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HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR $s-p-$CONVEX FUNCTIONS OF SEVERAL KINDS

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Abstract. Earlier, Mehmet has done work on Hermite-Hadamard-Fejer type inequality for $p-$convex functions in [6]. Now we generalized the work of Mehmet for $p-$convexity of Hermite-Hadamard-Fejer type inequalities for $s-p-$convex functions of 1^{st} kind and 2^{nd} kind.

1. Introduction

We introduce here some notation that would be used throughout the article. We use I to denote an interval on real line $\mathbb{R} = (-\infty, +\infty)$ and nonnegative real numbers are denoted by $\mathbb{R}_+ = [0, +\infty)$. I° denotes interior of I.
Here we recall some usefull related definitions and results.

Definition 1. [10]. *A function $f : I \to \mathbb{R}$ is a convex if whenever $x_1, y_1 \in I$ and $t_1 \in [0, 1]$ the following inequality holds*

$$f[t_1x_1 + (1-t_1)y_1] \leq t_1f(x_1) + (1-t_1)f(y_1)$$

Proposition 1. [12]. *If $f : [a_1, b_1] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(x_1)dx_1 \leq \frac{f(a_1)+f(b_1)}{2} \tag{1}$$

is known as Hermite-Hadamard's inequalities.

Theorem 1. [10]. *Let $f : [a_1, b_1] \to \mathbb{R}$ be a convex function and $g : [a_1, b_1] \to \mathbb{R}$ is a nonnegative, integrable and symmetric about $\frac{a_1+b_1}{2}$, then Fejer gave a generalization of the inequalities (1) as*

$$f\left(\frac{a_1+b_1}{2}\right)\int_{a_1}^{b_1} g(x_1)dx_1 \leq \int_{a_1}^{b_1} f(x_1)g(x_1)dx_1 \leq \frac{f(a_1)+f(b_1)}{2}\int_{a_1}^{b_1} g(x_1)dx_1$$

which is known as Hermite-Hadamard-Fejer inequalities.

Definition 2. [3]. *An interval I is said to be a $p-$convex set, if*

$$M_p(x_1, y_1; t_1) = [t_1{x_1}^p + (1-t_1){y_1}^p]^{\frac{1}{p}} \in I,$$

$\forall x_1, y_1 \in I, t_1 \in [0, 1]$, *where $p = 2k+1$ or $p = \frac{n}{m}, n = 2r+1, m = 2w+1$ and $k, r, w \in \mathbb{N}$.*

Definition 3. [2]. *Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be $p-$ convex, if*

$$f\left([t_1{x_1}^p + (1-t_1){y_1}^p]^{\frac{1}{p}}\right) \leq t_1f(x_1) + (1-t_1)f(y_1)$$

$\forall\ x_1, y_1 \in I$ *and* $t_1 \in [0, 1]$.

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Theorem 2. [3]. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $a_1, b_1 \in I$ with $a_1 < b_1$. If $f \in L[a_1, b_1]$ then the following inequalities hold:*

$$f\left(\left[\frac{a_1{}^p + b_1{}^p}{2}\right]^{\frac{1}{p}}\right) \le \int_{a_1}^{b_1} \frac{f(x_1)}{x_1{}^{1-p}} dx_1 \le \frac{f(a_1) + f(b_1)}{2}$$

For some results related to $p-$convex functions and its generalization refer to see [1, 2, 3, 4, 5, 9, 13].

Proposition 2. [6]. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a_1, b_1]$, where $a_1, b_1 \in I^\circ$ and $a_1 < b_1$. If $|f'|$ is p-convex function on $[a_1, b_1]$ for $p \in \mathbb{R}\backslash\{0\}$ and $\omega : [a_1, b_1] \to \mathbb{R}$ is continuous, then the following inequality holds:*

$$\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}} dx_1 - f\left(\left[\frac{a_1{}^p + b_1{}^p}{2}\right]^{\frac{1}{p}}\right) \int_{a_1}^{b_1} \frac{\omega(x_1)}{x_1{}^{1-p}} dx_1\right|$$
$$\le \left(\frac{b_1{}^p - a_1{}^p}{p}\right)^2 ||\omega||_\infty \left[c_1(p)|f'(a_1)| + c_2(p)|f'(b_1)|\right].$$

where

$$c_1(p) = \left[\int_0^{\frac{1}{2}} \frac{t_1{}^2}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{p}}} dt_1 + \int_{\frac{1}{2}}^1 \frac{t_1 - t_1{}^2}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{p}}} dt_1\right],$$
$$c_2(p) = \left[\int_0^{\frac{1}{2}} \frac{t_1 - t_1{}^2}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{p}}} dt_1 + \int_{\frac{1}{2}}^1 \frac{(1-t_1)^2}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{p}}} dt_1\right].$$

Proposition 3. [6]. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a_1, b_1]$, where $a_1, b_1 \in I^\circ$ and $a_1 < b_1$. If $|f'|^q, q \ge 1$, $p-$convex function on $[a_1, b_1]$ for $p \in \mathbb{R}\backslash\{0\}$ and $\omega : [a_1, b_1] \to \mathbb{R}$ is continuous, then the following inequality holds:*

$$\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}} dx_1 - f\left(\left[\frac{a_1{}^p + b_1{}^p}{2}\right]^{\frac{1}{p}}\right) \int_{a_1}^{b_1} \frac{\omega(x_1)}{x_1{}^{1-p}} dx_1\right|$$
$$\le \left(\frac{b_1{}^p - a_1{}^p}{p}\right)^2 \parallel \omega \parallel_\infty \Big[(C_3(p))^{1-\frac{1}{q}} \left[C_4(p)|f'(a_1)|^q + C_5(p)|f'(b_1)|^q\right]^{\frac{1}{q}}$$
$$(C_6(p))^{1-\frac{1}{q}} \left[C_7(p)|f'(a_1)|^q + C_8(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\Big]$$

where

$$C_3(P) = \int_0^{\frac{1}{2}} \frac{t_1}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{q}}} dt_1, \qquad C_4(p) = \int_0^{\frac{1}{2}} \frac{t_1{}^2}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{q}}} dt_1,$$
$$C_5(p) = \int_0^{\frac{1}{2}} \frac{t_1 - t_1{}^2}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{q}}} dt_1, \qquad C_6(p) = \int_{\frac{1}{2}}^1 \frac{1 - t_1}{[t_1 a_1^p + (1-t_1)b_1{}^p]^{1-\frac{1}{q}}} dt_1,$$
$$C_7(p) = \int_{\frac{1}{2}}^1 \frac{t_1 - t_1{}^2}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{q}}} dt_1, \qquad C_8(p) = \int_{\frac{1}{2}}^1 \frac{(1-t_1)^2}{[t_1 a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{q}}} dt_1.$$

Proposition 4. [6]. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a_1, b_1]$, where $a_1, b_1 \in I^\circ$ and $a_1 < b_1$. If $|f'|^q$, $q \ge 1$, $p-$convex function on $[a_1, b_1]$ for $p \in \mathbb{R}\backslash\{0\}$ and $\omega : [a_1, b_1] \to \mathbb{R}$ is continuous, then the following inequality holds:*

$$\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}} dx_1 - f\left(\left[\frac{a_1{}^p + b_1{}^p}{2}\right]^{\frac{1}{p}}\right) \int_{a_1}^{b_1} \frac{\omega(x_1)}{x_1{}^{1-p}} dx_1\right|$$

$$\leq \left(\frac{b_1{}^p - a_1{}^p}{p}\right)^2 \parallel \omega \parallel_\infty \left[C_9(p)\left(\frac{|f'(a_1)|^p + 3\,|f'(b_1)|^p}{8}\right)^{\frac{1}{p}} + C_{10}(p)\left(\frac{3\,|f'(a_1)|^p + |f'(b_1)|^p}{8}\right)^{\frac{1}{p}}\right]$$

where

$$C_9(p,r) = \left(\int_0^{\frac{1}{2}}\left(\frac{t_1}{[t_1a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}},$$

$$C_{10}(p,r) = \left(\int_{\frac{1}{2}}^1\left(\frac{1-t_1}{[t_1a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}$$

with $\dfrac{1}{p} + \dfrac{1}{z} = 1.$

Lemma 1. [6]. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $a_1, b_1 \in I^\circ$ with $a_1 < b_1$, $p \in \mathbb{R}\backslash\{0\}$. If $f' \in L[a_1, b_1]$ and $\omega : [a_1, b_1] \to \mathbb{R}$ is integrable, then the following equality holds:*

$$\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}}dx_1 - f\left(\left[\frac{a_1{}^p + b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1{}^{1-p}}dx_1$$

$$= \left(\frac{b_1{}^p - a_1{}^p}{p}\right)^2\int_0^1\frac{k(t_1)}{[t_1a_1{}^p + (1-t_1)b_1{}^p]^{1-\frac{1}{p}}}f'\left([t_1a_1{}^p + (1-t_1)b_1{}^p]^{\frac{1}{p}}\right)dt,$$

where

$$k(t_1) = \begin{cases} \displaystyle\int_0^{t_1}\omega\left([sa_1{}^p + (1-s)b_1{}^p]\right)ds, & t_1 \in [0, \frac{1}{2}), \\ \displaystyle -\int_{t_1}^1\omega\left([sa_1{}^p + (1-s)b_1{}^p]\right)ds, & t_1 \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Definition 4. [11]. *A function $f : I \to \mathbb{R}$ is said to be $s-$convex function of the first kind, if*

$$f\left[t_1x_1 + (1-t_1)y_1\right] \leq t_1{}^s f(x_1) + (1 - t_1{}^s)f(y_1)$$

$\forall x_1, y_1 \in I, t_1 \in [0,1]$ *and for some fixed* $s \in (0,1]$.

Definition 5. [8]. *A function $f : I \to [0,\infty)$ is said to be $s-$convex function of the second kind, if*

$$f\left[t_1x_1 + (1-t_1)y_1\right] \leq t_1{}^s f(x_1) + (1-t_1)^s f(y_1)$$

$\forall x_1, y_1 \in [0,\infty), t_1 \in [0,1],\ s \in (0,1]$.

Now we would like to define definition of $s-p-$ convex in first and second kind.

Definition 6. *We would call a function $f : I \to \mathbb{R}$ be $s-p-$convex function of the first kind, if*

$$f\left([t_1x_1{}^p + (1-t_1)y_1{}^p]^{\frac{1}{p}}\right) \leq t_1{}^s f(x_1) + (1 - t_1{}^s)f(y_1) \tag{2}$$

$\forall x_1, y_1 \in I, t_1 \in [0,1]$, $s \in (0,1]$.

Definition 7. *We would call a function $f : I \to (0,\infty)$ be $s-p-$convex function of the second kind, if*

$$f\left([t_1{x_1}^p + (1-t_1){y_1}^p]^{\frac{1}{p}}\right) \le {t_1}^s f(x_1) + (1-t_1)^s f(y_1) \tag{3}$$

$\forall x_1, y_1 \in I, t_1 \in [0,1]$ *and for some fixed* $s \in (0,1]$.

Now in this paper, we establish some new results related to Hermite-Hadamard-Fejer type integral inequalities for $s-p-$convex functions. The results presented here would provide extensions of those given results proved by Mehmet in [6].

2. Main results

Inequalities for $s-p-$convex function of first kind

By using Definition 6 of $s-p-$convex function we prove following results.

Theorem 3. *Let f: $I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a_1,b_1]$, where $a_1, b_1 \in I^\circ$ and $a_1 < b_1$. If $|f'|$ is $s-p-$convex function of first kind on $[a_1,b_1]$ for $p \in \mathbb{R}\backslash\{0\}$ and $\omega : [a_1,b_1] \to \mathbb{R}$ is continuous, then the following inequality holds*

$$\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{{x_1}^{1-p}}dx_1 - f\left(\left[\frac{{a_1}^p + {b_1}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{{x_1}^{1-p}}dx_1\right|$$
$$\le \left(\frac{{b_1}^p - {a_1}^p}{p}\right)^2 ||\omega||_\infty \left[c_1(p)\,|f'(a_1)| + c_2(p)\,|f'(b_1)|\right]$$

where

$$c_1(p) = \left[\int_0^{\frac{1}{2}} \frac{{t_1}^{s+1}}{[t_1{a_1}^p + (1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1 + \int_{\frac{1}{2}}^1 \frac{{t_1}^s - {t_1}^{s+1}}{[t_1{a_1}^p + (1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1\right],$$
$$c_2(p) = \left[\int_0^{\frac{1}{2}} \frac{t_1 - {t_1}^{s+1}}{[t_1{a_1}^p + (1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1 + \int_{\frac{1}{2}}^1 \frac{(1-t_1)(1-{t_1}^s)}{[t_1{a_1}^p + (1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1\right].$$

Proof. Using Lemma 1, it follows that

$$\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{{x_1}^{1-p}}dx_1 - f\left(\left[\frac{{a_1}^p + {b_1}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{{x_1}^{1-p}}dx_1\right|$$
$$\le \left(\frac{{b_1}^p - {a_1}^p}{p}\right)^2 ||\omega||_\infty \int_0^1 \frac{|k(t_1)|}{[t_1{a_1}^p + (1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p + (1-t_1){b_1}^p]^{\frac{1}{p}}\right)dt_1\right|$$
$$\le \left(\frac{{b_1}^p - {a_1}^p}{p}\right)^2 ||\omega||_\infty \left[\int_0^{\frac{1}{2}} \frac{t_1}{[t_1{a_1}^p + (1-t_1){b_1}^p]^{1-\frac{1}{p}}} f'\left([t_1{a_1}^p + (1-t_1){b_1}^p]^{\frac{1}{p}}\right)dt_1\right.$$
$$\left. + \int_{\frac{1}{2}}^1 \frac{(1-t_1)}{[t_1{a_1}^p + (1-t_1){b_1}^p]^{1-\frac{1}{p}}} f'\left([t_1{a_1}^p + (1-t_1){b_1}^p]^{\frac{1}{p}}\right)dt_1\right] \tag{4}$$

Since $|f'|$ is a $s-p-$convex function of 1^{st} Kind on $[a_1,b_1]$, we have

$$\left|f'\left([t_1{a_1}^p + (1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right| \le {t_1}^s|f'(a_1)| + (1-{t_1}^s)|f'(b_1)|. \tag{5}$$

using (5) in (4), we have

$$\begin{aligned}
&\left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{x_1^{1-p}}dx_1-f\left(\left[\frac{a_1{}^p+b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1^{1-p}}dx_1\right|\\
\leq&\left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2||\omega||_\infty\left[\int_0^{\frac{1}{2}}\frac{t_1}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{p}}}\left[t_1{}^s|f'(a_1)|\right.\right.\\
&\left.+(1-t_1{}^s)|f'(b_1)|\right]dt_1+\int_{\frac{1}{2}}^1\frac{(1-t_1)}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{p}}}\left[t_1{}^s|f'(a_1)|\right.\\
&\left.\left.+(1-t_1{}^s)|f'(b_1)|\right]dt_1\right]\\
\leq&\left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2||\omega||_\infty\\
&\times\left\{\left[\int_0^{\frac{1}{2}}\frac{t_1{}^{s+1}}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{p}}}dt_1+\int_{\frac{1}{2}}^1\frac{t_1{}^s-t_1{}^{s+1}}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{p}}}dt_1\right]|f'(a_1)|\right.\\
&\left.+\left[\int_0^{\frac{1}{2}}\frac{t_1-t_1{}^{s+1}}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{p}}}dt_1+\int_{\frac{1}{2}}^1\frac{(1-t_1)(1-t_1{}^s)}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{p}}}dt_1\right]|f'(b_1)|\right\}
\end{aligned}$$

Finally,

$$\begin{aligned}
&\left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{x_1^{1-p}}dx_1-f\left(\left[\frac{a_1{}^p+b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1^{1-p}}dx_1\right| \qquad (6)\\
\leq&\left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2||\omega||_\infty\left[c_1(p)\left|f'(a_1)\right|+c_2(p)\left|f'(b_1)\right|\right].
\end{aligned}$$

This completes the proof. □

Remark 1. *If we put $s=1$ under the assumption of Theorem 3, it gives us Proposition 2.*

Theorem 4. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a_1,b_1]$, where $a_1,b_1\in I^\circ$ and $a_1<b_1$. If $|f'|^q, q\geq 1$, $s-p-$convex function of first kind on $[a_1,b_1]$ for $p\in\mathbb{R}\backslash\{0\}$ and $\omega:[a_1,b_1]\to\mathbb{R}$ is continuous, then the following inequality holds:*

$$\begin{aligned}
&\left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{x_1^{1-p}}dx_1-f\left(\left[\frac{a_1{}^p+b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1^{1-p}}dx_1\right|\\
\leq&\left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2\parallel\omega\parallel_\infty\left[(C_3(p))^{1-\frac{1}{q}}\left[C_4(p)|f'(a_1)|^q+C_5(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\right.\\
&\left.(C_6(p))^{1-\frac{1}{q}}\left[C_7(p)|f'(a_1)|^q+C_8(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$C_3(P)=\int_0^{\frac{1}{2}}\frac{t_1}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{q}}}dt_1,\qquad C_4(p)=\int_0^{\frac{1}{2}}\frac{t_1{}^{s+1}}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{q}}}dt_1$$

$$C_5(p)=\int_0^{\frac{1}{2}}\frac{t_1-t_1{}^{s+1}}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{q}}}dt_1,\qquad C_6(p)=\int_{\frac{1}{2}}^1\frac{1-t_1}{\left[t_1a_1^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{q}}}dt_1$$

$$C_7(p)=\int_{\frac{1}{2}}^1\frac{t_1{}^s-t_1{}^{s+1}}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{q}}}dt_1,\qquad C_8(p)=\int_{\frac{1}{2}}^1\frac{(1-t_1)(1-t_1{}^s)}{\left[t_1a_1{}^p+(1-t_1)b_1{}^p\right]^{1-\frac{1}{q}}}dt_1.$$

Proof. Using Lemma 1, Power mean inequality and $s-p-$convexity of 1^{st} kind of $|f'|^q$ it follows that

$$\begin{aligned}
&\left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{{x_1}^{1-p}}dx_1 - f\left(\left[\frac{{a_1}^p+{b_1}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{{x_1}^{1-p}}dx_1\right| \\
\leq\ & \left(\frac{{b_1}^p-{a_1}^p}{p}\right)^2 ||\omega||_\infty \int_0^1 \frac{|k(t_1)|}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)dt_1\right| \\
\leq\ & \left(\frac{{b_1}^p-{a_1}^p}{p}\right)^2 ||\omega||_\infty \\
&\times\left[\int_0^{\frac{1}{2}}\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|dt_1\right. \\
&\left.+\int_{\frac{1}{2}}^1\frac{(1-t_1)}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|dt_1\right] \\
\leq\ & \left(\frac{{b_1}^p-{a_1}^p}{p}\right)^2 ||\omega||_\infty \left\{\left(\int_0^{\frac{1}{2}}\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\right. \\
&\times\left[\int_0^{\frac{1}{2}}\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|^q dt_1\right]^{\frac{1}{q}} \\
&+\left(\int_0^{\frac{1}{2}}\frac{(1-t_1)}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}} \\
&\left.\times\left[\int_{\frac{1}{2}}^1\frac{(1-t_1)}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|^q dt_1\right]^{\frac{1}{q}}\right\} \qquad (7)
\end{aligned}$$

Since $|f'|$ is a $s-p-$convex function in first Kind on $[a_1,b_1]$, we have

$$\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right| \leq {t_1}^s|f'(a_1)|+(1-{t_1}^s)|f'(b_1)|. \qquad (8)$$

Using (8) in (7), we have

$$\begin{aligned}
\leq\; & \left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2 ||\omega||_\infty \left\{\left(\int_0^{\frac{1}{2}} \frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\right.\\
& \times\left[\int_0^{\frac{1}{2}} \frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}[t_1{}^s|f'(a_1)|^q+(1-t_1{}^s)|f'(b_1)|^q]dt_1\right]^{\frac{1}{q}}\\
& +\left(\int_0^{\frac{1}{2}} \frac{(1-t_1)}{[t_1a_1^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\\
& \left.\times\left[\int_{\frac{1}{2}}^1 \frac{(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}[t_1{}^s|f'(a_1)|^q+(1-t_1{}^s)|f'(b_1)|^q]dt_1\right]^{\frac{1}{q}}\right\}\\
& \leq\left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2 \parallel\omega\parallel_\infty \left\{\left(\int_0^{\frac{1}{2}} \frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\right.\\
& \times\left[\int_0^{\frac{1}{2}} \frac{t_1{}^{s+1}}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}|f'(a_1)|^q dt_1\right.\\
& \left.+\int_0^{\frac{1}{2}} \frac{(t_1-t_1{}^{s+1})}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}|f'(b_1)|^q dt_1\right]^{\frac{1}{q}}\\
& +\left(\int_0^{\frac{1}{2}} \frac{(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\\
& \times\left[\int_{\frac{1}{2}}^1 \frac{t_1{}^s(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1|f'(a_1)|^q\right.\\
& \left.\left.+\int_{\frac{1}{2}}^1 \frac{(1-t_1{}^s)(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1|f'(b_1)|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

$$\begin{aligned}
& \left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}}dx_1 - f\left(\left[\frac{a_1{}^p+b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1{}^{1-p}}dx_1\right|\\
\leq\; & \left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2 \parallel\omega\parallel_\infty \left[(C_3(p))^{1-\frac{1}{q}}\left[C_4(p)|f'(a_1)|^q+C_5(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\right.\\
& \left.+(C_6(p))^{1-\frac{1}{q}}\left[C_7(p)|f'(a_1)|^q+C_8(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\right]. \qquad (9)
\end{aligned}$$

This completes the proof. □

Remark 2. *If one take $s=1$ under the assumption of Theorem 4, it gives us Proposition 3.*

Theorem 5. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a_1,b_1]$, where $a_1,b_1\in I^\circ$ and $a_1<b_1$. If $|f'|^q$, $q\geq 1$, $s-p-$convex function of first kind on $[a_1,b_1]$ for $p\in\mathbb{R}\setminus\{0\}$ and $\omega:[a_1,b_1]\to\mathbb{R}$ is continuous, then the following inequality*

holds:

$$\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}} dx_1 - f\left(\left[\frac{a_1{}^p + b_1{}^p}{2}\right]^{\frac{1}{p}}\right) \int_{a_1}^{b_1} \frac{\omega(x_1)}{x_1{}^{1-p}} dx_1\right|$$

$$\leq \left(\frac{b_1{}^p - a_1{}^p}{p}\right)^2 \parallel \omega \parallel_\infty \left\{C_9(p)\left(\frac{1}{(s+1)(2^{s+1})}|f'(a_1)|^q\right.\right.$$

$$\left.+\left(\frac{1}{2} - \frac{1}{(s+1)(2^{s+1})}\right)|f'(b_1)|^q\right)^{\frac{1}{q}} + C_{10}(p)\left[\left(\frac{1}{s+1} - \frac{1}{(s+1)(2^{s+1})}\right)\right.$$

$$\left.\left.|f'(a_1)|^q + \left(\frac{1}{2} - \frac{1}{s+1} + \frac{1}{(s+1)(2^{s+1})}\right)|f'(b_1)|^q dt_1\right]^{\frac{1}{q}}\right\}$$

where

$$C_9(p,r) = \left(\int_0^{\frac{1}{2}} \left(\frac{t_1}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}},$$

$$C_{10}(p,r) = \left(\int_{\frac{1}{2}}^1 \left(\frac{1-t_1}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}$$

with $\frac{1}{p} + \frac{1}{z} = 1.$

Proof. Using Lemma 1, Hölder's inequality and $s-p-$convexity of $|f'|^q$ it follows that

$$\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}} dx_1 - f\left(\left[\frac{a_1{}^p + b_1{}^p}{2}\right]^{\frac{1}{p}}\right) \int_{a_1}^{b_1} \frac{\omega(x_1)}{x_1{}^{1-p}} dx_1\right|$$

$$\leq \left(\frac{b_1{}^p - a_1{}^p}{p}\right)^2 ||\omega||_\infty$$

$$\times \int_0^1 \frac{|k(t_1)|}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} \left|f'\left([t_1 a_1{}^p + (1-t_1) b_1{}^p]^{\frac{1}{p}}\right) dt_1\right|$$

$$\leq \left(\frac{b_1{}^p - a_1{}^p}{p}\right)^2 ||\omega||_\infty$$

$$\times \left[\int_0^{\frac{1}{2}} \frac{t_1}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} \left|f'\left([t_1 a_1^p + (1-t_1) b_1{}^p]^{\frac{1}{p}}\right)\right| dt_1\right.$$

$$\left.+\int_{\frac{1}{2}}^1 \frac{(1-t_1)}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} \left|f'\left([t_1 a_1{}^p + (1-t_1) b_1{}^p]^{\frac{1}{p}}\right)\right| dt_1\right]$$

$$\leq \left(\frac{b_1{}^p - a_1{}^p}{p}\right)^2 ||\omega||_\infty \left\{\left(\int_0^{\frac{1}{2}} \left(\frac{t_1}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}\right.$$

$$\left(\int_0^{\frac{1}{2}} \left|f'\left([t_1 a_1{}^p + (1-t_1) b_1{}^p]^{\frac{1}{p}}\right)\right|^q dt_1\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1 \left(\frac{1-t_1}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}$$

$$\left(\int_{\frac{1}{2}}^{1}\left|f'\left([t_1a_1{}^p+(1-t_1)b_1{}^p]^{\frac{1}{p}}\right)\right|^q dt_1\right)^{\frac{1}{q}}\Bigg\} \tag{10}$$

Since $|f'|$ is a $s-p-$convex function of first kind on $[a_1,b_1]$, we have

$$\left|f'\left([t_1a_1{}^p+(1-t_1)b_1{}^p]^{\frac{1}{p}}\right)\right| \le t_1{}^s|f'(a_1)|+(1-t_1{}^s)|f'(b_1)|. \tag{11}$$

Using (11) in (10), we have

$$\begin{aligned}
\le\ & \left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2||\omega||_\infty\Bigg\{\left(\int_0^{\frac{1}{2}}\left(\frac{t_1}{[t_1a_1{}^p+(1-t)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}\\
&\left(\int_0^{\frac{1}{2}}[t_1{}^s|f'(a_1)|^q+(1-t_1{}^s)|f'(b_1)|^q]dt_1\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left(\frac{1-t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}\\
&\left(\int_{\frac{1}{2}}^{1}[t_1{}^s|f'(a_1)|^q+(1-t_1{}^s)|f'(b_1)|^q]dt_1\right)^{\frac{1}{q}}\Bigg\}\\
\le\ & \left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2||\omega||_\infty\Bigg\{\left(\int_0^{\frac{1}{2}}\left(\frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}\\
&\left(\int_0^{\frac{1}{2}}t_1{}^s|f'(a_1)|^q dt_1+\int_0^{\frac{1}{2}}(1-t_1{}^s)|f'(b_1)|^q dt_1\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left(\frac{1-t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}\\
&\left(\int_{\frac{1}{2}}^{1}t_1{}^s|f'(a_1)|^q dt_1+\int_{\frac{1}{2}}^{1}(1-t_1{}^s)|f'(b_1)|^q dt_1\right)^{\frac{1}{q}}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\left|\int_a^{b_1}\frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}}dx_1-f\left(\left[\frac{a_1{}^p+b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1{}^{1-p}}dx_1\right|\\
\le\ & \left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2||\omega||_\infty\Bigg\{C_9(p)\Bigg[\frac{1}{(s+1)(2^{s+1})}|f'(a_1)|^q\\
&+\left(\frac{1}{2}-\frac{1}{(s+1)(2^{s+1})}\right)|f'(b_1)|^q\Bigg]^{\frac{1}{q}}\\
&+C_{10}(p)\Bigg[\left(\frac{1}{s+1}-\frac{1}{(s+1)(2^{s+1})}\right)|f'(a_1)|^q\\
&+\left(\frac{1}{2}-\frac{1}{s+1}+\frac{1}{(s+1)(2^{s+1})}\right)|f'(b_1)|^q\Bigg]^{\frac{1}{q}}\Bigg\}
\end{aligned} \tag{12}$$

Which is our required result. □

Remark 3. *If $s=1$ under the assumption of Theorem 5, it gives us Proposition 4.*

Inequalities for $s-p-$convex functions of second kind

By using Definition 7 of $s-p-$convex function we prove following results.

Theorem 6. *Let f: $I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a_1, b_1]$, where $a_1, b_1 \in I^\circ$ and $a_1 < b_1$. If $|f'|$ is $s-p-$convex function of second Kind on $[a_1, b_1]$ for $p \in \mathbb{R}\setminus\{0\}$ and $\omega : [a_1, b_1] \to \mathbb{R}$ is continuous, then the following inequality holds:*

$$\begin{aligned} & \left| \int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}} dx_1 - f\left(\left[\frac{a_1{}^p + b_1{}^p}{2} \right]^{\frac{1}{p}} \right) \int_{a_1}^{b_1} \frac{\omega(x_1)}{x_1{}^{1-p}} dx_1 \right| \\ \leq & \left(\frac{b_1{}^p - a_1{}^p}{p} \right)^2 ||\omega||_\infty \left[c_1(p)|f'(a_1)| + c_2(p)|f'(b_1)| \right]. \end{aligned}$$

where

$$c_1(p) = \left[\int_0^{\frac{1}{2}} \frac{t_1{}^{s+1}}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} dt_1 + \int_{\frac{1}{2}}^1 \frac{t_1{}^s - t_1{}^{s+1}}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} dt_1 \right],$$

$$c_2(p) = \left[\int_0^{\frac{1}{2}} \frac{t_1(1-t_1)^s}{[t_1 a_1^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} dt_1 + \int_{\frac{1}{2}}^1 \frac{(1-t_1)^{s+1}}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} dt_1 \right].$$

Proof. Using Lemma 1, it follows that

$$\begin{aligned} & \left| \int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}} dx_1 - f\left(\left[\frac{a_1{}^p + b_1{}^p}{2} \right]^{\frac{1}{p}} \right) \int_{a_1}^{b_1} \frac{\omega(x_1)}{x_1{}^{1-p}} dx_1 \right| \\ \leq & \left(\frac{b_1{}^p - a_1{}^p}{p} \right)^2 ||\omega||_\infty \\ & \times \int_0^1 \frac{|k(t_1)|}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} \left| f'\left([t_1 a_1{}^p + (1-t_1) b_1{}^p]^{\frac{1}{p}} \right) dt_1 \right| \\ \leq & \left(\frac{b_1{}^p - a_1{}^p}{p} \right)^2 ||\omega||_\infty \\ & \times \left[\int_0^{\frac{1}{2}} \frac{t_1}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} \left| f'\left([t_1 a_1{}^p + (1-t_1) b_1{}^p]^{\frac{1}{p}} \right) dt_1 \right| \right. \\ & \left. + \int_{\frac{1}{2}}^1 \frac{(1-t_1)}{[t_1 a_1{}^p + (1-t_1) b_1{}^p]^{1-\frac{1}{p}}} \left| f'\left([t_1 a_1{}^p + (1-t_1) b_1{}^p]^{\frac{1}{p}} \right) dt_1 \right| \right] \end{aligned} \tag{13}$$

Since $|f'|$ is a $s-p-$convex function of second kind on $[a_1, b_1]$, we have

$$\left| f'\left([t_1 a_1{}^p + (1-t_1) b_1{}^p]^{\frac{1}{p}} \right) \right| \leq t_1{}^s |f'(a_1)| + (1-t_1)^s |f'(b_1)|. \tag{14}$$

using (14) in (13), we have

$$\left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{x_1^{1-p}}dx_1 - f\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1^{1-p}}dx_1\right|$$

$$\leq \left(\frac{b_1^p-a_1^p}{p}\right)^2 ||\omega||_\infty \left\{\int_0^{\frac{1}{2}}\frac{t_1}{[t_1a_1^p+(1-t_1)b_1^p]^{1-\frac{1}{p}}}\left[t_1^s|f'(a_1)|\right.\right.$$

$$\left.+(1-t_1)^s|f'(b_1)|\right]dt_1 + \int_{\frac{1}{2}}^1\frac{(1-t_1)}{[t_1a_1^p+(1-t_1)b_1^p]^{1-\frac{1}{p}}}\left[t_1^s|f'(a_1)|\right.$$

$$\left.\left.+(1-t_1)^s|f'(b_1)|\right]dt_1\right\}$$

$$\leq \left(\frac{b_1^p-a_1^p}{p}\right)^2 ||\omega||_\infty \left\{\left[\int_0^{\frac{1}{2}}\frac{t_1^{s+1}}{[t_1a_1^p+(1-t_1)b_1^p]^{1-\frac{1}{p}}}dt_1\right.\right.$$

$$\left.+\int_{\frac{1}{2}}^1\frac{t_1^s-t_1^{s+1}}{[t_1a_1^p+(1-t_1)b_1^p]^{1-\frac{1}{p}}}dt_1\right]|f'(a_1)|$$

$$+\left[\int_0^{\frac{1}{2}}\frac{t_1(1-t_1)^s}{[t_1a_1^p+(1-t_1)b_1^p]^{1-\frac{1}{p}}}dt_1\right.$$

$$\left.\left.+\int_{\frac{1}{2}}^1\frac{(1-t_1)^{s+1}}{[t_1a_1^p+(1-t_1)b_1^p]^{1-\frac{1}{p}}}dt_1\right]|f'(b_1)|\right\}$$

$$\left|\int_{a_1}^{p}\frac{f(x_1)\omega(x_1)}{x_1^{1-p}}dx_1 - f\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1^{1-p}}dx_1\right|$$

$$\leq \left(\frac{b_1^p-a_1^p}{p}\right)^2 ||\omega||_\infty \left[c_1(p)|f'(a_1)|+c_2(p)|f'(b_1)|\right]. \tag{15}$$

Which is our required result. □

Remark 4. *If* $s=1$ *using assumption of Theorem 6 it gives us Proposition 2.*

Theorem 7. *Let* $f: I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a_1,b_1]$, *where* $a_1,b_1\in I^\circ$ *and* $a_1<b_1$. *If* $|f'|^q$, $q\geq 1$, $s-p-$*convex function of second kind on* $[a_1,b_1]$ *for* $p\in\mathbb{R}\setminus\{0\}$ *and* $\omega:[a_1,b_1]\to\mathbb{R}$ *is continuous, then the following inequality holds:*

$$\left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{x_1^{1-p}}dx_1 - f\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1^{1-p}}dx_1\right|$$

$$\leq \left(\frac{b_1^p-a_1^p}{p}\right)^2 \parallel\omega\parallel_\infty \left[(C_3(p))^{1-\frac{1}{q}}\left[C_4(p)|f'(a_1)|^q+C_5(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+(C_6(p))^{1-\frac{1}{q}}\left[C_7(p)|f'(a_1)|^q+C_8(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\right].$$

where

$$C_3(P)=\int_0^{\frac{1}{2}}\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1,\qquad C_4(p)=\int_0^{\frac{1}{2}}\frac{{t_1}^{s+1}}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1$$

$$C_5(p)=\int_0^{\frac{1}{2}}\frac{t_1(1-t_1)^s}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1,\qquad C_6(p)=\int_{\frac{1}{2}}^1\frac{1-t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1$$

$$C_7(p)=\int_{\frac{1}{2}}^1\frac{{t_1}^s-{t_1}^{s+1}}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1,\qquad C_8(p)=\int_{\frac{1}{2}}^1\frac{(1-t_1)(1-t_1)^s}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1.$$

Proof. Using Lemma 1, Power mean inequality and $s-p-$convexity for second kind of $|f'|^q$ it follows that

$$\begin{aligned}
&\left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{{x_1}^{1-p}}dx_1-f\left(\left[\frac{{a_1}^p+{b_1}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{{x_1}^{1-p}}dx_1\right|\\
\le\ &\left(\frac{{b_1}^p-{a_1}^p}{p}\right)^2||\omega||_\infty\\
&\times\int_0^1\frac{|k(t_1)|}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)dt_1\right|\\
\le\ &\left(\frac{{b_1}^p-a^p}{p}\right)^2||\omega||_\infty\\
&\times\left[\int_0^{\frac{1}{2}}\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1a_1^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|dt_1\right.\\
&\left.+\int_{\frac{1}{2}}^1\frac{(1-t_1)}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|dt_1\right]\\
\le\ &\left(\frac{{b_1}^p-{a_1}^p}{p}\right)^2||\omega||_\infty\left\{\left(\int_0^{\frac{1}{2}}\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\right.\\
&\times\left[\int_0^{\frac{1}{2}}\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|^q dt_1\right]^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{1}{2}}\frac{(1-t_1)}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\\
&\left.\times\left[\int_{\frac{1}{2}}^1\frac{(1-t_1)}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|^q dt_1\right]^{\frac{1}{q}}\right\}
\end{aligned}\tag{16}$$

Since $|f'|$ is a $s-p-$convex function of second kind on $[a_1,b_1]$, we have

$$\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|\le{t_1}^s|f'(a_1)|+(1-t_1)^s|f'(b_1)|.\tag{17}$$

Using (17) in (16) then,

$$\leq \left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2||\omega||_\infty\left\{\left(\int_0^{\frac{1}{2}}\frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[\int_0^{\frac{1}{2}}\frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}[t_1{}^s|f'(a_1)|^q+(1-t_1)^s|f'(b_1)|^q]dt_1\right]^{\frac{1}{q}}$$
$$+\left(\int_0^{\frac{1}{2}}\frac{(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}$$
$$\left.\times\left[\int_{\frac{1}{2}}^1\frac{(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}[t_1{}^s|f'(a_1)|^q+(1-t_1)^s|f'(b_1)|^q]dt_1\right]^{\frac{1}{q}}\right\}$$

$$\leq \left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2\parallel\omega\parallel_\infty\left\{\left(\int_0^{\frac{1}{2}}\frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[\int_0^{\frac{1}{2}}\frac{t_1{}^{s+1}}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}|f'(a_1)|^qdt_1\right.$$
$$\left.+\int_0^{\frac{1}{2}}\frac{t_1(1-t_1)^s}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}|f'(b_1)|^qdt_1\right]^{\frac{1}{q}}$$
$$+\left(\int_0^{\frac{1}{2}}\frac{(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1\right)^{1-\frac{1}{q}}$$
$$\times\left[\int_{\frac{1}{2}}^1\frac{t_1{}^s(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1|f'(a_1)|^q\right.$$
$$\left.\left.+\int_{\frac{1}{2}}^1\frac{(1-t_1)^s(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}dt_1|f'(b_1)|^q\right]^{\frac{1}{q}}\right\}$$

$$\left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}}dx_1-f\left(\left[\frac{a_1{}^p+b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1{}^{1-p}}dx_1\right|$$
$$\leq \left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2\parallel\omega\parallel_\infty\left[(C_3(p))^{1-\frac{1}{q}}\left[C_4(p)|f'(a_1)|^q+C_5(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+(C_6(p))^{1-\frac{1}{q}}\left[C_7(p)|f'(a_1)|^q+C_8(p)|f'(b_1)|^q\right]^{\frac{1}{q}}\right]. \quad (18)$$

This completes the proof. □

Remark 5. *If $s=1$ under the assumption of Theorem 7, it gives us Proposition 3.*

Theorem 8. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a_1,b_1]$, where $a_1,b_1\in I^\circ$ and $a_1<b_1$. If $|f'|^q$, $q\geq 1$, $s-p-$convex function of second kind on $[a_1,b_1]$ for $p\in\mathbb{R}\backslash\{0\}$ and $\omega:[a_1,b_1]\to\mathbb{R}$ is continuous, then the following*

inequality holds:

$$\begin{aligned}
&\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}}dx_1 - f\left(\left[\frac{a_1{}^p+b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1{}^{1-p}}dx_1\right| \\
\leq\; &\left(\frac{b_1{}^p-a_1^p}{p}\right)^2 \parallel \omega \parallel_\infty \left\{C_9(p)\left(\frac{1}{(s+1)(2^{s+1})}|f'(a_1)|^q\right.\right. \\
&\left.+\left(\frac{1}{s+1}-\frac{1}{(s+1)(2^{s+1})}\right)|f'(b_1)|^q\right)^{\frac{1}{q}} \\
&\left.+C_{10}(p)\left(\left(\frac{1}{s+1}-\frac{1}{(s+1)(2^{s+1})}\right)|f'(a_1)|^q+\frac{1}{(s+1)(2^{s+1})}|f'(b_1)|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where

$$C_9(p,r)=\left(\int_0^{\frac{1}{2}}\left(\frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}},$$

$$C_{10}(p,r)=\left(\int_{\frac{1}{2}}^1\left(\frac{1-t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}$$

with $\frac{1}{p}+\frac{1}{z}=1.$

Proof. Using Lemma 1, Hölder's inequality and $s-p-$convexity for second kind of $|f'|^q$ it follows that

$$\begin{aligned}
&\left|\int_{a_1}^{b_1} \frac{f(x_1)\omega(x_1)}{x_1{}^{1-p}}dx_1 - f\left(\left[\frac{a_1{}^p+b_1{}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{x_1{}^{1-p}}dx_1\right| \\
\leq\; &\left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2 ||\omega||_\infty \int_0^1 \frac{|k(t_1)|}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1a_1{}^p+(1-t_1)b_1{}^p]^{\frac{1}{p}}\right)dt_1\right| \\
\leq\; &\left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2 ||\omega||_\infty \\
&\times\left[\int_0^{\frac{1}{2}}\frac{t_1}{[t_1a_1^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1a_1{}^p+(1-t_1)b_1{}^p]^{\frac{1}{p}}\right)\right|dt_1\right. \\
&\left.+\int_{\frac{1}{2}}^1\frac{(1-t_1)}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\left|f'\left([t_1a_1{}^p+(1-t_1)b_1{}^p]^{\frac{1}{p}}\right)\right|dt_1\right] \\
\leq\; &\left(\frac{b_1{}^p-a_1{}^p}{p}\right)^2 ||\omega||_\infty\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}\right. \\
&\left(\int_0^{\frac{1}{2}}\left|f'\left([t_1a_1{}^p+(1-t_1)b_1{}^p]^{\frac{1}{p}}\right)\right|^q dt_1\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1\left(\frac{1-t_1}{[t_1a_1{}^p+(1-t_1)b_1{}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}
\end{aligned}$$

$$\left(\int_{\frac{1}{2}}^{1}\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right|^q dt_1\right)^{\frac{1}{q}}\Bigg\} \tag{19}$$

Since $|f'|$ is a $s-p-$convex function of second kind on $[a_1, b_1]$, we have

$$\left|f'\left([t_1{a_1}^p+(1-t_1){b_1}^p]^{\frac{1}{p}}\right)\right| \leq {t_1}^s|f'(a_1)|+(1-t_1)^s|f'(b_1)|. \tag{20}$$

Using (20) in (19) then,

$$\begin{aligned}
\leq\ & \left(\frac{{b_1}^p-{a_1}^p}{p}\right)^2 ||\omega||_\infty \Bigg\{\left(\int_0^{\frac{1}{2}}\left(\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\right)^z dt\right)^{\frac{1}{z}} \\
& \left(\int_0^{\frac{1}{2}}[{t_1}^s|f'(a_1)|^q+(1-t_1)^s|f'(b_1)|^q]dt_1\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{1}{2}}^1\left(\frac{1-t_1}{[t_1a_1^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}} \\
& \left(\int_{\frac{1}{2}}^1[{t_1}^s|f'(a_1)|^q+(1-t_1)^s|f'(b_1)|^q]dt_1\right)^{\frac{1}{q}}\Bigg\} \\
\leq\ & \left(\frac{{b_1}^p-{a_1}^p}{p}\right)^2 ||\omega||_\infty \Bigg\{\left(\int_0^{\frac{1}{2}}\left(\frac{t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}} \\
& \left(\int_0^{\frac{1}{2}}{t_1}^s|f'(a_1)|^q dt_1+\int_0^{\frac{1}{2}}(1-t_1)^s|f'(b_1)|^q dt_1\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{1}{2}}^1\left(\frac{1-t_1}{[t_1{a_1}^p+(1-t_1){b_1}^p]^{1-\frac{1}{p}}}\right)^z dt_1\right)^{\frac{1}{z}}\left(\int_{\frac{1}{2}}^1{t_1}^s|f'(a_1)|^q dt_1\right. \\
& \left.+\int_{\frac{1}{2}}^1(1-t_1)^s|f'(b_1)|^q dt_1\right)^{\frac{1}{q}}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
& \left|\int_{a_1}^{b_1}\frac{f(x_1)\omega(x_1)}{{x_1}^{1-p}}dx_1-f\left(\left[\frac{{a_1}^p+{b_1}^p}{2}\right]^{\frac{1}{p}}\right)\int_{a_1}^{b_1}\frac{\omega(x_1)}{{x_1}^{1-p}}dx_1\right| \\
\leq\ & \left(\frac{{b_1}^p-{a_1}^p}{p}\right)^2 ||\omega||_\infty \Bigg\{C_9(p)\left(\frac{1}{(s+1)(2^{s+1})}|f'(a_1)|^q\right. \\
& \left.+\left(\frac{1}{s+1}-\frac{1}{(s+1)(2^{s+1})}\right)|f'(b_1)|^q\right)^{\frac{1}{q}} \\
& +C_{10}(p)\left(\left(\frac{1}{s+1}-\frac{1}{(s+1)(2^{s+1})}\right)|f'(a_1)|^q+\frac{1}{(s+1)(2^{s+1})}|f'(b_1)|^q\right)^{\frac{1}{q}}\Bigg\}
\end{aligned} \tag{21}$$

Which is our required result. □

Remark 6. *If $s=1$ under assumption of Theorem 8, it gives us Proposition 4.*

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