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A NOTE ON THE BESSEL FORM OF PARAMETER 3/2

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Abstract. In this manuscript, we consider a certain raising operator (with a nonzero free parameter) and we prove the following statement: up to normalization, the only orthogonal sequence that remain orthogonal after application of this raising operator is the one obtained by dilating the Bessel polynomials of parameter 3/2.

1. Introduction and preliminaries

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ be its dual. We note by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. For $n \geq 0$, $(u)_n = \langle u, x^n\rangle$ are the moments of the form u (linear functional). For any form u, any polynomial f, and any $(a, b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, let $Du = u'$, fu, $\tau_{-b}u$ and $h_a u$ be the forms defined by: for all $g \in \mathcal{P}$ [10]

$$\langle fu, g\rangle := \langle u, fg\rangle,\ \langle u', g\rangle := -\langle u, g'\rangle, \langle h_a u, g\rangle := \langle u, g(ax)\rangle,\ \langle \tau_{-b}u, g\rangle := \langle u, g(x-b)\rangle.$$

The form u is called regular if we can associate with it a monic orthogonal polynomials sequence $\{P_n\}_{n\geq 0}$, $\deg P_n = n$ (MOPS in short), such that [6, 10]

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m},\ n, m \geq 0,\ r_n \neq 0,\ n \geq 0. \tag{1}$$

In this paper, we suppose that any regular form u will be normalized that is to say $(u)_0 = 1$.
Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$ (MPS) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$. Notice that u_0 is said to be the canonical form associated with the MPS $\{P_n\}_{n\geq 0}$.

The MPS $\{P_n\}_{n\geq 0}$ is called D-classical when $\{P_n\}_{n\geq 0}$ and $\{P_n^{[1]}\}_{n\geq 0}$ are two MOPSs (Hahn property) [10, 11]. In this case the canonical form u_0 is also said D-classical. Moreover, When $\{P_n\}_{n\geq 0}$ is D-classical , then $\{\tilde{P}_n\}_{n\geq 0}$, where $\tilde{P}_n(x) = a^{-n}P_n(ax + b)$, $(a, b) \in \mathbb{C}^* \times \mathbb{C}$, is also D-classical [10, 11]. Equivalently, u_0 fulfils the Pearson equation [10, 11]

$$(\Phi(x)u_0)' + \Psi(x)u_0 = 0$$

where the polynomial Φ is monic, $\deg \Phi \leq 2$ and Ψ is a polynomial with $\deg \Psi = 1$; and its shifted $\widetilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$ fulfils the Pearson equation [10, 11]

$$(a^{-\deg \Phi}\Phi(ax + b)\widetilde{u}_0)' + a^{1-\deg \Phi}\Psi(ax + b)\widetilde{u}_0 = 0. \tag{2}$$

There are four D-classical MOPSs: Hermite, Laguerre, Bessel and Jacobi [10, 11].

Let us consider the D-classical Bessel polynomials $\{B_n^{(\alpha)}\}_{n\geq 0}$ and its canonical form $\mathcal{B}(\alpha)$, $\alpha \neq -\frac{n}{2}$, $n \geq 0$. The Bessel form $\mathcal{B}(\alpha)$, $\alpha \neq -\frac{n}{2}$, $n \geq 0$ satisfies the Pearson equation [10]

$$\big(x^2\mathcal{B}(\alpha)\big)' - 2(\alpha x + 1)\mathcal{B}(\alpha) = 0. \tag{3}$$

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Every polynomial $B_{n+1}^{(\alpha)},\ n\geq 0$ satisfies a second-order differential equation (Böchner property) [10]

$$x^2B_{n+1}^{(\alpha)''}(x)+2(\alpha x+1)B_{n+1}^{(\alpha)'}(x)=(n+1)(n+2\alpha)B_{n+1}^{(\alpha)}(x),\ n\geq 0. \tag{4}$$

In addition, These polynomials can be represented as a hypergeometric function [14]

$$B_n^{(\alpha)}(x)=\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{2^{n-\nu}\Gamma(n+2\alpha+\nu-1)}{\Gamma(2n+2\alpha-1)}\,x^{\nu},\ n\geq 0, \tag{5}$$

where Γ is the Gamma function. As consequence of (5), we get the following fundamental formula

$$(B_{n+1}^{(\alpha)})'(x)=(n+1)B_n^{(\alpha+1)}(x),\ n\geq 0. \tag{6}$$

Instead of the derivative operator D, we are going to consider the following raising operator $\mathfrak{B}_\xi$ defined by

$$\mathfrak{B}_\xi:=x(xD+I_{\mathcal{P}})+\xi I_{\mathcal{P}},\quad \xi\neq 0, \tag{7}$$

where $I_{\mathcal{P}}$ is the identity operator in $\mathcal{P}$. Denoting the MPS $\{Q_n\}_{n\geq 0}$ by

$$Q_0(x)=1,\quad Q_n(x):=\frac{\mathfrak{B}_\xi P_{n-1}(x)}{n},\ n\geq 1. \tag{8}$$

Our aim is then to find the MOPSs $\{P_n\}_{n\geq 0}$ such that the MPS $\{Q_n\}_{n\geq 0}$ is also orthogonal (the analog of Hahn's property); that is to say $\{P_n\}_{n\geq 0}$ is $\mathfrak{B}_\xi$-classical. The resulting is the scaled Bessel MOPS $\{a^{-n}B_n^{(3/2)}(ax)\}_{n\geq 0}$, where $a=2\xi^{-1}$. The main result will be proved in the next section (see Theorem 2.1 bellow). In fact, the concept of O-classical orthogonal polynomials, where O is an operator on $\mathcal{P}$, has been studied by many authors in the literature [1-5, 7-9, 11-13].

2. Main result

On account of (7), we may write

$$\begin{array}{rcl}\mathfrak{B}_\xi:\mathcal{P} & \longrightarrow & \mathcal{P}\\ f & \longmapsto & \mathfrak{B}_\xi(f)=x^2f'+(x+\xi)f.\end{array}$$

Clearly, the operator $\mathfrak{B}_\xi$ raises the degree of any polynomial. Such an operator is called *raising operator* [5, 8, 13].
Furthermore, (8) is equivalent to

$$Q_0(x)=1,\quad (n+1)Q_{n+1}(x)=x^2P_n'(x)+(x+\xi)P_n(x),\ n\geq 0. \tag{9}$$

Moreover, by transposition of the operator $\mathfrak{B}_\xi$, we get

$${}^t\mathfrak{B}_\xi=-\mathfrak{B}_{-\xi}$$

since ${}^tD=-D$ and the well known formula $(fu)'=f'u+fu',\ f\in\mathcal{P},\ u\in\mathcal{P}'$ [10].

Based on the Böchner characterization (4) of Bessel polynomials and the Pearson equation (3) satisfied by the Bessel form $\mathcal{B}(\alpha)$, we can state the following theorem.

Theorem 1. *For any nonzero complex number ξ and any MPS $\{P_n\}_{n\geq 0}$, the following statements are equivalent.*

(i) $\{P_n\}_{n\geq 0}$ *is an $\mathfrak{B}_\xi$-classical orthogonal sequence.*

(ii) *There exists $a\in\mathbb{C},\ a\neq 0$ such that $P_n(x)=a^{-n}B_n^{(3/2)}(ax),\ n\geq 0$.*

Proof. (i)$\Rightarrow$(ii).
Assume that $\{P_n\}_{n\geq 0}$ is a $\mathfrak{B}_\xi$-classical. Consequently, $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are two MOPSs. Denoting u_0 the canonical form of $\{P_n\}_{n\geq 0}$ and v_0 the one's of $\{Q_n\}_{n\geq 0}$. By virtue of (9) and (1), we get

$$\big\langle v_0, x^2P_n'(x)+(x+\xi)P_n(x)\big\rangle = \big\langle v_0,(n+1)Q_{n+1}(x)\big\rangle = 0,\ n\geq 0.$$

But the left hand side reads as

$$\big\langle -(x^2v_0)'+(x+\xi)v_0, P_n(x)\big\rangle = 0,\ n\geq 0.$$

In other words,

$$(x^2v_0)'-(x+\xi)v_0=0$$

that is to say that v_0 is the scaled Bessel form $\mathcal{B}(1/2)$ with a dilatation $a=2\xi^{-1}$ according to (3) and (2).
Furthermore, writing for all $n\geq 0$

$$P_n(x)=x^n+\sum_{k=0}^{n-1}a_{n,k}x^k \quad \text{and} \quad Q_{n+1}(x)=x^{n+1}+\sum_{k=0}^{n}b_{n+1,k}x^k,$$

then, the equality in (9) becomes

$$\begin{aligned}\sum_{k=0}^{n}(n+1)b_{n+1,k}x^k &= \xi x^n+\sum_{k=0}^{n-1}(k+1)a_{n,k}x^{k+1}+\sum_{k=0}^{n-1}\xi a_{n,k}x^k\\ &= (\xi+na_{n,n-1})x^n+\sum_{k=1}^{n-1}(ka_{n,k-1}+\xi a_{n,k})x^k+\xi a_{n,0}.\end{aligned}$$

Therefore,

$$(n+1)b_{n+1,k}=ka_{n,k-1}+\xi a_{n,k}, \quad 0\leq k\leq n, \tag{10}$$

with the convention $a_{n,-1}:=0$.
Equivalently, one may write (10) as

$$a_{n,k+1}=-\xi^{-1}(k+1)a_{n,k}+\xi^{-1}(n+1)b_{n+1,k+1},\ k\geq 0, \tag{11}$$

with $a_{n,0}=\xi^{-1}(n+1)b_{n+1,0}$.
On account of (11) and by using Lemma 4.1 p. 439 in [12] we get in a straightforward way

$$a_{n,k}=(-1)^k k!\,\xi^{-k-1}(n+1)\sum_{l=0}^{k}\frac{(-1)^l\xi^l}{l!}b_{n+1,l},\ 0\leq k\leq n. \tag{12}$$

Moreover, for all $n\geq 0$

$$\begin{aligned}Q_{n+1}(x) &= (2\xi^{-1})^{-n-1}B_{n+1}^{(1/2)}(2\xi^{-1}x)\\ &\underset{by\ (5)}{=} \xi^{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{\Gamma(n+k+1)}{\Gamma(2n+2)}\,\xi^{-k}x^k\end{aligned}$$

since $v_0=h_{(2\xi^{-1})^{-1}}\mathcal{B}(1/2)$.
Consequently,

$$b_{n+1,k}=\binom{n+1}{k}\frac{\Gamma(n+k+1)}{\Gamma(2n+2)}\,\xi^{n+1-k}, \quad 0\leq k\leq n+1, \quad n\geq 0. \tag{13}$$

Due to (13), (12) becomes

$$a_{n,k}=(-1)^k k!\,\xi^{n-k}\frac{(n+1)!}{(2n+1)!}\sum_{\nu=0}^{k}(-1)^\nu\binom{n+1}{\nu}\binom{n+\nu}{\nu}, \quad 0\leq k\leq n.$$

By induction, it is easily seen that

$$\sum_{\nu=0}^{k}(-1)^{\nu}\binom{n+1}{\nu}\binom{n+\nu}{\nu}=(-1)^{k}\,\frac{(k+1)^2}{(n+1)^2}\binom{n+1}{k+1}\binom{n+k+1}{k+1}.$$

Therefore,

$$a_{n,k}=\binom{n}{k}\xi^{n-k}\,\frac{\Gamma(n+k+1)}{\Gamma(2n+2)},\qquad 0\le k\le n. \tag{14}$$

On the other hand,

$$\begin{aligned}(2\xi^{-1})^{-n}B_n^{(3/2)}(2\xi^{-1}x) &\underset{by\ (5)}{=} 2^{-n}\xi^{n}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\,\frac{\Gamma(n+k+2)}{\Gamma(2n+2)}\,2^{k}\,\xi^{-k}\,x^{k}\\ &= \sum_{k=0}^{n}\binom{n}{k}\xi^{n-k}\,\frac{\Gamma(n+k+2)}{\Gamma(2n+2)}\,x^{k}\\ &\underset{by\ (14)}{=} \sum_{k=0}^{n}a_{n,k}x^{k}\\ &= P_n(x).\end{aligned}$$

Thus, P_n are the scaled polynomials orthogonal with respect to the form $\mathcal{B}(3/2)$ with a dilatation $a=2\xi^{-1}$.

(ii)$\Rightarrow$(i).

Let a in $\mathbb{C}$, with $a\neq 0$ and let $P_n(x)=a^{-n}B_n^{(3/2)}(ax),\ n\ge 0$. It is clear that $\{P_n\}_{n\ge 0}$ is a MOPS. By using (4) which is satisfied by $B_{n+1}^{(1/2)}$ ($\alpha=1/2$), we have

$$x^2B_{n+1}^{(1/2)''}(x)+(x+2)B_{n+1}^{(1/2)'}(x)=(n+1)^2B_{n+1}^{(1/2)}(x),\ n\ge 0,$$

and the relation (6), we have

$$x^2B_n^{(3/2)'}(x)+(x+2)B_n^{(3/2)}(x)=(n+1)B_{n+1}^{(1/2)}(x),\ n\ge 0. \tag{15}$$

Besides, from (15) where x is replaced by ax it comes that

$$x^2\big(B_n^{(3/2)}(ax)\big)'+(x+2a^{-1})B_n^{(3/2)}(ax)=(n+1)a^{-1}B_{n+1}^{(1/2)}(ax),\ n\ge 0,$$

or, equivalently,

$$\mathfrak{B}_{\xi}P_n(x)=(n+1)a^{-(n+1)}B_{n+1}^{(1/2)}(ax),\ n\ge 0,$$

where $\xi=2a^{-1}$. Hence, (i) holds since $\{a^{-n}B_n^{(1/2)}(ax)\}_{n\ge 0}$ is a MOPS. □

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