TJMM
11 (2019), No. 1-2, 01-08

CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS INVOLVING q-RUSCHEWEYH DIFFERENTIAL OPERATOR

ABDULLAH ALSOBOH AND MASLINA DARUS

Abstract. In this paper, we introduce a new q-analogue of differential operator involving q-Ruscheweyh operator with a new subclass of meromorphic functions. We obtain coefficient conditions and show some properties for function f belonging to this subclass such as convolution conditions, closure and convex combinations. Finally, the neighbourhoods problem is solved.

1. Introduction

Quantum calculus (or q-calculus) has attracted the interest of many researchers due to its several applications in different branches of mathematics and physics, especially the geometric function theory. The q-calculus was believed to be initiated by Euler and Jacobi in 18^{th} century, and the application was given and developed by Jackson [10, 11]. Other applications of q-operator are studied by many other authors recently, for example see ([4, 5]) and Elhaddad et.al ([7, 8]). Many different problems related to q-calculus can also be seen in [2, 3, 14].

Throughout this article, we will assume $0 < q < 1$. We begin with some concepts of q-calculus, for any non-negative integer i, the q-factorial $[i]_q!$ is defined by:

$$[i]_q! = \begin{cases} [i]_q[i-1]_q...[2]_q[1]_q & , i = 2, 3, ... \\ \\ 1 & , i = 1 \end{cases} \tag{1}$$

where $[i]_q = \frac{1-q^n}{1-q}$, when $q \to 1^-$ then $[i]_q$ tends to i.

The q-derivative of a function f is defined by:

$$\partial_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (z \neq 0),$$

and $\partial_q f(0) = f^{'}(0)$, when $q \to 1^-$ then $\partial_q f(z)$ tends to $f^{'}(z)$.

Let Σ denotes a family of all meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{i=1}^{\infty} a_i z^i, z \in \mathbb{U}^*, \tag{2}$$

which are analytic in punctured open unit disk $\mathbb{U}^* = \mathbb{U}\backslash\{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For a function f of the form (2), one can observe that

$$\partial_q f(z) = \frac{-1}{qz^2} + \sum_{i=1}^{\infty} [i]_q a_i z^{i-1}.$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic function, Meromorphic function, Starlike function, Ruscheweyh operator, q- calculus.

Definition 1. *Let f be as in (2) and $h(z) = \frac{1}{z} + \sum_{i=1}^{\infty} b_i z^i$, then the convolution or Hadamard product of f and h is given by*

$$(f * h)(z) = (h * f)(z) = \frac{1}{z} + \sum_{i=1}^{\infty} a_i b_i z^i.$$

Definition 2. *For any two analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f(z) \prec g(z)$, if there exist a Schwarz function $\omega(z)$ with $\omega(0) = 0$, $|\omega(z)| \leq 1$ such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$* [13].

For $n, \lambda \in \mathbb{N}_0$ and $0 < q < 1$, and using the idea of convolutions, we define new q-differential operator $D_q^{n,\lambda} f(z) : \Sigma \to \Sigma$ by

$$D_q^{n,\lambda} f(z) = \mathcal{R}_q^{\lambda} f(z) * \mathfrak{D}_q^n f(z) = \frac{(-1)^n}{z} + \sum_{i=1}^{\infty} \frac{q^n [i]_q^n [\lambda - 1 + i]_q!}{[\lambda]_q! [i-1]_q!} a_i z^i \tag{3}$$

where $\mathcal{R}_q^{\lambda}$ denotes the q-Ruscheweyh differential operator introduced in [1] as

$$\mathcal{R}_q^{\lambda} f(z) = \frac{1}{z} + \sum_{i=1}^{\infty} \frac{[\lambda - 1 + i]_q!}{[\lambda]_q! [i-1]_q!} a_i z^i, \quad (\lambda \in \mathbb{N}),$$

and we introduce the differential operator $\mathfrak{D}_q^n f(z)$ by

$$\mathfrak{D}_q^n f(z) = qz\partial_q (D_q^{n-1} f(z)) = \frac{(-1)^n}{z} + \sum_{j=1}^{\infty} a_j q^n ([j]_q)^n z^j, \quad (n \in \mathbb{N}).$$

It is clear that

$$\partial_q D_q^{n,\lambda} f(z) = \frac{(-1)^{n+1}}{qz^2} + \sum_{i=1}^{\infty} \frac{q^n [i]_q^{n+1} [\lambda - 1 + i]_q!}{[\lambda]_q! [i-1]_q!} a_i z^{i-1}, z \in \mathbb{U}^*. \tag{4}$$

Note that:

i) When $n = 0$ and $q \to 1^-$ then $D_q^{n,\lambda}$ is reduced to $D^n f(z)$ [17].
ii) When $n = 0$, then $D_q^{n,\lambda}$ is reduced to $\mathcal{L}_q^{\mu-1}$ [1].
iii) When $\lambda = 0$, and $q \to 1^-$ we get *Sãlãgean* differential operator introduced in [16].

Finally, by using the introduced differential operator $D_q^{n,\lambda}$, we define a subfamily $\Sigma_q^*(\alpha, n, \lambda)$ of analytic meromorphic functions in $\mathbb{U}^*$, as follows:

Definition 3. *For $0 \leq \alpha < 1$ and $n, \lambda \in \mathbb{N}_0$. A function f of the form (2) is said to be in the class ${\Sigma_q}^*(\alpha, n, \lambda)$ of meromorphic starlike function of order α, if it satisfies the condition*

$$Re\left\{ \frac{-qz\partial_q (D_q^{n,\lambda} f(z))}{D_q^{n,\lambda} f(z)} \right\} > \alpha, z \in \mathbb{U}^*. \tag{5}$$

In this paper, we obtain coefficient conditions and show some properties for function f belonging to this subclass. These include the convolution conditions, closure theorem, convex combinations and the neighbourhoods problem.

2. Some Properties of the Class ${\Sigma_q}^*(\alpha, n, \lambda)$

In the following theorem, we get the sufficient condition for functions belonging to the class ${\Sigma_q}^*(\alpha, n, \lambda)$.

Theorem 1. *For $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$. Let $f \in \Sigma$ be given by (2). Then*

$$\sum_{i=1}^{\infty}(q[i]_q + \alpha)\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}|a_i| \leq 1 - \alpha, \tag{6}$$

if and only if $f \in {\Sigma_q}^*(\alpha, n, \lambda)$.

Proof. 'If' part: Let $f \in \Sigma$ as in (2) and the condition (6) holds $\forall z \in \mathbb{U}^*$, we will show that $f \in {\Sigma_q}^*(\alpha, n, \lambda)$, according to definition of the class ${\Sigma_q}^*(\alpha, n, \lambda)$, and for $0 \leq \alpha < 1$, we want to show $Re\Big\{\frac{-qz\partial_q(D_q^{n,\lambda}f(z))}{D_q^{n,\lambda}f(z)}\Big\} \geq \alpha$. By using the fact that $Re(\Delta) \geq \alpha$ if $|1 - \alpha + \Delta| \geq |1 + \alpha - \Delta|$, it suffices to show $\big|\Phi(z)\big| - \big|\Omega(z)\big| \geq 0$, where $\Phi(z) = -qz\partial_q(D_q^{n,\lambda}f(z)) + (1-\alpha)D_q^{n,\lambda}f(z)$ and $\Omega(z) = qz\partial_q(D_q^{n,\lambda}f(z)) + (1+\alpha)D_q^{n,\lambda}f(z)$.

Now

$$\begin{aligned}
\Big|\Phi(z)\Big| - \Big|\Omega(z)\Big| &= \Big|\frac{2-\alpha}{z} - \sum_{i=1}^{\infty}(q[i]_q - 1 + \alpha)\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}a_i z^i\Big| \\
&-\Big|\frac{\alpha}{z} - \sum_{i=1}^{\infty}(q[i]_q + \alpha + 1)\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}a_i z^{i-1}\Big|, \\
&\geq \Big|\frac{2-2\alpha}{z} - \sum_{i=1}^{\infty}2(q[i]_q + \alpha)\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}a_i z^i\Big|, \\
&\geq \frac{2(1-\alpha)}{|z|}\Big(1 - \sum_{i=1}^{\infty}\frac{(q[i]_q + \alpha)}{1-\alpha}\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}|a_i||z^{i+1}|\Big), \\
&\geq \frac{2(1-\alpha)}{|z|}\Big(1 - \sum_{i=1}^{\infty}\frac{(q[i]_q + \alpha)}{1-\alpha}\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}|a_i|\Big).
\end{aligned}$$

By condition (6), then the last inequality is non-negative.

Conversely, suppose that $f \in {\Sigma_q}^*(\alpha, n, \lambda)$, then the inequality (6) is true for all $z \in \mathbb{U}^*$, and we have $Re\left\{\frac{-qz\partial_q(D_q^{n,\lambda}f(z))}{D_q^{n,\lambda}f(z)}\right\} \geq 0$, equivalently to,

$$Re\left\{\frac{1 - \sum_{i=1}^{\infty}\frac{q^{n+1}[i]_q^{n+1}[\lambda-1+i]_q!}{[\lambda]_q![i-1]_q!}a_i z^{i+1}}{1 + \sum_{i=1}^{\infty}\frac{q^n[i]_q^n[\lambda-1+i]_q!}{[\lambda]_q![i-1]_q!}a_i z^{i+1}}\right\} \geq \alpha,$$

choosing a value of $z = r \in (0,1)$ on the real axis such that $r \to 1^-$, we have

$$1 - \sum_{i=1}^{\infty}\frac{q^{n+1}[i]_q^{n+1}[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}a_i \geq \alpha + \alpha\sum_{i=1}^{\infty}\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}a_i$$

we obtain inequality (6), Thus by the maximum modulus theorem, we have $f \in {\Sigma_q}^*(\alpha, n, \lambda)$.
□

Corollary 1. *If $f \in {\Sigma_q}^*(\alpha, n, \lambda)$, then*

$$\sum_{i=1}^{\infty}a_i \leq \frac{[2]_q(1-\alpha)}{(q+\alpha)q^n[\lambda+1]_q[\lambda+2]_q}, \qquad 0 \leq \alpha < 1.$$

In the next theorem, we show the main result of closure of such functions belonging to the class ${\Sigma_q}^*(\alpha, n, \lambda)$.

Theorem 2. *Let $f_k(z)$ be in the class ${\Sigma_q}^*(\alpha, n, \lambda)$ for every $k = 1, 2, ..., m$, where*

$$f_k(z) = \frac{1}{z} + \sum_{i=1}^{\infty} a_{i,k} z^i, \quad (a_{i,k} \geq 0).$$

Then the function

$$\beta_k(z) = \frac{1}{z} + \sum_{i=1}^{\infty} b_i z^i, \quad (b_i \geq 0),$$

is also in the class ${\Sigma_q}^*(\alpha, n, \lambda)$, where $b_i = \frac{1}{m}\sum_{k=1}^{m} a_{i,k}$.

Proof. To show that $\beta_k(z) \in {\Sigma_q}^*(\alpha, n, \lambda)$, it is enough to show that condition (2.1) holds, such that

$$\sum_{i=1}^{\infty}(q[i]_q + \alpha)\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}|b_i| = \sum_{i=1}^{\infty}(q[i]_q + \alpha)\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}\frac{1}{m}\sum_{k=1}^{m}|a_{i,k}|$$

$$= \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{\infty}(q[i]_q + \alpha)\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}|a_{i,k}|,$$

as $f_k(z) \in {\Sigma_q}^*(\alpha, n, \lambda)$ for all $k = 1, 2, ..., m$, then it satisfies condition (2.1), therefore

$$\sum_{i=1}^{\infty}(q[i]_q + \alpha)\frac{q^n[i]_q^n[\lambda - 1 + i]_q!}{[\lambda]_q![i-1]_q!}|b_i| \leq \frac{1}{m}\sum_{k=1}^{m}(1 - \alpha) \leq 1 - \alpha,$$

therefore, we have $\beta_k(z) \in {\Sigma_q}^*(\alpha, n, \lambda)$ and this completes the proof. □

In next theorem, we proved the class ${\Sigma_q}^*(\alpha, n, \lambda)$ is closed under convex combination.

Theorem 3. *Let f as in (2). Then $f \in {\Sigma_q}^*(\alpha, n, \lambda)$ if and only if*

$$f(z) = \sum_{\iota=0}^{\infty} \upsilon_\iota f_\iota, \tag{7}$$

where

$$f_0(z) = \frac{1}{z}, \quad f_\iota(z) = \frac{1}{z} + \left(\frac{[\lambda]_q![\iota - 1]_q!(1-\alpha)}{q^n[\iota]_q^n[\lambda - 1 + \iota]_q!(q[\iota]_q + \alpha)}\right) z^\iota, \quad \iota = 1, 2, ... \tag{8}$$

where $0 \leq \upsilon_\iota \leq 1$ and $\sum_{\iota=0}^{\infty} \upsilon_\iota = 1$.

Proof. Let

$$f(z) = \sum_{\iota=0}^{\infty} \upsilon_\iota f_\iota = \upsilon_0 f_0 + \sum_{\iota=1}^{\infty} \upsilon_\iota f_\iota = \frac{\upsilon_0}{z} + \sum_{\iota=1}^{\infty} \upsilon_\iota \left\{\frac{1}{z} + \left(\frac{[\lambda]_q![\iota - 1]_q!(1-\alpha)}{q^n[\iota]_q^n[\lambda - 1 + \iota]_q!(q[\iota]_q + \alpha)}\right) z^\iota\right\},$$

by applying condition (6), we get

$$\sum_{\iota=1}^{\infty}(q[\iota]_q + \alpha)\frac{q^n[\iota]_q^n[\lambda - 1 + \iota]_q!}{[\lambda]_q![\iota - 1]_q!}\left(\frac{[\lambda]_q![\iota - 1]_q!(1-\alpha)}{q^n[\iota]_q^n[\lambda - 1 + \iota]_q!(q[i]_q + \alpha)}\upsilon_\iota\right)$$

$$= (1 - \alpha)\sum_{\iota=1}^{\infty}\upsilon_\iota = (1 - \alpha)(1 - \upsilon_1) \leq 1 - \alpha,$$

so $f \in {\Sigma_q}^*(\alpha, n, \lambda)$.

Conversely, suppose that $f \in {\Sigma_q}^*(\alpha, n, \lambda)$. Set

$$\upsilon_\iota = \frac{(q[\iota]_q + \alpha)q^n[\iota]_q^n[\lambda - 1 + \iota]_q!}{[\lambda]_q![\iota - 1]_q!(1-\alpha)} a_\iota, \quad 0 \leq \upsilon_\iota \leq 1,$$

$$\upsilon_0 = 1 - \sum_{\iota=1}^{\infty} \upsilon_\iota.$$

Therefore, f can be expressed as

$$f(z) = \frac{1}{z} + \sum_{\iota=1}^{\infty} a_\iota z^\iota = \frac{1}{z} + \sum_{\iota=1}^{\infty} \frac{[\lambda]_q![\iota-1]_q!(1-\alpha)}{(q[\iota]_q+\alpha)q^n[\iota]_q^n[\lambda-1+\iota]_q!}\upsilon_\iota z^\iota$$

$$= \frac{\upsilon_0}{z} + \sum_{\iota=1}^{\infty}\left(\frac{1}{z} + \frac{[\lambda]_q![\iota-1]_q!(1-\alpha)}{(q[\iota]_q+\alpha)q^n[\iota]_q^n[\lambda-1+\iota]_q!}z^\iota\right)\upsilon_\iota = \sum_{\iota=0}^{\infty}\upsilon_\iota f_\iota.$$

And this completes the proof. □

The convolution conditions is obtained in the next theorem.

Theorem 4. *Let f of the form (1.1) and $g(z) = \frac{1}{z} + \sum_{i=1}^{\infty} b_i z^i$ be in class ${\Sigma_q}^*(\alpha, n, \lambda)$. Then $(f * g) \in {\Sigma_q}^*(\delta, n, \lambda)$, where*

$$\delta = \frac{(q[i]_q+\alpha)^2 q^n[i]_q^n[\lambda-1+i]_q! - q[i]_q(1-\alpha)^2[\lambda]_q![i-1]_q!}{(q[i]_q+\alpha)^2 q^n[i]_q^n[\lambda-1+i]_q! + (1-\alpha)^2[\lambda]_q![i-1]_q!)}.$$

Proof. Since $f, g \in {\Sigma_q}^*(\alpha, n, \lambda)$, then

$$\sum_{i=1}^{\infty} \frac{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\alpha)[\lambda]_q![i-1]_q!}a_i \le 1,$$

and

$$\sum_{i=1}^{\infty} \frac{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\alpha)[\lambda]_q![i-1]_q!}b_i \le 1.$$

We have to find the largest δ such that

$$\sum_{i=1}^{\infty} \frac{(q[i]_q+\delta)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\delta)[\lambda]_q![i-1]_q!}a_i b_i \le 1, \tag{9}$$

by using Cauchy-Schwartz inequality, we have

$$\sum_{i=1}^{\infty} \frac{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\alpha)[\lambda]_q![i-1]_q!}\sqrt{a_i b_i} \le 1, \tag{10}$$

it is enough to show that

$$\sum_{i=1}^{\infty} \frac{(q[i]_q+\delta)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\delta)[\lambda]_q![i-1]_q!}a_i b_i \le \sum_{i=1}^{\infty} \frac{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\alpha)[\lambda]_q![i-1]_q!}\sqrt{a_i b_i}. \tag{11}$$

This is equivalent to

$$\sqrt{a_i b_i} \le \frac{(1-\delta)(q[i]_q+\alpha)}{(q[i]_q+\delta)(1-\alpha)}. \tag{12}$$

From (2.9) we have

$$\sqrt{a_i b_i} \le \frac{(1-\alpha)[\lambda]_q![i-1]_q!}{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}. \tag{13}$$

Thus it is enough to show that

$$\frac{(1-\alpha)[\lambda]_q![i-1]_q!}{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!} \le \frac{(1-\delta)(q[i]_q+\alpha)}{(q[i]_q+\delta)(1-\alpha)}, \tag{14}$$

then

$$\delta \leq \frac{(q[i]_q+\alpha)^2 q^n [i]_q^n [\lambda-1+i]_q! - q[i]_q(1-\alpha)^2[\lambda]_q![i-1]_q!}{(q[i]_q+\alpha)^2 q^n [i]_q^n [\lambda-1+i]_q! + (1-\alpha)^2[\lambda]_q![i-1]_q!)}, \tag{15}$$

□

Theorem 5. *Let f of the form (2) and $g(z) = \frac{1}{z} + \sum_{i=1}^{\infty} b_i z^i$ be in class ${\Sigma_q}^*(\alpha, n, \lambda)$. Then the function $\beta(z) = \frac{1}{z} + \sum_{i=1}^{\infty}(a_i^2 + b_i^2)z^i$ is also in class ${\Sigma_q}^*(\sigma, n, \lambda)$. where*

$$\sigma = 1 - \frac{2(1-\alpha)^2(1+q)}{(q+\alpha)^2 q^n + 2(1-\alpha)^2}.$$

Proof. We want to find the largest σ such that

$$\sum_{i=1}^{\infty} \frac{(q[i]_q+\sigma)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\sigma)[\lambda]_q![i-1]_q!}(a_i^2+b_i^2) \leq 1, \tag{16}$$

Since $f, g \in {\Sigma_q}^*(\alpha, n, \lambda)$, then

$$\sum_{i=1}^{\infty} \left\{ \frac{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\alpha)[\lambda]_q![i-1]_q!} \right\}^2 a_i^2 \leq 1, \tag{17}$$

and

$$\sum_{i=1}^{\infty} \left\{ \frac{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\alpha)[\lambda]_q![i-1]_q!} \right\}^2 b_i^2 \leq 1. \tag{18}$$

Combining the inequalities (17) and (18), we get

$$\sum_{i=1}^{\infty} \frac{1}{2} \left\{ \frac{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\alpha)[\lambda]_q![i-1]_q!} \right\}^2 (a_i^2+b_i^2) \leq 1. \tag{19}$$

But, $\beta(z) \in {\Sigma_q}^*(\alpha, n, \lambda)$ if and only if

$$\sum_{i=1}^{\infty} \left\{ \frac{(q[i]_q+\sigma)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\sigma)[\lambda]_q![i-1]_q!} \right\} (a_i^2+b_i^2) \leq 1. \tag{20}$$

This is true if

$$\left\{ \frac{(q[i]_q+\sigma)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\sigma)[\lambda]_q![i-1]_q!} \right\} \leq \frac{1}{2} \left\{ \frac{(q[i]_q+\alpha)q^n[i]_q^n[\lambda-1+i]_q!}{(1-\alpha)[\lambda]_q![i-1]_q!} \right\}^2.$$

$$\frac{(1-\sigma)}{(q[i]_q+\sigma)} \geq \frac{2(1-\alpha)^2[\lambda]_q![i-1]_q!}{(q[i]_q+\alpha)^2 q^n[i]_q^n[\lambda-1+i]_q!} = \Phi(i).$$

Where $\Phi(i)$ is decreasing of i and has a maximum value $\Phi(1) = \frac{2(1-\alpha)^2}{(q+\alpha)^2 q^n}$ attains at $i = 1$.

$$\frac{(1-\sigma)}{(q+\sigma)} \geq \frac{2(1-\alpha)^2}{(q+\alpha)^2 q^n}.$$

simplify the last inequality we obtain

$$\sigma \leq 1 - \frac{2(1-\alpha)^2(1+q)}{(q+\alpha)^2 q^n + 2(1-\alpha)^2}.$$

This completes the proof. □

3. Neighborhoods on $\Sigma_q^{\sigma*}(\alpha, n, \lambda)$

Following the earlier works by Goodman [9], Ruscheweyh [15] and Liu and Srivastava [12], we define the δ-neighborhood of a function $f \in \Sigma$ by

$$N_\delta(f) = \left\{ h \in \Sigma : h(z) = \frac{1}{z} + \sum_{i=1}^{\infty} \gamma_i z^i \quad and \quad \sum_{i=1}^{\infty} i|a_i - \gamma_i| \le \delta, \quad 0 \le \delta < 1 \right\}. \tag{21}$$

For the identity function $I(z) = z$, we have

$$N_\delta(I) = \left\{ h \in \Sigma : h(z) = \frac{1}{z} + \sum_{i=1}^{\infty} \gamma_i z^i \quad and \quad \sum_{i=1}^{\infty} i|\gamma_i| \le \delta \right\}. \tag{22}$$

Definition 4. *A function $f \in \Sigma$ is said to be in the class $\Sigma_q^{\sigma*}(\alpha, n, \lambda)$ if there exist $h \in \Sigma_q^*(\alpha, n, \lambda)$ such that*

$$\left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \sigma, \quad (z \in \mathbb{U}^*, \ 0 \le \sigma < 1).$$

Theorem 6. *If $h \in {\Sigma_q}^*(\alpha, n, \lambda)$ and*

$$\sigma = 1 - \frac{\delta(q+\alpha)q^n[\lambda+1]_q[\lambda+2]_q}{(q+\alpha)q^n[\lambda+1]_q[\lambda+2]_q - [2]_q(1-\alpha)} \tag{23}$$

then $N_\delta(h) \subset \Sigma_q^{\sigma*}(\alpha, n, \lambda)$.

Proof. Consider $f \in N_\delta(h)$, then from equality (21), we have

$$\sum_{i=1}^{\infty} i|a_i - \gamma_i| \le \delta \Rightarrow \sum_{i=1}^{\infty} |a_i - \gamma_i| \le \delta, \ (i \in \mathbb{N}).$$

Since $h \in \Sigma_q^*(\alpha, n, \lambda)$, then from Corollary 1 we have

$$\sum_{i=1}^{\infty} \gamma_i \le \frac{[2]_q(1-\alpha)}{(q+\alpha)q^n[\lambda+1]_q[\lambda+2]_q}.$$

And hence

$$\left| \frac{f(z)}{h(z)} - 1 \right| < \frac{\sum_{i=1}^{\infty} |a_i - \gamma_i|}{1 - \sum_{i=1}^{\infty} \gamma_i} \le \frac{\delta(q+\alpha)q^n[\lambda+1]_q[\lambda+2]_q}{(q+\alpha)q^n[\lambda+1]_q[\lambda+2]_q - [2]_q(1-\alpha)} = 1 - \sigma.$$

Thus, for given σ in (22) and by definition (21) we have $f \in \Sigma_q^*(\alpha, n, \lambda)$, and the proof is complete.

□

Acknowledgement: The work here is partially supported by UKM grant: GUP-2019-032.

References

[1] B. Ahmad, Arif, M., *New Subfamily of Meromorphic Convex Functions in Circular Domain Involving q-Operator*, International Journal of Analysis and Applications, **16**(1), 75-82, 2018.

[2] Aldweby, H., Darus, M., *A Note On q-Integral Operators*, Electronic Notes in Discrete Mathematics, **67**, 25-30, 2018.

[3] Aldweby, H., Darus, M., *Some subordination results on-analogue of Ruscheweyh differential operator*, Abstract and Applied Analysis, **2014**, 6 pages, 2014.

[4] Alsoboh, A., Darus, M., *New subclass of analytic functions defined by q-differential operator with respect to k-symmetric points*, International Journal of Mathematics and Computer Science, **14** (4), 761-773, 2019.

[5] Alsoboh, A., Darus, M., *On Fekete-Szego problem associated with q-derivative operator*, J. Phys.: Conf. Ser., **1212** (2019), 012003, 2019.

[6] Atshan, W., Khalaf, A., Mahdi, M., *On a new class of meromorphic univalent function associated with Dziok-Srivastava operator*, Journal of Kufa for Mathematics and Computer, **2** (2), 56-63, 2014.

[7] Elhaddad, S., Aldweby, H., Darus, M., *Some properties on a class of harmonic univalent functions defined by q-analogue of Ruscheweyh operator*, Journal of Mathematical Analysis, **9** (4), 28-35, 2018.

[8] Elhaddad, S., Darus, M., *On meromorphic functions defined by a new operator containing the Mittag-Leffler function*, Symmetry, **11** (2), 210, 2019.

[9] Goodman, A.,W., *Univalent functions and nonanalytic curves*, Proceedings of the American Mathematical Society, **8** (3),598-601, 1957.

[10] Jackson, F.,H., *On q-functions and a certain difference operator*, Transactions of the Royal Society of Edinburgh, **46** (2), 253-281, 1909.

[11] Jackson, F.,H., *On q-definite integrals*, The Quarterly Journal of Pure and Applied Mathematics, **41**, 193-203, 1910.

[12] Liu, J., L., Srivastava, H., M., *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl., 259, 566- 581, 2001.

[13] Miller, S., S., Mocanu, P., T., *Differential subordinations: theory and applications*, CRC Press. 2000.

[14] Mohammed, A., Darus, M., *A generalized operator involving the q-hypergeometric function*, Matematicki Vesnik, **65**(4) , 454-465, 2014.

[15] Ruscheweyh, S., *Neighborhoods of univalent functions*, Proceedings of the American Mathematical Society, **81** (4), 521-527, 1981.

[16] *Sãlãgean*, G., S., *Subclasses of univalent functions*, Lecture Notes in Math, Springer-Verlag, Heidelberg, 362-372, 1983.

[17] Yuan, S., M., Liu, Z., M., Srivastava, H., M., *Some inclusion relationships and integral-preserving properties of certain subclasses of meromorphic functions associated with a family of integral operators*, Journal of Mathematical Analysis and Applications, **337** (1), 505-515, 2008.

Universiti Kebangsaan Malaysia
Centre of Modelling and Data Science
43600, UKM Bangi, Selangor, Malaysia
E-mail address: p92712@siswa.ukm.edu.my

Universiti Kebangsaan Malaysia
Centre of Modelling and Data Science
43600, UKM Bangi, Selangor, Malaysia
E-mail address: maslina@ukm.edu.my(Corresponding Author)