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POSITIVE PERIODIC SOLUTIONS FOR SECOND-ORDER NEUTRAL DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. In this article, we obtain sufficient conditions for the existence of positive periodic solutions for a second-order neutral difference equation with variable coefficients. The main tool employed here is the Krasnoselskii fixed point theorem dealing with a sum of two mappings, one is a contraction and the other is a completely continuous.

1. Introduction

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential and difference equations, see the references [1]–[17], [19]–[22], [24]–[27] and the references therein.

In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of delay difference equations. Motivated by the papers [1]–[17], [19]–[22], [24]–[27] and the references therein, we concentrate on the existence of positive periodic solutions for the second-order neutral difference equation

$$\triangle^2 \left(x\left(t\right) - c\left(t\right) x\left(t-\tau\right)\right) = a\left(t\right) x\left(t+1\right) - f\left(t, x\left(t-\tau\right)\right), \tag{1}$$

where a and τ are positive ω-periodic sequences, $c : \mathbb{Z} \to \mathbb{R}$ is ω-periodic sequence and $f : \mathbb{Z} \times \mathbb{R} \to [0, \infty)$ is continuous in x and ω-periodic in t with ω is a positive integer constant. Throughout this paper $\triangle$ denotes the forward difference operator $\triangle x\left(t\right) = x\left(t+1\right) - x\left(t\right)$ for any sequence $\{x\left(t\right),\ t \in \mathbb{Z}\}$. For more on the calculus of difference equations, we refer the reader to [18].

The purpose of this paper is to use Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions for equation (1) with $1 < c\left(t\right) < \infty$, $-\infty < c\left(t\right) < -1$, $0 \leq c\left(t\right) < 1$ and $-1 < c\left(t\right) \leq 0$ as four different ranges for the variable coefficient c. To apply Krasnoselskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is a completely continuous.

The organization of this paper is as follows. In Section 2, we introduce some notations and lemmas, and state some preliminary results needed in later sections, then we give the Green's function of (1), which plays an important role in this paper. Also, we present the inversion of (1), and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [23]. In Sections 3 and 4, we present our main results on existence of positive periodic solutions of (1).

2. Preliminaries

Let ω be an integer such that $\omega \geq 1$. Define $C_\omega = \{x \in C\left(\mathbb{Z}, \mathbb{R}\right) : x\left(t+\omega\right) = x\left(t\right)\}$ where $C\left(\mathbb{Z}, \mathbb{R}\right)$ is the space of all real valued functions. Then $\left(C_\omega, \|.\|\right)$ is a Banach space

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with the maximum norm
$$\|x\| = \max_{t\in[0,\omega-1]\cap\mathbb{Z}} |x(t)|.$$
Define
$$C_\omega^+ = \{x \in C_\omega : x > 0\}, \; C_\omega^- = \{x \in C_\omega : x < 0\}.$$
Denote
$$M = \max\{a(t) : t \in [0, \omega - 1] \cap \mathbb{Z}\}, \; m = \min\{a(t) : t \in [0, \omega - 1] \cap \mathbb{Z}\}, \; \beta = \sqrt{M}.$$

Lemma 1 ([1]). *The equation*
$$\triangle^2 y(t) - My(t+1) = h(t), \; h \in C_\omega^-,$$
has a ω-periodic solution
$$y(t) = \sum_{s=t}^{t+\omega-1} G(t,s)(-h(s)), \; G(t,s) = \frac{r^{2t+\omega} + r^{2s+2}}{r^{t+s}(r^2-1)(r^\omega - 1)},$$
where
$$r = \frac{1}{2}\left(2 + \beta^2 + \beta\sqrt{\beta^2+4}\right).$$

Lemma 2. *$G(t,s) > 0$ and $\sum_{s=t}^{t+\omega-1} G(t,s) = \frac{1}{M}$ for all $t \in [0,\omega-1] \cap \mathbb{Z}$ and $s \in [t, t+\omega-1] \cap \mathbb{Z}$.*

Proof. By the definition of $G(t,s)$, it is clear that $G(t,s) > 0$ and
$$\begin{aligned}
\sum_{s=t}^{t+\omega-1} G(t,s) &= \sum_{s=t}^{t+\omega-1} \frac{r^{2t+\omega} + r^{2s+2}}{r^{t+s}(r^2-1)(r^\omega-1)} \\
&= \frac{1}{(r^2-1)(r^\omega-1)} \sum_{s=t}^{t+\omega-1} \left(r^{t-s+\omega} + r^{-t+s+2}\right) \\
&= \frac{1}{(r^2-1)(r^\omega-1)} \left\{ r^{t+\omega} \sum_{s=t}^{t+\omega-1} \left(\frac{1}{r}\right)^s + r^{-t+2} \sum_{s=t}^{t+\omega-1} (r)^s \right\} \\
&= \frac{1}{(r^2-1)(r^\omega-1)} \left\{ \frac{r(r^\omega-1)}{r-1} + \frac{r^2(r^\omega-1)}{r-1} \right\} \\
&= \frac{r}{(r-1)(r^2-1)} + \frac{r^2}{(r-1)(r^2-1)} \\
&= \frac{r}{(r-1)^2} = \frac{1}{\beta^2} = \frac{1}{M}.
\end{aligned}$$
□

Corollary 1. *$G(t+\tau,s) > 0$ and $\sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau,s) = \frac{1}{M}$ for all $t \in [0,\omega-1] \cap \mathbb{Z}$ and $s \in [t+\tau, t+\tau+\omega-1] \cap \mathbb{Z}$.*

Lemma 3. *The equation*
$$\triangle^2 y(t) - a(t)y(t+1) = h(t), \; h \in C_\omega^-, \tag{2}$$
has a positive ω-periodic solution
$$y(t) = (Ph)(t) = (I - TB)^{-1} Th(t),$$

where

$$(Th)(t) = \sum_{s=t}^{t+\omega-1} G(t,s)(-h(s)), \ (By)(t) = [-M + a(t)] y(t+1).$$

Proof. Clearly T and B are completely continuous, $(Th)(t) > 0$ for $h(t) < 0$ and $\|B\| \leq (M-m)$. By Lemma 1, the solution of (2) can be written in the form

$$y(t) = (Th)(t) + (TBy).$$

Since

$$\|TB\| \leq \|T\| \|B\| \leq \frac{1}{M}(M-m) = 1 - \frac{m}{M} < 1,$$

then

$$y(t) = (I - TB)^{-1}(Th)(t) = (Ph)(t).$$

It is obvious that y is a positive ω-periodic solution of (2) for any $h \in C_\omega^-$. □

Lemma 4. *P is completely continuous and satisfies*

$$0 < (Th)(t) \leq (Ph)(t) \leq \frac{M}{m}\|Th\|, \ h \in C_\omega^-.$$

Proof. By Neumann expansions of P, we have

$$\begin{aligned} P &= (I - TB)^{-1} T \\ &= \left(I + TB + (TB)^2 + \cdots + (TB)^n + \cdots\right) T \\ &= T + TBT + (TB)^2 T + \cdots + (TB)^n T + \cdots . \end{aligned} \tag{3}$$

Since T and B are completely continuous, so is P. Moreover, by (3), and recalling that $\|TB\| \leq 1 - \frac{m}{M}$ and $(Th)(t) > 0$ for $h(t) < 0$, we get

$$(Th)(t) \leq (Ph)(t) \leq \frac{M}{m}\|Th\|, \ h \in C_\omega^-.$$

□

The following lemma is essential for our results on existence of positive periodic solution of (1).

Lemma 5. *If $x \in C_\omega$ then x is a solution of equation (1) if and only if*

$$x(t) = c(t)x(t-\tau) + P(-f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau)). \tag{4}$$

Proof. Let $x \in C_\omega$ be a solution of (1). Rewrite (1) as

$$\begin{aligned} &\triangle^2 (x(t) - c(t)x(t-\tau)) - M(x(t+1) - c(t+1)x(t+1-\tau)) \\ &= (-M + a(t))(x(t+1) - c(t+1)x(t+1-\tau)) \\ &- f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau) \\ &= B(x(t) - c(t)x(t-\tau)) - f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau). \end{aligned}$$

From Lemma 1, we have

$$\begin{aligned} x(t) - c(t)x(t-\tau) &= TB(x(t) - c(t)x(t-\tau)) \\ &\quad + T(-f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau)). \end{aligned}$$

This yields

$$\begin{aligned} &(I - TB)(x(t) - c(t)x(t-\tau)) \\ &= T(-f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau)). \end{aligned}$$

Therefore

$$\begin{aligned}
&x(t) - c(t)x(t-\tau) \\
&= (I - TB)^{-1} T(-f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau)) \\
&= P(-f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau)).
\end{aligned}$$

Obviously

$$x(t) = c(t)x(t-\tau) + P(-f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau)).$$

This completes the proof. □

Corollary 2. *Suppose that $c(t) \neq 0$ for all $t \in \mathbb{Z}$. If $x \in C_\omega$ then x is a solution of equation (1) if and only if*

$$x(t) = \frac{x(t+\tau)}{c(t+\tau)} + \frac{1}{c(t+\tau)} P(-f(t+\tau, x(t)) + a(t+\tau)c(t+\tau+1)x(t+1)). \quad (5)$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1). For its proof we refer the reader to [23].

Theorem 1 (Krasnoselskii)**.** *Let $\mathbb{D}$ be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{D}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,

(ii) $\mathcal{A}$ is completely continuous,

(iii) $\mathcal{B}$ is a contraction mapping.

Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3. Existence of positive periodic solutions in the case $|c(t)| > 1$

To apply Theorem 1, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is a completely continuous. So, we let $(\mathbb{B}, \|.\|) = (C_\omega, \|.\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : L \leq \varphi \leq K\}$, where L and K are positive constants. In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases; $1 < c(t) < \infty$ and $-\infty < c(t) < -1$ for all $t \in \mathbb{Z}$.

In the case $1 < c(t) < \infty$, we assume that there exist positive constants c_1 and c_2 such that

$$c_1 \leq c(t) \leq c_2, \text{ for all } t \in [0, \omega - 1] \cap \mathbb{Z}, \quad (6)$$

$$c_1 > 1, \quad (7)$$

and for all $t \in [0, \omega - 1] \cap \mathbb{Z}$, $x \in \mathbb{D}$,

$$(c_2 - 1)ML \leq f(t+\tau, x(t)) - a(t+\tau)c(t+\tau+1)x(t+1) \leq (c_1 - 1)mK. \quad (8)$$

We express equation (5) as

$$\varphi(t) = (\mathcal{B}_1\varphi)(t) + (\mathcal{A}_1\varphi)(t) := (H_1\varphi)(t),$$

where $\mathcal{A}_1, \mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_1\varphi)(t) = \frac{1}{c(t+\tau)} P(-f(t+\tau, \varphi(t)) + a(t+\tau)c(t+\tau+1)\varphi(t+1)), \quad (9)$$

and

$$(\mathcal{B}_1\varphi)(t) = \frac{\varphi(t+\tau)}{c(t+\tau)}. \quad (10)$$

Lemma 6. *Suppose that (6) and (7) hold. If $\mathcal{B}_1$ is given by (10), then $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction.*

Proof. Let $\mathcal{B}_1$ be defined by (10). It is easy to show that $(\mathcal{B}_1\varphi)(t+\omega) = (\mathcal{B}_1\varphi)(t)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$|(\mathcal{B}_1\varphi)(t) - (\mathcal{B}_1\psi)(t)| \leq \left| \frac{\varphi(t+\tau)}{c(t+\tau)} - \frac{\psi(t+\tau)}{c(t+\tau)} \right| \leq \frac{1}{c_1} \|\varphi - \psi\|.$$

Then $\|\mathcal{B}_1\varphi - \mathcal{B}_1\psi\| \leq \dfrac{1}{c_1} \|\varphi - \psi\|$. Thus $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction by (7). $\square$

Besides, by the complete continuity of P, it is easy to verify the following lemma.

Lemma 7. *Suppose that the conditions (6)-(8) hold. Then $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$ is completely continuous.*

Theorem 2. *Suppose (6)-(8) hold. Then equation (1) has a positive ω-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 6, the operator $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction. Also, from Lemma 7, the operator $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$ is completely continuous. Moreover, we claim that $\mathcal{B}_1\psi + \mathcal{A}_1\varphi \in \mathbb{D}$ for all $\varphi, \psi \in \mathbb{D}$. Since

$$f(t+\tau, \varphi(t)) - a(t+\tau)\,c(t+\tau+1)\,\varphi(t+1) \geq (c_2 - 1)\,ML > 0$$

which implies

$$-f(t+\tau, \varphi(t)) + a(t+\tau)\,c(t+\tau+1)\,\varphi(t+1) < 0,$$

then for any $\varphi, \psi \in \mathbb{D}$, by Lemma 2 and Lemma 4, we have

$$\begin{aligned}
&(\mathcal{B}_1\psi)(t) + (\mathcal{A}_1\varphi)(t) \\
&= \frac{\psi(t+\tau)}{c(t+\tau)} + \frac{1}{c(t+\tau)} P\left(-f(t+\tau, \varphi(t)) + a(t+\tau)\,c(t+\tau+1)\,\varphi(t+1)\right) \\
&\leq \frac{1}{c_1}\psi(t+\tau) + \frac{M}{c_1 m} \left\|T\left(-f(t+\tau, \varphi(t)) + a(t+\tau)\,c(t+\tau+1)\,\varphi(t+1)\right)\right\| \\
&\leq \frac{K}{c_1} + \frac{M}{c_1 m} \max_{t\in[0,\omega-1]\cap\mathbb{Z}} \left| \sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau, s)\left(f(s+\tau, \varphi(s))\right.\right. \\
&\left.\left. -a(s+\tau)\,c(s+\tau+1)\,\varphi(s+1)\right)\right| \\
&\leq \frac{K}{c_1} + \frac{M}{c_1 m} \max_{t\in[0,\omega-1]\cap\mathbb{Z}} \sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau, s)\left(f(s+\tau, \varphi(s))\right. \\
&\left. -a(s+\tau)\,c(s+\tau+1)\,\varphi(s+1)\right) \\
&\leq \frac{K}{c_1} + \frac{M}{c_1 m} \sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau, s)\,(c_1 - 1)\,mK \\
&= \frac{K}{c_1} + \frac{M}{c_1 m}(c_1 - 1)\,mK\frac{1}{M} \\
&= K.
\end{aligned}$$

On the other hand, Lemma 2 and Lemma 4,

$$\begin{aligned}
&(\mathcal{B}_1\psi)(t) + (\mathcal{A}_1\varphi)(t) \\
&= \frac{\psi(t+\tau)}{c(t+\tau)} + \frac{1}{c(t+\tau)} P\left(-f(t+\tau,\varphi(t)) + a(t+\tau)c(t+\tau+1)\varphi(t+1)\right) \\
&\geq \frac{1}{c_2}\psi(t+\tau) + \frac{1}{c_2}\sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau,s)\left(f(s+\tau,\varphi(s)) - a(s+\tau)c(s+\tau+1)\varphi(s+1)\right) \\
&\geq \frac{L}{c_2} + \frac{1}{c_2}\sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau,s)(c_2-1)ML \\
&= \frac{L}{c_2} + \frac{1}{c_2}(c_2-1)ML\frac{1}{M} \\
&= L.
\end{aligned}$$

Then $\mathcal{B}_1\psi + \mathcal{A}_1\varphi \in \mathbb{D}$ for all $\varphi, \psi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}_1x + \mathcal{B}_1x$. By Lemma 5 this fixed point is a solution of (1) and the proof is complete. □

In the case $-\infty < c(t) < -1$, we substitute conditions (6)-(8) with the following conditions respectively. We assume that there exist negative constants c_3 and c_4 such that

$$c_3 \leq c(t) \leq c_4, \text{ for all } t \in [0,\omega-1] \cap \mathbb{Z},\ x \in \mathbb{D}, \tag{11}$$

$$c_4 < -1, \tag{12}$$

and for all $t \in [0,\omega-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$\left(-c_3L + \frac{c_3}{c_4}K\right)M \leq -f(t+\tau,x(t)) + a(t+\tau)c(t+\tau+1)x(t+1) \leq -c_4mK. \tag{13}$$

We express equation (5) as

$$\varphi(t) = (\mathcal{B}_2\varphi)(t) + (\mathcal{A}_2\varphi)(t) := (H_2\varphi)(t),$$

where $\mathcal{A}_2, \mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_2\varphi)(t) = \frac{-1}{c(t+\tau)} P\left(f(t+\tau,\varphi(t)) - a(t+\tau)c(t+\tau+1)\varphi(t+1)\right), \tag{14}$$

and

$$(\mathcal{B}_2\varphi)(t) = \frac{\varphi(t+\tau)}{c(t+\tau)}. \tag{15}$$

Lemma 8. *Suppose that (11) and (12) hold. If $\mathcal{B}_2$ is given by (15), then $\mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ is a contraction.*

Proof. Let $\mathcal{B}_2$ be defined by (15). It is easy to show that $(\mathcal{B}_2\varphi)(t+\omega) = (\mathcal{B}_2\varphi)(t)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$|(\mathcal{B}_2\varphi)(t) - (\mathcal{B}_2\psi)(t)| \leq \left|\frac{\varphi(t+\tau)}{c(t+\tau)} - \frac{\psi(t+\tau)}{c(t+\tau)}\right| \leq \frac{-1}{c_4}\|\varphi-\psi\|.$$

Then $\|\mathcal{B}_2\varphi - \mathcal{B}_2\psi\| \leq \dfrac{-1}{c_4}\|\varphi-\psi\|$. Thus $\mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ is a contraction by (12). □

Besides, by the complete continuity of P, it is easy to verify the following lemma.

Lemma 9. *Suppose that the conditions (11)-(13) hold. Then $\mathcal{A}_2 : \mathbb{D} \to \mathbb{B}$ is completely continuous.*

Theorem 3. *Suppose (11)-(13) hold. Then equation (1) has a positive ω-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 8, the operator $\mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ is a contraction. Also, from Lemma 9, the operator $\mathcal{A}_2 : \mathbb{D} \to \mathbb{B}$ is completely continuous. Moreover, we claim that $\mathcal{B}_2\psi + \mathcal{A}_2\varphi \in \mathbb{D}$ for all $\varphi, \psi \in \mathbb{D}$. In fact, for any $\varphi, \psi \in \mathbb{D}$, by Lemma 2 and Lemma 4, we have

$$
\begin{aligned}
&(\mathcal{B}_2\psi)(t) + (\mathcal{A}_2\varphi)(t) \\
&= \frac{\psi(t+\tau)}{c(t+\tau)} - \frac{1}{c(t+\tau)} P\left(f(t+\tau,\varphi(t)) - a(t+\tau)c(t+\tau+1)\varphi(t+1)\right) \\
&\leq -\frac{M}{c_4 m}\left\|T\left(f(t+\tau,\varphi(t)) - a(t+\tau)c(t+\tau+1)\varphi(t+1)\right)\right\| \\
&\leq -\frac{M}{c_4 m}\max_{t\in[0,\omega-1]\cap\mathbb{Z}}\left|\sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau,s)\left(-f(s+\tau,\varphi(s))\right.\right. \\
&\left.\left.+a(s+\tau)c(s+\tau+1)\varphi(s+1)\right)\right| \\
&\leq -\frac{M}{c_4 m}\sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau,s)\left(-f(s+\tau,\varphi(s)) + a(s+\tau)c(s+\tau+1)\varphi(s+1)\right) \\
&\leq -\frac{M}{c_4 m}\sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau,s)(-c_4 m K) \\
&= -\frac{M}{c_4 m}(-c_4 m K)\frac{1}{M} \\
&= K.
\end{aligned}
$$

On the other hand, Lemma 2 and Lemma 4,

$$
\begin{aligned}
&(\mathcal{B}_2\psi)(t) + (\mathcal{A}_2\varphi)(t) \\
&= \frac{\psi(t+\tau)}{c(t+\tau)} - \frac{1}{c(t+\tau)} P\left(f(t+\tau,\varphi(t)) - a(t+\tau)c(t+\tau+1)\varphi(t+1)\right) \\
&\geq \frac{K}{c_4} - \frac{1}{c_3}\sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau,s)\left(-f(t+\tau,\varphi(t)) + a(t+\tau)c(t+\tau+1)\varphi(t+1)\right) \\
&\geq \frac{K}{c_4} - \frac{1}{c_3}\sum_{s=t+\tau}^{t+\tau+\omega-1} G(t+\tau,s)\left(-c_3 L + \frac{c_3}{c_4}K\right)M \\
&= \frac{K}{c_4} - \frac{1}{c_3}\left(-c_3 L + \frac{c_3}{c_4}K\right)M\frac{1}{M} \\
&= L.
\end{aligned}
$$

Then $\mathcal{B}_2\psi + \mathcal{A}_2\varphi \in \mathbb{D}$ for all $\varphi, \psi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}_2 x + \mathcal{B}_2 x$. By Lemma 5 this fixed point is a solution of (1) and the proof is complete. □

4. Existence of positive periodic solutions in the case $|c(t)| < 1$

In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases; $0 \leq c(t) < 1$ and $-1 < c(t) \leq 0$ for all $t \in \mathbb{Z}$.

In the case $0 \leq c(t) < 1$, we assume that there exists positive constant c_5 such that

$$0 \leq c(t) \leq c_5, \text{ for all } t \in [0,\omega-1]\cap\mathbb{Z}, \tag{16}$$

$$c_5 < 1, \tag{17}$$

and for all $t \in [0, \omega - 1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$ML \leq f(t, x(t-\tau)) - a(t)c(t+1)x(t+1-\tau) \leq (1-c_5)mK. \tag{18}$$

We express equation (4) as

$$\varphi(t) = (\mathcal{B}_3\varphi)(t) + (\mathcal{A}_3\varphi)(t) := (H_3\varphi)(t),$$

where $\mathcal{A}_3, \mathcal{B}_3 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_3\varphi)(n) = P(-f(t, \varphi(t-\tau)) + a(t)c(t+1)\varphi(t+1-\tau)), \tag{19}$$

and

$$(\mathcal{B}_3\varphi)(n) = c(t)\varphi(t-\tau). \tag{20}$$

Theorem 4. *Suppose (16)-(18) hold. Then equation (1) has a positive ω-periodic solution x in the subset $\mathbb{D}$.*

Proof. The proof is similar to that Theorem 2, so it is omitted. □

In the case $-1 < c(t) \leq 0$, we substitute conditions (16)-(18) with the following conditions respectively. We assume that there exists negative constant c_6 such that

$$c_6 \leq c(t) \leq 0, \text{ for all } n \in [0, \omega-1] \cap \mathbb{Z}, \tag{21}$$

$$c_6 > -1, \tag{22}$$

and for all $t \in [0, \omega - 1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$(L - c_6K)M \leq -f(t, x(t-\tau)) + a(t)c(t+1)x(t+1-\tau) \leq mK. \tag{23}$$

We express equation (4) as

$$\varphi(t) = (\mathcal{B}_4\varphi)(t) + (\mathcal{A}_4\varphi)(t) := (H_4\varphi)(t),$$

where $\mathcal{A}_4, \mathcal{B}_4 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_4\varphi)(n) = -P(f(t, \varphi(t-\tau)) - a(t)c(t+1)\varphi(t+1-\tau)), \tag{24}$$

and

$$(\mathcal{B}_4\varphi)(n) = c(t)\varphi(t-\tau). \tag{25}$$

Theorem 5. *Suppose (21)-(23) hold. Then equation (1) has a positive ω-periodic solution x in the subset $\mathbb{D}$.*

Proof. The proof is similar to that Theorem 3, so it is omitted. □

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