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EXISTENCE AND UNIQUENESS THEOREMS FOR FOURTH-ORDER EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract. The purpose of this paper is to prove the existence and uniqueness theorem of the boundary value problem for fourth-order differential equations

$$u^{(4)} + q(x)u(x) = g(x),\ 0 < x < 1,$$

subject to the BC:

$$u(0) = u'(0) = u''(1) = u'''(1) = 0$$

in the Sobolev space $\mathbb{H}^4([0,1])$ by using an a priori estimate. We also investigate the Schauder's fixed point theorem for proving the existence theorem of the boundary value problem for fourth-order nonlinear differential equations

$$u^{(4)} = f(x, u', u', u'', u'''),\ 0 < x < 1,$$

under the above boundary conditions (BC), where $f : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a continuous function and satisfies

$$|f(x,u,v,w,z)| \leq a + a_0|u| + a_1|v| + a_2|w| + a_3|z|,$$

where $a, a_i > 0$, $i = 0, ..., 3$.

1. Introduction

In [1], the author considered the following linear boundary value problem for fourth-order differential equation:

$$u^{(4)}(x) + q(x)u(x) = g(x),\ 0 < x < 1, \tag{1}$$

$$u(0) = a,\ u(1) = b,\ u''(0) = c,\ u''(1) = d, \tag{2}$$

where q and g are continuous functions on $[0,1]$, and established a sufficient condition $\sup_{0\leq x\leq 1} \mid q(x) \mid < \pi^4$ that guarantees a unique solution for Pr.(1)-(2). This problem is used in different areas of physics, engineering and mathematics such as plate deflection theory. The analytical solution of Pr.(1)-(2) is given by Timoshenko and Woinowsky-Krieger [2] provided the functions $q(x)$ and $g(x)$ are constants. Also, the use of a matrix and power series methods for solving this problem are given in [3].
Yang [4] extended Pr.(1)-(2) and considered the following nonlinear problem

$$u^{(4)} = g(x,\ u,\ u''),\ 0 < x < 1,$$

subject to (2), and established a result on the existence and uniqueness theorem under a suitable condition

$$\mid g(x,\ y,\ z) \mid \leq a \mid y \mid + b \mid z \mid + c,\ a, b, c > 0,\ \frac{a}{\pi^4} + \frac{b}{\pi^2} < 1.$$

In this paper, we consider the beam equation (1) under various boundary conditions:

$$u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{3}$$

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which correspond to a beam clamped at $x = 0$ and free at $x = 1$ and prove some results on the existence and uniqueness theorems for Eq.(1) and Eq.(3) in the Sobolev space $\mathbb{H}^4([0,1])$. The proof is based on an a priori estimate and the density of the range of the linear operator generated by the studied problem. We also consider the following general nonlinear problem

$$u^{(4)} = f(x, u, u', u'', u'''),\ 0 < x < 1, \tag{4}$$

subject to (3), where $f : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a continuous function. A result on the existence theorem is proved under the condition

$$|f(x, u, v, w, z)| \leq a + a_0|u| + a_1|v| + a_2|w| + a_3|z|,$$

where $a, a_i > 0,\ i = 0, ..., 3$.

2. The linear BVP problem Eq.(1) with Eq.(3)

2.1. A result on the existence and uniqueness theorem in $\mathbb{H}^4([0,1])$. Rewrite the above boundary value problem Eq. (1) and Eq. (3) in the linear equation of the form

$$\mathcal{F}u = g,\ u \in U,$$

where

$$\mathcal{F} : D(\mathcal{F}) \subset U \to \mathbb{L}_2 :\ \mathcal{F}u = u^{(4)} + q(x)u$$

and U is a Hilbert space:

$$U = \left\{ u : u,\ \frac{d^i u}{dx^i} \in \mathbb{L}_2(0,1),\ i = 1, ..., 4. \right\}$$

with respect to the norm

$$||u||_U^2 = \int_0^1 \left[u^2 + \left(\frac{du}{dx}\right)^2 + \left(\frac{d^2u}{dx^2}\right)^2 + \left(\frac{d^3u}{dx^3}\right)^2 + \left(\frac{d^4u}{dx^4}\right)^2 \right] dx < \infty,$$

$$D(\mathcal{F}) = \{ u \in U :\ u(0) = u'(0) = u''(1) = u'''(1) = 0 \}.$$

The purpose of this section is to establish the uniqueness solution in U.

Lemma 1 (Wirtinger's Inequality). *Suppose $u \in \mathbb{C}^1[a,b]$ with $u(a) = 0$ or $u(b) = 0$. Then*

$$\int_a^b u^2(x)dx \leq \frac{4(b-a)^2}{\pi^2} \int_a^b (u'(x))^2 dx.$$

2.2. A priori estimate. For $u \in U$, we define the operator M by $Mu \equiv u$ and consider the scalar product $(\mathcal{F}u,\ Mu)_{L_2}$.
Employing integration by parts, and taking into account that the BC (3), we obtain

$$(\mathcal{F}u, u)_{L_2} = \int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 dx + \int_0^1 q(x)u^2 dx.$$

The scalar product $(\mathcal{F}u,\ Mu)_{L_2}$ can be estimated by means of the Cauchy-Schwarz-Bunyakovski inequality and the $\epsilon-$ inequality

$$2uv \leq \epsilon u^2 + \frac{1}{\epsilon}v^2,\ u,\ v \geq 0,\ \epsilon > 0,$$

$$|\ (\mathcal{F}u,\ Mu)_{L_2}\ | \quad \leq \quad \frac{1}{2\epsilon_1} \int_0^1 g^2(x)dx + \frac{\epsilon_1}{2} \int_0^1 u^2(x)dx,\ \epsilon_1 > 0.$$

If we suppose that $0 < \alpha \leq q(x) \leq \beta$ for all $x \in [0,1]$, then

$$(\alpha - \frac{\epsilon_1}{2}) \int_0^1 u^2 dx + \int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 dx \leq \frac{1}{2\epsilon_1} \int_0^1 g^2(x)dx. \tag{5}$$

From Eq.(1), we have

$$\frac{d^4u}{dx^4} = g(x) - q(x)u$$

and therefore it follows that

$$\epsilon_2 \int_0^1 \left(\frac{d^4u}{dx^4}\right)^2 dx \leq 2\epsilon_2 \left(\int_0^1 g^2(x)dx + \beta^2 \int_0^1 u^2 dx\right), \ \epsilon_2 > 0.$$

Adding this estimate with inequality (5), we obtain

$$(\alpha - \frac{\epsilon_1}{2} - 2\epsilon_2\beta^2) \int_0^1 u^2 dx + \int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 dx + \epsilon_2 \int_0^1 \left(\frac{d^4u}{dx^4}\right)^2 dx \leq (2\epsilon_2 + \frac{1}{2\epsilon_1}) \int_0^1 g^2(x)dx. \tag{6}$$

Adding also the following Wirtinger type inequalities [5]

$$\int_0^1 \left(\frac{du}{dx}\right)^2 dx \leq \frac{4}{\pi^2} \int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 dx, \ u'(0) = 0$$

and

$$\int_0^1 \left(\frac{d^3u}{dx^3}\right)^2 dx \leq \frac{4}{\pi^2} \int_0^1 \left(\frac{d^4u}{dx^4}\right)^2 dx, \ u'''(0) = 0 \text{ or } u'''(1) = 0$$

to (6) and choosing $\epsilon_i, \ i = 1,2$ to be sufficiently small so that $\alpha - \frac{\epsilon_1}{2} - 2\epsilon_2\beta^2 > 0$ and $\frac{4}{\pi^2} < \epsilon_2 < 1$. Since $0 < \alpha < \beta$, we have $0 < \epsilon_1 < 2\alpha - \frac{16}{\pi^2}\alpha^2$. Hence

$$\int_0^1 \left[u^2 + \left(\frac{du}{dx}\right)^2 + \left(\frac{d^2u}{dx^2}\right)^2 + \left(\frac{d^3u}{dx^3}\right)^2 + \left(\frac{d^4u}{dx^4}\right)^2\right] dx \leq C \int_0^1 g^2(x)dx,$$

that is

$$||u||_U \leq C_1 ||g||_{L_2}, \tag{7}$$

where $C_1 = C^{\frac{1}{2}}$ and $C = \frac{2\epsilon_2 + \frac{1}{2\epsilon_1}}{\min(\alpha - \frac{\epsilon_1}{2} - 2\epsilon_2\beta^2, \epsilon_2 - \frac{4}{\pi^2})}$.
Thus we have proved the following statement.

Theorem 1. *Suppose that $0 < \alpha \leq q(x) \leq \beta$ for all $x \in [0,1]$. Then for $g \in L_2[0,1]$, there exists a constant $C_1 > 0$ such that the obtained a priori estimate (7) holds.*

2.3. **Existence of the solution.** Notice that the uniqueness of the solution follows immediately from the estimate (7), however the existence of the solution is equivalent to the property $R(L) = \mathbb{L}_2$, where $R(L)$ is the range of L.
We need the following lemmas:

Lemma 2. *L is a closed operator.*

Proof. To show that L is a closed operator, let $\{u_n\}$ be a convergent sequence, that is $u_n \to u$ in $\mathbb{L}_2$ such that $\{Lu_n\}$ converges to w in $\mathbb{L}_2$. From,

$$\int_0^1 Lu_n\varphi(x)dx = \int_0^1 u_n^{(4)}(x)\varphi(x)dx + \int_0^1 q(x)u_n(x)\varphi(x)dx, \ \varphi \in \mathbb{C}_0^\infty(0,1),$$

that is φ and its derivatives $\varphi^{(k)}, \ \forall \ k \in \mathbb{N}$ vanish outside $[0,1]$, we have

$$\int_0^1 Lu_n\varphi(x)dx = \int_0^1 u_n^{(4)}(x)\varphi(x)dx + \int_0^1 q(x)u_n(x)\varphi(x)dx.$$

Taking the limit $n \to \infty$ to get

$$\int_0^1 w\varphi(x)dx = \int_0^1 (u^{(4)} + q(x)u)\varphi(x)dx.$$

Thus $Lu = w \in \mathbb{L}_2$. $\square$

Lemma 3. *For the linear operator $L : D(L) \subset U \to \mathbb{L}_2$, we have $R(L) = \mathbb{L}_2$ if and only if $R(L)$ is closed and $R(L)^{\perp} = \{0\}$.*

Thus we have

Theorem 2. *Under the hypotheses of Proposition 1, and if we assume $R(L)^{\perp} = \{0\}$. Then for each $g \in \mathbb{L}_2$, the problem Eq. (1) and Eq. (3) has a unique solution.*

Proof. Let $\{u_n\} \subset D(L)$ be a sequence with $Lu_n = g_n$. Using the a priori estimate (7), we obtain

$$||u_n - u_m||_U \leq C_1||g_n - g_m||_{L_2}.$$

Thus $\{u_n\}$ is a Cauchy sequence in U. Since U is a Hilbert space and L is closed, the sequence $\{u_n\}$ converges, that is $u_n \to u \in U$, we conclude that $u \in D(L)$ and $Lu = g \in R(L)$. Consequently, $R(L)$ is closed. So we obtain the existence of the solution. $\square$

3. The nonlinear BVP problem Eq.(4) with (3)

In this section, we will reformulate the nonlinear boundary value problem Eq.(4) subject to the BC (3) as a fixed point problem for integral equation.
We will use the following Lemmas.

Lemma 4. *Let $g : [0, 1] \to \mathbb{R}$ be a continuous function. The unique solution u of the following boundary value problem*

$$u^{(4)} = g(x) \tag{8}$$

subject to the boundary conditions (3) is given by

$$u(x) = \int_0^1 G(x, y)g(y)dy,$$

where $G(x, y)$ is the Green function given by

$$G(x,y) = \begin{cases} \dfrac{-y^3}{6} + \dfrac{xy^2}{2}, & \text{if } 0 \leq y \leq x \leq 1, \\ \dfrac{-x^3}{6} + \dfrac{x^2y}{2}, & \text{if } 0 \leq x \leq y \leq 1. \end{cases}$$

Proof. Integrating (8) twice, we obtain:

$$u''(x) = C_1x + C_2 + \int_0^x \int_0^x g(s)dsdy,$$

where C_1 and C_2 are constants.
Thus,

$$u''(x) = C_1x + C_2 + \int_0^x (x - y)g(y)dy.$$

Integrating again both sides of this equation, we obtain

$$u'(x) = \frac{C_1x^2}{2} + C_2x + C_3 + \frac{1}{2}\int_0^x (x - y)^2 g(y)dy.$$

where C_i, $i = 1, 2, 3$ are also constants. Hence,

$$u(x) = \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4 + \frac{1}{6}\int_0^x (x-y)^3 g(y)dy.$$

We determine C_i from $u(0) = u'(0) = u''(1) = u'''(1) = 0$, we obtain

$$C_1 = -\int_0^1 g(y)dy,\ C_2 = \int_0^1 yg(y)dy,\ C_3 = C_4 = 0.$$

Consequently,

$$u(x) = \int_0^x \left(\frac{-y^3}{6} + \frac{xy^2}{2}\right)g(y)dy + \int_x^1 \left(\frac{-x^3}{6} + \frac{x^2 y}{2}\right)g(y)dy.$$

□

Lemma 5. *The Green function and its derivatives satisfy the following properties:*

(1)
$$\max_{0\le x\le 1}\int_0^1 |G(x,y)|dy \le 0.220026174 = K_0,$$

(2)
$$\max_{0\le x\le 1}\int_0^1 |G_x(x,y)|dy \le \frac{1}{6} = K_1,$$

(3)
$$\max_{0\le x\le 1}\int_0^1 |G_{xx}(x,y)|dy \le \frac{2}{3} = K_2,$$

(4)
$$\max_{0\le x\le 1}\int_0^1 |G_{xxx}(x,y)|dy \le 1 = K_3.$$

The problem (4)-(3) can be converted into the following system:

$$\begin{cases} u''' &= v,\ u(0) = u'(0) = u''(1) = 0, \\ v' &= f(x, u, u', u'', v),\ v(1) = 0. \end{cases} \tag{9}$$

Proposition 1. *Suppose that* $f : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ *satisfies*

$$|f(x, u, v, w, z)| \le a + a_0|u| + a_1|v| + a_2|w| + a_3|z|,$$

where $a, a_i > 0$, $i = 0, ..., 3$ *and* $\frac{16a_0}{\pi^3} + \frac{8a_1}{\pi^2} + \frac{4a_2}{\pi} + \frac{2a_3}{\pi} < 1 - \frac{\epsilon}{2}$, *where* $\epsilon > 0$ *is sufficiently small. Then there exist a constant* $M > 0$, *such that*

$$\|u\|_E \le M,\ \text{where } \|u\|_E = \max_{0\le x\le 1}\sum_{i=0}^{3} a_i|u^{(i)}(x)| \ \text{ and } E = \mathbb{C}^3([0,1]).$$

Proof. Multiplying both sides of the first equation of (9) by u'' and integrating from x to 1, we get

$$\frac{-1}{2}(u'')^2(x) = \int_x^1 v(x)u''(x)dx.$$

Then

$$(u'')^2(x) \le 2\int_x^1 |v(x)||u''(x)|dx.$$

Using Cauchy-Schwarz inequality, we get

$$(u''(x))^2 dx \le 2\left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 (u''(x))^2 dx\right)^{\frac{1}{2}}.$$

Thus

$$\int_0^1 (u''(x))^2 dx \leq 2\left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 (u''(x))^2 dx\right)^{\frac{1}{2}}.$$

Hence

$$\int_0^1 (u''(x))^2 dx \leq 4\int_0^1 v^2(x)dx. \tag{10}$$

Applying Wirtinger's inequality Lemma to the R.H.S of (10), we get

$$\int_0^1 (u''(x))^2 dx \leq \frac{16}{\pi^2}\int_0^1 (v'(x))^2 dx. \tag{11}$$

Multiplying now both sides of the second equation of Eq.(9) by v' and integrating from 0 to 1, we get

$$\int_0^1 (v'(x))^2 dx = \int_0^1 f(x,u,u',u'',v)v'(x)dx.$$

Thus, from the given condition on f, we obtain

$$\begin{aligned}\int_0^1 (v'(x))^2 dx \quad &\leq \quad a\int_0^1 |v'(x)|dx + a_0\int_0^1 |u(x)||v'(x)|dx\\ &+ \quad a_1\int_0^1 |u'(x)||v'(x)|dx + a_2\int_0^1 |u''(x)||v'(x)|dx\\ &+ \quad a_3\int_0^1 |v(x)||v'(x)|dx.\end{aligned}$$

Therefore, by using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_0^1 (v'(x))^2 dx \leq a\int_0^1 |v'(x)|dx \quad &+ \quad a_0\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}}\\ +a_1\left(\int_0^1 (u'(x))^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}} \quad &+ \quad a_2\left(\int_0^1 (u''(x))^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}}\\ +a_3\left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

The term $a\int_0^1 |v'(x)|dx$ can be estimated by means of the ϵ- inequality. Thus

$$\begin{aligned}\int_0^1 (v'(x))^2 dx \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}\int_0^1 (v'(x))^2 dx \quad &+ \quad a_0\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}}\\ +a_1\left(\int_0^1 (u'(x))^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}} \quad &+ \quad a_2\left(\int_0^1 (u''(x))^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}}\\ &+ \quad a_3\left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

From the following Wirtinger's inequalities:

$$\begin{aligned}
\int_0^1 v^2(x)dx &= \frac{4}{\pi^2}\int_0^1 (v'(x))^2 dx, \quad v(1)=0,\\
\int_0^1 u^2(x)dx &= \frac{4}{\pi^2}\int_0^1 (u'(x))^2 dx, \quad u(0)=0,\\
\int_0^1 (u'(x))^2 dx &= \frac{4}{\pi^2}\int_0^1 (u''(x))^2 dx, \quad u'(0)=0,
\end{aligned}$$

we have,

$$\begin{aligned}
\int_0^1 (v'(x))^2 dx &\le \frac{a^2}{2\epsilon} + \frac{\epsilon}{2}\int_0^1 (v'(x))^2 dx + \frac{16a_0}{\pi^3}\int_0^1 (v'(x))^2 dx + \frac{8a_1}{\pi^2}\int_0^1 (v'(x))^2 dx\\
&+ \frac{4a_2}{\pi}\int_0^1 (v'(x))^2 dx + \frac{2a_3}{\pi}\int_0^1 (v'(x))^2 dx.
\end{aligned}$$

Hence,

$$(1-\frac{\epsilon}{2}-\frac{16a_0}{\pi^3}-\frac{8a_1}{\pi^2}-\frac{4a_2}{\pi}-\frac{2a_3}{\pi})\int_0^1 (v'(x))^2 dx \le \frac{a^2}{2\epsilon}.$$

Since ϵ is sufficiently small, we can choose

$$1-\frac{\epsilon}{2}-\frac{16a_0}{\pi^3}-\frac{8a_1}{\pi^2}-\frac{4a_2}{\pi}-\frac{2a_3}{\pi} = K,\ K>0.$$

Thus

$$\int_0^1 (v'(x))^2 dx \le \frac{a^2}{2\epsilon K} \tag{12}$$

since

$$-v(x) = \int_x^1 v'(x)dx, \quad v(1)=0,$$

we have

$$|v(x)| \le \left(\int_0^1 (v'(x))^2 dx\right)^{\frac{1}{2}} \le \frac{a}{\sqrt{2\epsilon K}}.$$

Hence

$$|u'''(x)| \le M_3 = \frac{a}{\sqrt{2\epsilon K}}. \tag{13}$$

From (11) and (12), we have:

$$\int_0^1 (u''(x))^2 dx \le \frac{8a^2}{\epsilon\pi^2 K}$$

since

$$u'(x) = \int_0^x u''(x)dx,\ u'(0)=0.$$

Thus

$$|u'(x)| \le \left(\int_0^1 (u''(x))^2 dx\right)^{\frac{1}{2}} \le \frac{2\sqrt{2}a}{\pi\sqrt{\epsilon K}} = M_1. \tag{14}$$

In view of

$$u(x) = \int_0^x u'(x)dx, \quad u(0)=0,$$

we obtain

$$|u(x)| \le \left(\int_0^1 (u'(x))^2 dx\right)^{\frac{1}{2}} \le M_1. \tag{15}$$

Also, from

$$u''(x) = \int_1^x u'''(x)dx, \quad u''(1) = 0,$$

we obtain

$$|u''(x)| \leq \int_0^1 |u'''(x)|dx \leq M_3. \tag{16}$$

From (13), (14), (15) and (16), we get the required result. □

Theorem 3. *Suppose f satisfies the condition of Proposition (1). Then problem (3) with (4) has a solution.*

Proof. Define the operator $T : \mathbb{C}^{(3)}[0,1] \longrightarrow \mathbb{C}^{(3)}[0,1]$ by

$$Tu = \int_0^1 G(x,y)f(s,u(s),u'(s),u''(s),u'''(s))ds.$$

By differentiation with respect to x, we get

$$\begin{aligned}
(Tu)'(x) &= \int_0^1 G_x(x,s)f(s,u(s),u'(s),u''(s)u'''(s))ds, \\
(Tu)''(x) &= \int_0^1 G_{xx}(x,s)f(s,u(s),u'(s),u''(s),u'''(s))ds, \\
(Tu)'''(x) &= \int_0^1 G_{xxx}(x,s)f(s,u(s),u'(s),u''(s),u'''(s))ds.
\end{aligned}$$

Thus

$$\begin{aligned}
|Tu(x)| &\leq \int_0^1 |G(x,s)||f(s,u(s),u'(s),u''(s),u'''(s))|ds, \\
|(Tu)'(x)| &\leq \int_0^1 |G_x(x,s)||f(s,u(s),u'(s),u''(s),u'''(s))|ds, \\
|(Tu)''(x)| &\leq \int_0^1 |G_{xx}(x,s)||f(s,u(s),u'(s),u''(s),u'''(s))|ds, \\
|(Tu)'''(x)| &\leq \int_0^1 |G_{xxx}(x,s)||f(s,u(s),u'(s),u''(s),u'''(s))|ds.
\end{aligned}$$

Using Lemma (5) and the condition of $f(s,u(s),u'(s),u''(s),u'''(s))$, we obtain

$$\begin{aligned}
|Tu(x)| &\leq K_0\left[a + \int_0^1 [a_0|u| + a_1|u'| + a_2|u''| + a_3|u'''|]\, ds\right], \\
|(Tu)'(x)| &\leq K_1\left[a + \int_0^1 [a_0|u| + a_1|u'| + a_2|u''| + a_3|u'''|]\, ds\right], \\
|(Tu)''(x)| &\leq K_2\left[a + \int_0^1 [a_0|u| + a_1|u'| + a_2|u''| + a_3|u'''|]\, ds\right], \\
|(Tu)'''(x)| &\leq K_3\left[a + \int_0^1 [a_0|u| + a_1|u'| + a_2|u''| + a_3|u'''|]\, ds\right].
\end{aligned}$$

Since $|u^{(i)}(x)| \leq \max_{0\leq x\leq 1}|u^{(i)}(x)|$, then

$$\begin{aligned} |Tu(x)| &\leq K_0\left[a+\|u\|_E\right], \\ |(Tu)'(x)| &\leq K_1\left[a+\|u\|_E\right], \\ |(Tu)''(x)| &\leq K_2\left[a+\|u\|_E\right], \\ |(Tu)'''(x)| &\leq K_3\left[a+\|u\|_E\right]. \end{aligned}$$

Multiplying the last forth inequalities by a_0, a_1, a_2, a_3, respectively, we obtain

$$\begin{aligned} a_0|Tu(x)| &\leq a_0K_0\left[a+\|u\|_E\right], \\ a_1|(Tu)'(x)| &\leq a_1K_1\left[a+\|u\|_E\right], \\ a_2|(Tu)''(x)| &\leq a_2K_2\left[a+\|u\|_E\right], \\ a_3|(Tu)'''(x)| &\leq a_3K_3\left[a+\|u\|_E\right]. \end{aligned}$$

Hence

$$\|Tu\|_E \leq [a_0K_0+a_1K_1+a_2K_2+a_3K_3][a+\|u\|_E].$$

Using Proposition (1), we obtain

$$\|Tu\|_E \leq M_1,$$

where $M_1 = (a+M)[a_0K_0+a_1K_1+a_2K_2+a_3K_3]$.
Hence T maps the closed, bounded and convex set

$$S = \left\{u \in \mathbb{C}^{(3)}[0,1]: \ |u^{(i)}(x)| \leq M_i, i=0,...3\right\}$$

into itself. T is completely continuous on $\mathbb{C}^{(3)}[0,1]$ and so is a compact operator by Ascoli's theorem. The Schauder's fixed point theorem then yields the fixed point of T, which is a solution of the given nonlinear boundary value problem.

□

Another result similar to [6] is that

Theorem 4. *Let $f : [a,b]\times\mathbb{R}^4 \longrightarrow \mathbb{R}$ satisfy a Lipshitz condition*

$$\begin{aligned} |f(x,u(x),u'(x),u''(x),u'''(x)) - f(x,v(x),v'(x),v''(x),v'''(x))| &< a_0|u(x)-v(x)| \\ +a_1|u'(x)-v'(x)| + a_2|u''(x)-v''(x)| + a_3|u'''(x)-v'''(x)|, \end{aligned}$$

where $a_i > 0$, $i = 0,...,3$. Assume also that $a_0K_0+a_1K_1+a_2K_2+a_3K_3 < 1$. Thus there exists a unique solution to the boundary value problem Pr.(4)-(3).

Proof. It must be shown that T is a contraction map. Indeed,

$$\begin{aligned} |Tu(x)-Tv(x)| &\leq K_0\|u-v\|_3, \\ |(Tu)'(x)-(Tv)'(x)| &\leq K_1\|u-v\|_3 \\ |(Tu)''(x)-(Tv)''(x)| &\leq K_2\|u-v\|_3 \\ |(Tu)'''(x)-(Tv)'''(x)| &\leq K_3\|u-v\|_3. \end{aligned}$$

Thus

$$\|Tu-Tv\|_E \leq [a_0K_0+a_1K_1+a_2K_2+a_3K_3]\|u-v\|_3.$$

By hypothesis, $a_0K_0+a_1K_1+a_2K_2+a_3K_3 < 1$. Therefore T is a contraction. Hence there exists a unique solution u such that $Tu = u$. □

Example 1. *Let*

$$u^{(4)} = \frac{1}{10}e^x u + \frac{1}{5}\sin(u') - \frac{1}{8}u'' + \frac{x}{10}u'''.$$

Here $f(x,u,u',u'',u''') = \frac{1}{10}e^x u + \frac{1}{5}\sin(u') - \frac{1}{8}u'' + \frac{x}{10}u'''$, $x \in [0,1]$. *Thus, the mean value theorem gives us*

$$\mid \frac{\partial f}{\partial u} \mid = \mid \frac{1}{10}e^x \mid \leq \frac{1}{10} = a_0, \ \mid \frac{\partial f}{\partial u'} \mid = \frac{1}{5} \mid \cos \mid \leq \frac{1}{5} = a_1,$$

$$\mid \frac{\partial f}{\partial u''} \mid = \frac{1}{8} = a_2, \ \mid \frac{\partial f}{\partial u'''} \mid = \frac{x}{10} \leq \frac{1}{10} = a_3.$$

Since $a_0K_0 + a_1K_1 + a_2K_2 + a_3K_3 < 1$. *Thus the given problem has a unique solution.*

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