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ESTIMATION OF UPPER BOUNDS INVOLVING KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. The main objective of this article is to give some further generalizations of Hermite-Hadamard type inequalities via harmonic convex functions. The inequalities involve Katugampola's fractional integral. We also give new estimation to upper bounds essentially using Katugampola's fractional integrals. Special cases are also discussed.

1. Introduction and Preliminaries

Convexity in relation with inequalities has attracted many researchers. Many inequalities are direct consequences of the applications of classical convexity. An interesting and fascinating result in this regard is Hermite-Hadamard's inequality which is considered as an equivalent property with convexity. This result reads as:
Let $T : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be an integrable convex function, then

$$T\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b T(x)\mathrm{d}x \leq \frac{T(a)+T(b)}{2}.$$

For some useful details on generalized convexity and integral inequalities interested readers are referred to [4, 5, 6]. Recently Iscan [7] introduced the class of harmonic convex functions, which reads as follows:
A function $T : \Omega \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic convex function, if

$$T\left(\frac{ab}{ta+(1-t)b}\right) \leq (1-t)T(a) + tT(b), \quad \forall a, b \in \Omega, t \in [0, 1].$$

Iscan also gave a new extension for Hermite-Hadamard type inequality via harmonic convex functions. Noor et al. [14] generalized the class of harmonic convex functions with the introduction of harmonic φ-convex functions.

Definition 1 ([14])**.** *Let $\varphi : [0, 1] \subseteq J \to \mathbb{R}$ be a real function. A function $T : [a, b] \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically φ-convex function, if*

$$T\left(\frac{xy}{tx+(1-t)y}\right) \leq \varphi(1-t)T(x) + \varphi(t)T(y), \quad \forall x, y \in [a, b], t \in (0, 1). \tag{1}$$

Note that for different suitable choices of the real function $\varphi(.)$ in Definition 1, we have some other classes of harmonic convexity. For example, for $\varphi(t) = t, t^s, t^{-s}$ and for $\varphi(t) = 1$, we have classical harmonic convex functions, Breckner type of harmonic s-convex functions, Godunova-Levin type of harmonic s-convex functions and P-harmonic functions respectively. For some recent studies on harmonic convex functions and its

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generalizations, see [1, 2, 10, 11, 12, 13, 14].
We now recall some concepts from fractional calculus. Classical Riemann-Liouville fractional integrals are defined as:

Definition 2 ([9]). *Let $T \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}T$ and $J^{\alpha}_{b^-}T$ of order $\alpha > 0$ are defined by*

$$J^{\alpha}_{a^+}T(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} T(t)\mathrm{d}t, \quad x > a,$$

and

$$J^{\alpha}_{b^-}T(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} T(t)\mathrm{d}t, \quad x < b,$$

Sarikaya et al. [15] used Riemann-Liouville integrals and obtained fractional version of Hermite-Hadamard type inequality via convex functions. Wu et al. [8] using the techniques of Sarikaya et al. [15] improved the Hermite-Hadamard inequality via harmonic convex functions for fractional integrals.
Recently Katugampola [3] gave a new generalized approach to fractional integrals.

Definition 3. *Let $[a,b] \in \mathbb{R}$ be a finite interval. Then, the left and right side Katugampola fractional integrals of order $\alpha > 0$ are defined by:*

$${}^{\varrho}I^{\alpha}_{a^+}T(x) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\varrho-1}}{(x^{\varrho}-t^{\varrho})^{1-\alpha}} T(t)\mathrm{d}t,$$

and

$${}^{\varrho}I^{\alpha}_{b^-}T(x) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\varrho-1}}{(t^{\varrho}-x^{\varrho})^{1-\alpha}} T(t)\mathrm{d}t,$$

with $0 < a < b$ and $\varrho > 0$ if integrals exist.

Chen et al. [3] improved the Hermite-Hadamard type inequalities via Katugampola fractional integrals.
For the reader's convenience, we recall the definitions of the Gamma function $\Gamma(.)$ and Beta function $B(.,.)$ respectively.

$$\Gamma(x) = \int_0^{\infty} e^{-x} t^{x-1} \mathrm{d}t,$$

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, \mathrm{d}t.$$

It is known that [9]

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2\mathfrak{F}_1(x,y;c;z) = \frac{1}{\mathrm{B}(y,c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\mathrm{d}t$$

for $|z| < 1, c > y > 0$.
The main motivation of this paper is to derive some new fractional versions of Hermite-Hadamard like inequalities involving Katugampola fractional integrals via harmonic φ-convex functions. We also discuss some new special cases of the main results.

2. Results and Discussions

In this section, we obtain our main results. Our first result is a new refinement of fractional Hermite-Hadamard type inequality via harmonic φ-convex functions.

Theorem 1. *Let $\alpha, \varrho > 0$. Let $T : [a^\varrho, b^\varrho] \to \mathbb{R}$ be a positive function with $0 \le a < b$. If T is harmonic φ-convex function on $[a, b]$, then for $\varphi\big(\frac{1}{2}\big) \ne 0$, we have*

$$\begin{aligned}
&\frac{1}{\varphi\big(\frac{1}{2}\big)} T\left(\frac{2a^\varrho b^\varrho}{a^\varrho + b^\varrho}\right)\\
&\le \varrho^\alpha \Gamma(\alpha+1)\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a}^-}(T\circ W)\big(\frac{1}{b^\varrho}\big) + {}^\varrho I^\alpha_{\frac{1}{b}^+}(T\circ W)\big(\frac{1}{a^\varrho}\big)\right]\\
&\le [T(a^\varrho) + T(b^\varrho)]\alpha\varrho^{2\alpha-1}\int_0^1 t^{\alpha\varrho-1}[\varphi(t^\varrho) + \varphi(1-t^\varrho)]\mathrm{d}t,
\end{aligned}$$

where $W(x) = \frac{1}{x}$.

Proof. Since T is harmonic φ-convex function, then

$$T\Big(\frac{2x^\varrho y^\varrho}{x^\varrho + y^\varrho}\Big) \le \varphi\Big(\frac{1}{2}\Big)[T(x^\varrho) + T(y^\varrho)].$$

Consider $x, y \in [a, b]$, defined by $x^\varrho = \frac{a^\varrho b^\varrho}{(1-t^\varrho)a^\varrho + t^\varrho b^\varrho}$ and $y^\varrho = \frac{a^\varrho b^\varrho}{t^\varrho a^\varrho + (1-t^\varrho)b^\varrho}$. Using this in the above inequality, we get

$$\frac{1}{\varphi\big(\frac{1}{2}\big)} T\left(\frac{2a^\varrho b^\varrho}{a^\varrho + b^\varrho}\right) \le T\left(\frac{a^\varrho b^\varrho}{(1-t^\varrho)a^\varrho + t^\varrho b^\varrho}\right) + T\left(\frac{a^\varrho b^\varrho}{t^\varrho a^\varrho + (1-t^\varrho)b^\varrho}\right).$$

Multiplying both sides of above inequality with $t^{\alpha\varrho-1}$ and then integrating with respect to t on $[0, 1]$, we have

$$\begin{aligned}
&\frac{1}{\varphi\big(\frac{1}{2}\big)} T\left(\frac{2a^\varrho b^\varrho}{a^\varrho + b^\varrho}\right)\int_0^1 t^{\alpha\varrho-1}\mathrm{d}t\\
&\le \int_0^1 t^{\alpha\varrho-1} T\left(\frac{a^\varrho b^\varrho}{(1-t^\varrho)a^\varrho + t^\varrho b^\varrho}\right)\mathrm{d}t + \int_0^1 t^{\alpha\varrho-1} T\left(\frac{a^\varrho b^\varrho}{t^\varrho a^\varrho + (1-t^\varrho)b^\varrho}\right)\mathrm{d}t.
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{\alpha\varrho\varphi\big(\frac{1}{2}\big)} T\left(\frac{2a^\varrho b^\varrho}{a^\varrho + b^\varrho}\right)\\
&\le \left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[\int_{\frac{1}{b}}^{\frac{1}{a}} \left(x^\varrho - \frac{1}{b^\varrho}\right)^{\alpha-1} T(x^\varrho)x^{\varrho-1}\mathrm{d}x + \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a^\varrho} - y^\varrho\right)^{\alpha-1} T(y^\varrho)y^{\varrho-1}\mathrm{d}y\right].
\end{aligned}$$

Thus

$$\frac{1}{\alpha\varrho^\alpha\varphi\big(\frac{1}{2}\big)} T\left(\frac{2a^\varrho b^\varrho}{a^\varrho + b^\varrho}\right) \le \Gamma(\alpha)\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a}^-}(T\circ W)\big(\frac{1}{b^\varrho}\big) + {}^\varrho I^\alpha_{\frac{1}{b}^+}(T\circ W)\big(\frac{1}{a^\varrho}\big)\right]. \tag{2}$$

In order to prove the right hand side of the theorem, we again use the fact that T is harmonic φ-convex function, then

$$T\left(\frac{a^{\varrho}b^{\varrho}}{(1-t^{\varrho})a^{\varrho}+t^{\varrho}b^{\varrho}}\right)+T\left(\frac{a^{\varrho}b^{\varrho}}{t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho}}\right)\leq[T(a^{\varrho})+T(b^{\varrho})][\varphi(t^{\varrho})+\varphi(1-t^{\varrho})].$$

Multiplying both sides of above inequality with $t^{\alpha\varrho-1}$ and then integrating with respect to t on $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{\alpha\varrho-1}T\left(\frac{a^{\varrho}b^{\varrho}}{(1-t^{\varrho})a^{\varrho}+t^{\varrho}b^{\varrho}}\right)\mathrm{d}t+\int_0^1 t^{\alpha\varrho-1}T\left(\frac{a^{\varrho}b^{\varrho}}{t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho}}\right)\mathrm{d}t\\&\leq[T(a^{\varrho})+T(b^{\varrho})]\int_0^1 t^{\alpha\varrho-1}[\varphi(t^{\varrho})+\varphi(1-t^{\varrho})]\mathrm{d}t.\end{aligned}$$

This implies

$$\begin{aligned}&\Gamma(\alpha)\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^-}(T\circ W)(\frac{1}{b^{\varrho}})+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^+}(T\circ W)(\frac{1}{a^{\varrho}})\right]\\&\leq[T(a^{\varrho})+T(b^{\varrho})]\varrho^{\alpha-1}\int_0^1 t^{\alpha\varrho-1}[\varphi(t^{\varrho})+\varphi(1-t^{\varrho})]\mathrm{d}t.\end{aligned}\tag{3}$$

Combining (2) and (3) and after suitable rearrangements completes the proof. □

We now discuss some special cases of Theorem 1.

Corollary 1. *Under the assumptions of Theorem 1, if $\varphi(t)=t$, then we have the following known result. For more details, see* [11].

$$\begin{aligned}&T\left(\frac{2a^{\varrho}b^{\varrho}}{a^{\varrho}+b^{\varrho}}\right)\\&\leq\frac{\varrho^{\alpha}\Gamma(\alpha+1)}{2}\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^-}(T\circ W)(\frac{1}{b^{\varrho}})+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^+}(T\circ W)(\frac{1}{a^{\varrho}})\right]\leq\frac{T(a^{\varrho})+T(b^{\varrho})}{2}.\end{aligned}$$

Corollary 2. *Under the assumptions of Theorem 1, if $\varphi(t)=t^s$, then we have*

$$\begin{aligned}&T\left(\frac{2a^{\varrho}b^{\varrho}}{a^{\varrho}+b^{\varrho}}\right)\\&\leq\frac{\varrho^{\alpha}\Gamma(\alpha+1)}{2^s}\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^-}(T\circ W)(\frac{1}{b^{\varrho}})+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^+}(T\circ W)(\frac{1}{a^{\varrho}})\right]\\&\leq\frac{\alpha\varrho^{2\alpha-1}}{2^s}\left[\frac{1}{\alpha\varrho+s}+\frac{1}{\varrho}B(\alpha,1+s)\right][T(a^{\varrho})+T(b^{\varrho})].\end{aligned}$$

The above result is for Breckner type of harmonic s-convex functions. As far as we concern the result is new in the literature.

Corollary 3. *Under the assumptions of Theorem 1, if $\varphi(t)=t^{-s}$, then we have*

$$\begin{aligned}&T\left(\frac{2a^{\varrho}b^{\varrho}}{a^{\varrho}+b^{\varrho}}\right)\\&\leq 2^s\varrho^{\alpha}\Gamma(\alpha+1)\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^-}(T\circ W)(\frac{1}{b^{\varrho}})+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^+}(T\circ W)(\frac{1}{a^{\varrho}})\right]\\&\leq 2^s\alpha\varrho^{2\alpha-1}\left[\frac{1}{\alpha\varrho-s}+\frac{1}{\varrho}B(\alpha,1-s)\right][T(a^{\varrho})+T(b^{\varrho})].\end{aligned}$$

The above result is for Godunova-Levin-Dragomir type of harmonic s-convex functions. As far as we concern the result is new in the literature.

Corollary 4. *Under the assumptions of Theorem 1, if $\varphi(t)=1$, then we have*

$$\begin{aligned}&\frac{1}{\alpha\varrho^{\alpha}}T\left(\frac{2a^{\varrho}b^{\varrho}}{a^{\varrho}+b^{\varrho}}\right)\\ &\leq \Gamma(\alpha)\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^{-}}(T\circ W)\Big(\frac{1}{b^{\varrho}}\Big)+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^{+}}(T\circ W)\Big(\frac{1}{a^{\varrho}}\Big)\right]\\ &\leq \frac{2\varrho^{\alpha-1}}{\alpha\varrho}[T(a^{\varrho})+T(b^{\varrho})]\varrho^{\alpha-1}.\end{aligned}$$

The above result is for harmonic P-convex functions. As far as we concern the result is new in the literature.
The following is a new generalized fractional integral identity. This identity will serve as an auxiliary result for the rest of results. While writing this paper we came to know that the similar identity has been obtained by Mumcu et al. [11] independently. So we skip the proof of the result as readers can find it in the said reference.

Lemma 1. *Let $T:[a^{\varrho},b^{\varrho}]\to\mathbb{R}$ be a differentiable function on $(a^{\varrho},b^{\varrho})$. Then following fractional integral identity holds:*

$$\begin{aligned}&\frac{T(a^{\varrho})+T(b^{\varrho})}{2}-\frac{\varrho^{\alpha}\Gamma(\alpha+1)}{2}\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^{-}}(T\circ W)\Big(\frac{1}{b^{\varrho}}\Big)+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^{+}}(T\circ W)\Big(\frac{1}{a^{\varrho}}\Big)\right]\\ &=\frac{\varrho a^{\varrho}b^{\varrho}(b^{\varrho}-a^{\varrho})}{2}\int_0^1\frac{[t^{\varrho\alpha}-(1-t^{\varrho})^{\alpha}]t^{\varrho-1}}{(t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho})^2}T'\left(\frac{a^{\varrho}b^{\varrho}}{t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho}}\right)\mathrm{d}t,\end{aligned}$$

where $W(x)=\frac{1}{x}$.

Now using Lemma 1, we derive following results.

Theorem 2. *Let $T:[a^{\varrho},b^{\varrho}]\to\mathbb{R}$ be a differentiable function on $(a^{\varrho},b^{\varrho})$. If $|T'|^q$ is harmonic φ-convex function, where $q\geq 1$, then*

$$\begin{aligned}&\left|\frac{T(a^{\varrho})+T(b^{\varrho})}{2}-\frac{\varrho^{\alpha}\Gamma(\alpha+1)}{2}\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^{-}}(T\circ W)\Big(\frac{1}{b^{\varrho}}\Big)+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^{+}}(T\circ W)\Big(\frac{1}{a^{\varrho}}\Big)\right]\right|\\ &\leq\frac{\varrho a^{\varrho}b^{\varrho}(b^{\varrho}-a^{\varrho})}{2}\Psi_1^{1-\frac{1}{q}}(a,b;\varrho,\alpha)\\ &\times\left(\int_0^1 G(a,b;\varrho\alpha)[\varphi(1-t^{\varrho})|T'(a^{\varrho})|^q+\varphi(t^{\varrho})|T'(b^{\varrho})|^q]\mathrm{d}t\right)^{\frac{1}{q}},\end{aligned}$$

where $G(a,b;\varrho\alpha):=\frac{[t^{\varrho\alpha}-(1-t^{\varrho})^{\alpha}]t^{\varrho-1}}{(t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho})^2}$.

Proof. Using Lemma 1 and taking modulus on both sides, we have

$$\begin{aligned}&\left|\frac{T(a^{\varrho})+T(b^{\varrho})}{2}-\frac{\varrho^{\alpha}\Gamma(\alpha+1)}{2}\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^{-}}(T\circ W)\Big(\frac{1}{b^{\varrho}}\Big)+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^{+}}(T\circ W)\Big(\frac{1}{a^{\varrho}}\Big)\right]\right|\\ &=\left|\frac{\varrho a^{\varrho}b^{\varrho}(b^{\varrho}-a^{\varrho})}{2}\int_0^1\frac{[t^{\varrho\alpha}-(1-t^{\varrho})^{\alpha}]t^{\varrho-1}}{(t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho})^2}T'\left(\frac{a^{\varrho}b^{\varrho}}{t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho}}\right)\mathrm{d}t\right|.\end{aligned}$$

Using the property of modulus, we have

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2}-\frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho-a^\varrho}\right)^\alpha\left[{}^\varrho I^\alpha_{\frac{1}{a}-}(T\circ W)\Big(\frac{1}{b^\varrho}\Big)+{}^\varrho I^\alpha_{\frac{1}{b}+}(T\circ W)\Big(\frac{1}{a^\varrho}\Big)\right]\right|$$
$$\leq\frac{\varrho a^\varrho b^\varrho(b^\varrho-a^\varrho)}{2}\int_0^1\frac{|t^{\varrho\alpha}-(1-t^\varrho)^\alpha||t^{\varrho-1}|}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^2}\left|T'\left(\frac{a^\varrho b^\varrho}{t^\varrho a^\varrho+(1-t^\varrho)b^\varrho}\right)\right|\mathrm{d}t.$$

Utilizing power mean inequality and the fact that $|T'|$ is harmonic φ-convex function, we have

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2}-\frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho-a^\varrho}\right)^\alpha\left[{}^\varrho I^\alpha_{\frac{1}{a}-}(T\circ W)\Big(\frac{1}{b^\varrho}\Big)+{}^\varrho I^\alpha_{\frac{1}{b}+}(T\circ W)\Big(\frac{1}{a^\varrho}\Big)\right]\right|$$
$$\leq\frac{\varrho a^\varrho b^\varrho(b^\varrho-a^\varrho)}{2}\left(\int_0^1\frac{|t^{\varrho\alpha}-(1-t^\varrho)^\alpha||t^{\varrho-1}|}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^2}\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{[t^{\varrho\alpha}-(1-t^\varrho)^\alpha]t^{\varrho-1}}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^2}[\varphi(1-t^\varrho)|T'(a^\varrho)|^q+\varphi(t^\varrho)|T'(b^\varrho)|^q]\mathrm{d}t\right)^{\frac{1}{q}}$$
$$=\frac{\varrho a^\varrho b^\varrho(b^\varrho-a^\varrho)}{2}\Psi_1^{1-\frac{1}{q}}(a,b;\varrho,\alpha)$$
$$\times\left(\int_0^1\frac{[t^{\varrho\alpha}-(1-t^\varrho)^\alpha]t^{\varrho-1}}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^2}[\varphi(1-t^\varrho)|T'(a^\varrho)|^q+\varphi(t^\varrho)|T'(b^\varrho)|^q]\mathrm{d}t\right)^{\frac{1}{q}},$$

where

$$\Psi_1(a,b;\varrho,\alpha)=\int_0^1\frac{|t^{\varrho\alpha}-(1-t^\varrho)^\alpha||t^{\varrho-1}|}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^2}$$
$$=\frac{b^{-2\varrho}}{\varrho(\alpha+1)}\left[{}_2\mathfrak{F}_1\left(2,\alpha+1;\alpha+2;1-\frac{a^\varrho}{b^\varrho}\right)+{}_2\mathfrak{F}_1\left(2,1;\alpha+2;1-\frac{a^\varrho}{b^\varrho}\right)\right].\tag{4}$$

□

Now we shall discuss some special cases of Theorem 2.

I. If we take $\varphi(t)=t$ in Theorem 2, then we get Theorem 3.1 [11].

II. If we take $\varphi(t)=t^s$ in Theorem 2, then we get a new result for Breckner type harmonic s-convex functions.

Corollary 5. *Let $T:[a^\varrho,b^\varrho]\to\mathbb{R}$ be a differentiable function on (a^ϱ,b^ϱ). If $|T'|^q$ is Breckner type of harmonic s-convex function, where $q\geq 1$, then*

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2}-\frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho-a^\varrho}\right)^\alpha\left[{}^\varrho I^\alpha_{\frac{1}{a}-}(T\circ W)\Big(\frac{1}{b^\varrho}\Big)+{}^\varrho I^\alpha_{\frac{1}{b}+}(T\circ W)\Big(\frac{1}{a^\varrho}\Big)\right]\right|$$
$$\leq\frac{\varrho a^\varrho b^\varrho(b^\varrho-a^\varrho)}{2}\Psi_1^{1-\frac{1}{q}}(a,b;\varrho,\alpha)\left(G_1(a,b;\varrho,\alpha)|T'(a^\varrho)|^q+G_2(a,b;\varrho,\alpha)|T'(b^\varrho)|^q\right)^{\frac{1}{q}},$$

where $\Psi_1(a,b;\varrho,\alpha)$ *is given by (4),*

$$G_1(a,b;\varrho,\alpha) := \int_0^1 \frac{[t^{\varrho\alpha}-(1-t^{\varrho})^{\alpha}]t^{\varrho-1}(1-t^{\varrho})^{s}}{(t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho})^2}$$
$$= \frac{b^{-2\varrho}}{\varrho}\left\{B(1+\alpha,1+s)\,{}_2\mathfrak{F}_1\left(2,\alpha+1;2+\alpha+s;1-\frac{a^{\varrho}}{b^{\varrho}}\right)\right.$$
$$\left.+B(1,1+\alpha+s)\,{}_2\mathfrak{F}_1\left(2,1;2+\alpha+s;1-\frac{a^{\varrho}}{b^{\varrho}}\right)\right\}$$

and

$$G_2(a,b;\varrho,\alpha) := \int_0^1 \frac{[t^{\varrho\alpha}-(1-t^{\varrho})^{\alpha}]t^{\varrho-1}t^{\varrho s}}{(t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho})^2}$$
$$= \frac{b^{-2\varrho}}{\varrho}\left\{B(1+\alpha+s,1)\,{}_2\mathfrak{F}_1\left(2,1+\alpha+s;2+\alpha+s;1-\frac{a^{\varrho}}{b^{\varrho}}\right)\right.$$
$$\left.+B(1+s,1+\alpha)\,{}_2\mathfrak{F}_1\left(2,1+s;2+\alpha+s;1-\frac{a^{\varrho}}{b^{\varrho}}\right)\right\}.$$

III. If we take $\varphi(t)=t^{-s}$ in Theorem 2, then we get a new result for Godunova-Levin-Dragomir type harmonic s-convex functions.

Corollary 6. *Let* $T:[a^{\varrho},b^{\varrho}]\to\mathbb{R}$ *be a differentiable function on* $(a^{\varrho},b^{\varrho})$. *If* $|T'|^q$ *is Godunova-Levin-Dragomir type of harmonic s-convex function, where* $q\geq 1$, *then*

$$\left|\frac{T(a^{\varrho})+T(b^{\varrho})}{2}-\frac{\varrho^{\alpha}\Gamma(\alpha+1)}{2}\left(\frac{a^{\varrho}b^{\varrho}}{b^{\varrho}-a^{\varrho}}\right)^{\alpha}\left[{}^{\varrho}I^{\alpha}_{\frac{1}{a}^-}(T\circ W)\Big(\frac{1}{b^{\varrho}}\Big)+{}^{\varrho}I^{\alpha}_{\frac{1}{b}^+}(T\circ W)\Big(\frac{1}{a^{\varrho}}\Big)\right]\right|$$
$$\leq \frac{\varrho a^{\varrho}b^{\varrho}(b^{\varrho}-a^{\varrho})}{2}\Psi_1^{1-\frac{1}{q}}(a,b;\varrho,\alpha)\left(G_3(a,b;\varrho,\alpha)|T'(a^{\varrho})|^q+G_4(a,b;\varrho,\alpha)|T'(b^{\varrho})|^q\right)^{\frac{1}{q}},$$

where $\Psi_1(a,b;\varrho,\alpha)$ *is given by (4),*

$$G_3(a,b;\varrho,\alpha) := \int_0^1 \frac{[t^{\varrho\alpha}-(1-t^{\varrho})^{\alpha}]t^{\varrho-1}(1-t^{\varrho})^{-s}}{(t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho})^2}$$
$$= \frac{b^{-2\varrho}}{\varrho}\left\{B(1+\alpha,1-s)\,{}_2\mathfrak{F}_1\left(2,\alpha+1;2+\alpha-s;1-\frac{a^{\varrho}}{b^{\varrho}}\right)\right.$$
$$\left.+B(1,1+\alpha-s)\,{}_2\mathfrak{F}_1\left(2,1;2+\alpha-s;1-\frac{a^{\varrho}}{b^{\varrho}}\right)\right\}$$

and

$$G_2(a,b;\varrho,\alpha) := \int_0^1 \frac{[t^{\varrho\alpha}-(1-t^{\varrho})^{\alpha}]t^{\varrho-1}t^{-\varrho s}}{(t^{\varrho}a^{\varrho}+(1-t^{\varrho})b^{\varrho})^2}$$
$$= \frac{b^{-2\varrho}}{\varrho}\left\{B(1+\alpha-s,1)\,{}_2\mathfrak{F}_1\left(2,1+\alpha-s;2+\alpha-s;1-\frac{a^{\varrho}}{b^{\varrho}}\right)\right.$$
$$\left.+B(1-s,1+\alpha)\,{}_2\mathfrak{F}_1\left(2,1-s;2+\alpha-s;1-\frac{a^{\varrho}}{b^{\varrho}}\right)\right\}.$$

IV. If we take $\varphi(t)=1$ in Theorem 2, then we get a new result for harmonic P-convex functions.

Corollary 7. *Let $T : [a^\varrho, b^\varrho] \to \mathbb{R}$ be a differentiable function on (a^ϱ, b^ϱ). If $|T'|^q$ is harmonic P-convex function, where $q \geq 1$, then*

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2} - \frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a^-}}(T\circ W)\Big(\frac{1}{b^\varrho}\Big) + {}^\varrho I^\alpha_{\frac{1}{b^+}}(T\circ W)\Big(\frac{1}{a^\varrho}\Big)\right]\right|$$
$$\leq \frac{\varrho a^\varrho b^\varrho(b^\varrho - a^\varrho)}{2}\Psi_1(a,b;\varrho,\alpha)\left(G_1(a,b;\varrho,\alpha)|T'(a^\varrho)|^q + G_2(a,b;\varrho,\alpha)|T'(b^\varrho)|^q\right)^{\frac{1}{q}},$$

where $\Psi_1(a,b;\varrho,\alpha)$ is given by (4).

Theorem 3. *Let $T : [a^\varrho, b^\varrho] \to \mathbb{R}$ be a differentiable function on (a^ϱ, b^ϱ). If $|T'|^p$ is harmonic φ-convex function, where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, then*

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2} - \frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a^-}}(T\circ W)\Big(\frac{1}{b^\varrho}\Big) + {}^\varrho I^\alpha_{\frac{1}{b^+}}(T\circ W)\Big(\frac{1}{a^\varrho}\Big)\right]\right|$$
$$\leq \frac{\varrho a^\varrho b^\varrho(b^\varrho - a^\varrho)}{2}\left(M_1(a,b;\varrho;\alpha)^{\frac{1}{p}} + M_2(a,b;\varrho;\alpha)^{\frac{1}{p}}\right)$$
$$\times\left(\int_0^1[\varphi(1-t^\varrho)|T'(a^\varrho)|^q + \varphi(t^\varrho)|T'(b^\varrho)|^q]\mathrm{d}t\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 1 and taking modulus on both sides, we have

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2} - \frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a^-}}(T\circ W)\Big(\frac{1}{b^\varrho}\Big) + {}^\varrho I^\alpha_{\frac{1}{b^+}}(T\circ W)\Big(\frac{1}{a^\varrho}\Big)\right]\right|$$
$$= \left|\frac{\varrho a^\varrho b^\varrho(b^\varrho - a^\varrho)}{2}\int_0^1 \frac{[t^{\varrho\alpha} - (1-t^\varrho)^\alpha]t^{\varrho-1}}{(t^\varrho a^\varrho + (1-t^\varrho)b^\varrho)^2}T'\left(\frac{a^\varrho b^\varrho}{t^\varrho a^\varrho + (1-t^\varrho)b^\varrho}\right)\mathrm{d}t\right|.$$

Using the property of modulus, we have

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2} - \frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a^-}}(T\circ W)\Big(\frac{1}{b^\varrho}\Big) + {}^\varrho I^\alpha_{\frac{1}{b^+}}(T\circ W)\Big(\frac{1}{a^\varrho}\Big)\right]\right|$$
$$\leq \frac{\varrho a^\varrho b^\varrho(b^\varrho - a^\varrho)}{2}\int_0^1 \frac{[t^{\varrho\alpha} - (1-t^\varrho)^\alpha]t^{\varrho-1}}{(t^\varrho a^\varrho + (1-t^\varrho)b^\varrho)^2}\left|T'\left(\frac{a^\varrho b^\varrho}{t^\varrho a^\varrho + (1-t^\varrho)b^\varrho}\right)\right|\mathrm{d}t.$$

Utilizing Hölder's inequality and the fact that $|T'|$ is harmonic φ-convex function, we have

$$\begin{aligned}
&\left|\frac{T(a^\varrho)+T(b^\varrho)}{2}-\frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho-a^\varrho}\right)^\alpha\left[{}^\varrho I^\alpha_{\frac{1}{a}-}(T\circ W)\Big(\frac{1}{b^\varrho}\Big)+{}^\varrho I^\alpha_{\frac{1}{b}+}(T\circ W)\Big(\frac{1}{a^\varrho}\Big)\right]\right|\\
&\le\frac{\varrho a^\varrho b^\varrho(b^\varrho-a^\varrho)}{2}\left\{\left(\int_0^1\frac{t^{\varrho\alpha p}t^{p(\varrho-1)}}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1[\varphi(1-t^\varrho)|T'(a^\varrho)|^q+\varphi(t^\varrho)|T'(b^\varrho)|^q]\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^1\frac{(1-t^\varrho)^{\alpha p}t^{p(\varrho-1)}}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1[\varphi(1-t^\varrho)|T'(a^\varrho)|^q+\varphi(t^\varrho)|T'(b^\varrho)|^q]\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\
&=\frac{\varrho a^\varrho b^\varrho(b^\varrho-a^\varrho)}{2}\left(M_1(a,b;\varrho;\alpha)^{\frac{1}{p}}+M_2(a,b;\varrho;\alpha)^{\frac{1}{p}}\right)\\
&\qquad\times\left(\int_0^1[\varphi(1-t^\varrho)|T'(a^\varrho)|^q+\varphi(t^\varrho)|T'(b^\varrho)|^q]\mathrm{d}t\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
M_1(a,b;\varrho;\alpha)&=\int_0^1\frac{t^{\varrho\alpha p}t^{p(\varrho-1)}}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^{2p}}\mathrm{d}t\\
&=b^{-2p\varrho}\left(\alpha p+p+\frac{1-p}{\varrho}\right)\,{}_2\mathfrak{F}_1\left(2p,\alpha p+p+\frac{1-p}{\varrho};\alpha p+p+1+\frac{1-p}{\varrho};1-\frac{a^\varrho}{b^\varrho}\right),
\end{aligned}\tag{5}$$

and

$$\begin{aligned}
M_2(a,b;\varrho;\alpha)&=\int_0^1\frac{(1-t^\varrho)^{\alpha p}t^{p(\varrho-1)}}{(t^\varrho a^\varrho+(1-t^\varrho)b^\varrho)^{2p}}\mathrm{d}t\\
&=\frac{b^{-2p\varrho}}{B\big(\frac{\varrho p-p+1}{p},\alpha p+1\big)}\,{}_2\mathfrak{F}_1\left(2p,\frac{\varrho p-p+1}{\varrho};\alpha p+p+1+\frac{1-p}{\varrho};1-\frac{a^\varrho}{b^\varrho}\right).
\end{aligned}\tag{6}$$

This completes the proof. □

Now we shall discuss some special cases of Theorem 3.

I. If we take $\varphi(t)=t$ in Theorem 3, then we get Theorem 3.3 [11].

II. If we take $\varphi(t)=t^s$ in Theorem 3, then we get a new result for Breckner type harmonic s-convex functions.

Corollary 8. *Let $T : [a^\varrho, b^\varrho] \to \mathbb{R}$ be a differentiable function on (a^ϱ, b^ϱ). If $|T'|^p$ is Breckner type of harmonic s-convex, where $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$, then*

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2} - \frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a}^-}(T\circ W)\left(\frac{1}{b^\varrho}\right) + {}^\varrho I^\alpha_{\frac{1}{b}^+}(T\circ W)\left(\frac{1}{a^\varrho}\right)\right]\right|$$
$$\leq \frac{\varrho a^\varrho b^\varrho (b^\varrho - a^\varrho)}{2}\left(M_1(a,b;\varrho;\alpha)^{\frac{1}{p}} + M_2(a,b;\varrho;\alpha)^{\frac{1}{p}}\right)$$
$$\times\left(\frac{1}{\varrho}B\left(\frac{1}{\varrho}, s+1\right)|T'(a^\varrho)|^q + \frac{1}{\varrho s+1}|T'(b^\varrho)|^q\right)^{\frac{1}{q}},$$

where $M_1(a,b;\varrho;\alpha)^{\frac{1}{p}}$ and $M_2(a,b;\varrho;\alpha)^{\frac{1}{p}}$ are given by (5) and (6) respectively.

III. If we take $\varphi(t) = t^{-s}$ in Theorem 2, then we get a new result for Godunova-Levin-Dragomir type harmonic s-convex functions.

Corollary 9. *Let $T : [a^\varrho, b^\varrho] \to \mathbb{R}$ be a differentiable function on (a^ϱ, b^ϱ). If $|T'|^p$ is Godunova-Levin-Dragomir type of harmonic s-convex, where $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$, then*

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2} - \frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a}^-}(T\circ W)\left(\frac{1}{b^\varrho}\right) + {}^\varrho I^\alpha_{\frac{1}{b}^+}(T\circ W)\left(\frac{1}{a^\varrho}\right)\right]\right|$$
$$\leq \frac{\varrho a^\varrho b^\varrho (b^\varrho - a^\varrho)}{2}\left(M_1(a,b;\varrho;\alpha)^{\frac{1}{p}} + M_2(a,b;\varrho;\alpha)^{\frac{1}{p}}\right)$$
$$\times\left(\frac{1}{\varrho}B\left(\frac{1}{\varrho}, 1-s\right)|T'(a^\varrho)|^q + \frac{1}{\varrho+1-s}|T'(b^\varrho)|^q\right)^{\frac{1}{q}},$$

where $M_1(a,b;\varrho;\alpha)^{\frac{1}{p}}$ and $M_2(a,b;\varrho;\alpha)^{\frac{1}{p}}$ are given by (5) and (6) respectively.

IV. If we take $\varphi(t) = 1$ in Theorem 2, then we get a new result for harmonic P-convex functions.

Corollary 10. *Let $T : [a^\varrho, b^\varrho] \to \mathbb{R}$ be a differentiable function on (a^ϱ, b^ϱ). If $|T'|^p$ is harmonic P-convex, where $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$, then*

$$\left|\frac{T(a^\varrho)+T(b^\varrho)}{2} - \frac{\varrho^\alpha\Gamma(\alpha+1)}{2}\left(\frac{a^\varrho b^\varrho}{b^\varrho - a^\varrho}\right)^\alpha \left[{}^\varrho I^\alpha_{\frac{1}{a}^-}(T\circ W)\left(\frac{1}{b^\varrho}\right) + {}^\varrho I^\alpha_{\frac{1}{b}^+}(T\circ W)\left(\frac{1}{a^\varrho}\right)\right]\right|$$
$$\leq \frac{\varrho a^\varrho b^\varrho (b^\varrho - a^\varrho)}{2}\left(M_1(a,b;\varrho;\alpha)^{\frac{1}{p}} + M_2(a,b;\varrho;\alpha)^{\frac{1}{p}}\right)\left(|T'(a^\varrho)|^q + |T'(b^\varrho)|^q\right)^{\frac{1}{q}},$$

where $M_1(a,b;\varrho;\alpha)^{\frac{1}{p}}$ and $M_2(a,b;\varrho;\alpha)^{\frac{1}{p}}$ are given by (5) and (6) respectively.

3. Conclusion

A new approach using Katugampola fractional integrals has been utilized in obtaining Hermite-Hadamard like inequalities via harmonic φ-convexity. New and known special cases of the main results have also been discussed in detail. We hope that the ideas and techniques of the paper will inspire interested readers.

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