f_{12}(t,s)|^p \, dsdt + (d-c) \int_a^b \int_c^d P(t) |f_{12}(t,s)|^p \, dsdt \right)^{\frac{1}{p}} \\
& \times \left((b-a) \int_a^b \int_c^d \left| \frac{c+d}{2} - s \right| |f_{12}(t,s)|^q \, dsdt + (d-c) \int_a^b \int_c^d \left| \frac{a+b}{2} - t \right| |f_{12}(t,s)|^q \, dsdt \right)^{\frac{1}{q}} \\
\le\ & \frac{1}{4p}\left[(b-a) \int_a^b \int_c^d Q(s) |f_{12}(t,s)|^p \, dsdt + (d-c) \int_a^b \int_c^d P(t) |f_{12}(t,s)|^p \, dsdt \right] \\
& + \frac{1}{4q}\left[(b-a) \int_a^b \int_c^d \left| \frac{c+d}{2} - s \right| |f_{12}(t,s)|^q \, dsdt + (d-c) \int_a^b \int_c^d \left| \frac{a+b}{2} - t \right| |f_{12}(t,s)|^q \, dsdt \right].
\end{aligned}
$$

Remark 1. *If we choose $p = q = 2$ in Corollary 2, then the inequality (23) reduces to the inequality (3).*

Theorem 5. *Let $f(s,t)$, $f_1(s,t)$, $f_{12}(s,t)$ and $h(t,s)$ be continuous on $[a,b]\times[c,d]$ and let $f_{12}\in L_p([a,b]\times[c,d])$. If $h\in L_q([a,b]\times[c,d])$ with $\int_a^b h(t,.)dt=\int_c^d h(.,s)ds=0$, then we have the following inequality*

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(t,s)h(t,s)dsdt\right| \\
&\le \frac{1}{4}\left((b-a)\int_a^b\int_c^d Q(s,y)\left|f_{12}(t,s)\right|^p dsdt+(d-c)\int_a^b\int_c^d P(t,x)\left|f_{12}(t,s)\right|^p dsdt\right)^{\frac{1}{p}} \\
&\quad\times\left((b-a)\int_a^b\int_c^d |y-s|\left|h(t,s)\right|^q dsdt+(d-c)\int_a^b\int_c^d |x-t|\left|h(t,s)\right|^q dsdt\right)^{\frac{1}{q}} \\
&\le \frac{1}{4p}\left[(b-a)\int_a^b\int_c^d Q(s,y)\left|f_{12}(t,s)\right|^p dsdt+(d-c)\int_a^b\int_c^d P(t,x)\left|f_{12}(t,s)\right|^p dsdt\right] \\
&\quad+\frac{1}{4q}\left[(b-a)\int_a^b\int_c^d |y-s|\left|h(t,s)\right|^q dsdt+(d-c)\int_a^b\int_c^d |x-t|\left|h(t,s)\right|^q dsdt\right]
\end{aligned}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $Q(s,y)$ and $P(t,x)$ are defined by as in Theorem 4.

Proof. If we choose $g(t,s)=\int_a^t\int_c^s h(u,v)dvdu$ in Theorem 4, then we have $g_{12}(t,s)=h(t,s)$ and from the inequality (4), we get

$$\begin{aligned}
&\int_a^b\int_c^d \left|f_{12}(t,s)\left(\int_a^t\int_c^s h(u,v)dvdu\right)\right|dsdt \\
&\le \frac{1}{4}\left((b-a)\int_a^b\int_c^d Q(s,y)\left|f_{12}(t,s)\right|^p dsdt+(d-c)\int_a^b\int_c^d P(t,x)\left|f_{12}(t,s)\right|^p dsdt\right)^{\frac{1}{p}} \\
&\quad\times\left((b-a)\int_a^b\int_c^d |y-s|\left|h(t,s)\right|^q dsdt+(d-c)\int_a^b\int_c^d |x-t|\left|h(t,s)\right|^q dsdt\right)^{\frac{1}{q}} \\
&\le \frac{1}{4p}\left[(b-a)\int_a^b\int_c^d Q(s,y)\left|f_{12}(t,s)\right|^p dsdt+(d-c)\int_a^b\int_c^d P(t,x)\left|f_{12}(t,s)\right|^p dsdt\right] \\
&\quad+\frac{1}{4q}\left[b-a)\int_a^b\int_c^d |y-s|\left|h(t,s)\right|^q dsdt+(d-c)\int_a^b\int_c^d |x-t|\left|h(t,s)\right|^q dsdt\right].
\end{aligned}$$

By using the property modulus and then using the integration by parts, we have

$$\begin{aligned}
&\int_a^b\int_c^d \left| f_{12}(t,s)\left(\int_a^t\int_c^s h(u,v)dvdu\right)\right| dsdt \\
\geq\ & \left|\int_a^b\int_c^d f_{12}(t,s)\left(\int_a^t\int_c^s h(u,v)dvdu\right) dsdt\right| \\
=\ & \left|\int_a^b \left[\left(\int_a^t\int_c^s h(u,v)dvdu\right) f_1(t,s)\Big|_c^d - \int_c^d f_1(t,s)\left(\int_a^t h(u,s)dv\right) ds\right] dt\right| \\
=\ & \left|\int_a^b\int_c^d f_1(t,s)\left(\int_a^t h(u,s)dv\right) dsdt\right| \\
=\ & \left|\int_c^d \left[\left(\int_a^t h(u,s)dv\right) f(t,s)\Big|_a^b - \int_a^b f(t,s)h(t,s)dt\right] ds\right| \\
=\ & \left|\int_a^b\int_c^d |f(t,s)h(t,s)|\, dsdt\right|
\end{aligned}$$

which completes the proof. □

Corollary 3. *If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Theorem 5, then we have the following inequality*

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(t,s)h(t,s)dsdt\right| \\
\leq\ & \frac{1}{4}\left((b-a)\int_a^b\int_c^d Q(s)\,|f_{12}(t,s)|^p\, dsdt + (d-c)\int_a^b\int_c^d P(t)\,|f_{12}(t,s)|^p\, dsdt\right)^{\frac{1}{p}} \\
&\times\left((b-a)\int_a^b\int_c^d \left|\frac{c+d}{2}-s\right| |h(t,s)|^q\, dsdt + (d-c)\int_a^b\int_c^d \left|\frac{a+b}{2}-t\right| |h(t,s)|^q\, dsdt\right)^{\frac{1}{q}} \\
\leq\ & \frac{1}{4p}\left[(b-a)\int_a^b\int_c^d Q(s)\,|f_{12}(t,s)|^p\, dsdt + (d-c)\int_a^b\int_c^d P(t)\,|f_{12}(t,s)|^p\, dsdt\right] \\
&+\frac{1}{4q}\left[b-a)\int_a^b\int_c^d \left|\frac{c+d}{2}-s\right| |h(t,s)|^q\, dsdt + (d-c)\int_a^b\int_c^d \left|\frac{a+b}{2}-t\right| |h(t,s)|^q\, dsdt\right]
\end{aligned}$$

where $Q(s)$ and $P(t)$ are defined by as in Corollary 1.

Theorem 6. *Let $g(s,t)$, $g_1(s,t)$ and $g_{12}(s,t)$ be continuous on $[a,b]\times[c,d]$ and let $g_{12} \in L_q([a,b]\times[c,d])$. If $h \in L_p([a,b]\times[c,d])$ and $g(a,s) = g(b,s) = g_1(t,c) = g_1(t,d) = 0$ for*

$(t,s)\in[a,b]\times[c,d]$, then for all $(x,y)\in[a,b]\times[c,d]$ we have the following inequality

$$\begin{aligned}
&\int_a^b\int_c^d |h(t,s)g(t,s)|\,dsdt\\
\leq\;& \frac{1}{4}\left((b-a)\int_a^b\int_c^d Q(s,y)\,|h(t,s)|^p\,dsdt+(d-c)\int_a^b\int_c^d P(t,x)\,|h(t,s)|^p\,dsdt\right)^{\frac{1}{p}}\\
&\times\left((b-a)\int_a^b\int_c^d |y-s|\,|g_{12}(t,s)|^q\,dsdt+(d-c)\int_a^b\int_c^d |x-t|\,|g_{12}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}\\
\leq\;& \frac{1}{4p}\left[(b-a)\int_a^b\int_c^d Q(s,y)\,|h(t,s)|^p\,dsdt+(d-c)\int_a^b\int_c^d P(t,x)\,|h(t,s)|^p\,dsdt\right]\\
&+\frac{1}{4q}\left[(b-a)\int_a^b\int_c^d |y-s|\,|g_{12}(t,s)|^q\,dsdt+(d-c)\int_a^b\int_c^d |x-t|\,|g_{12}(t,s)|^q\,dsdt\right]
\end{aligned}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $Q(s,y)$ and $P(t,x)$ are defined by as in Theorem 4.

Proof. The proof is obvious by choosing $f(t,s)=\int_a^t\int_c^s h(u,v)dvdu$ in Theorem 4. □

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