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A NUMERICAL ALGORITHM FOR GENERATING THE SET OF RIGHT TRIANGLES THAT SIMULTANEOUSLY HAVE THEIR SIDES, AREA AND HEIGHTS EXPRESSED BY NATURAL NUMBERS

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Abstract. The article presents the mathematical model based on which we designed the numerical algorithm that generates the set of all right triangles that simultaneously satisfy the following requirements: the sides, the area and the heights are expressed by natural numbers (i.e. integers).

1. Introduction

Gladly browsing the book of the mathematician Viorel Gh.Vodă, titled **Surprize în matematica elementară**, published by Editura Albatros, Bucureşti, 1981, and refereed by prof.dr.doc Solomon Marcus and dr. Şerban Grigorescu, in Chapter 2, titled **Mic tratat de geometrie demodată**, I have arrived at page 80, paragraph 2.2, titled **Magia numerelor naturale (triunghiurile VUX şi altele)**, and in Chapter 4 , page 124 , titled **Surprize şi mai mari**, I have found the following problems:

(1) Find right triangles with sides expressed by natural numbers that also have an equal area. This problem was formulated in 1898, by reverend Charles Lutwidge Dodgson, professor of mathematics at Oxford College, and he found the triangle in which: $a = 20$, $b = 21$, and $c = 29$ (hypotenuse), as well as the triangle in which $a = 12$, $b = 35$, $c = 37$. Both triangles have area equal to 210. Moreover, he also discovered that by doubling the found sides, one also obtains two rectangular triangles with the same area, namely 840. The author of the problem also raised another question: are there three rectangular triangles such that the above properties hold? While he never managed to find a third solution, it nevertheless exists: $a = 15$, $b = 112$, $c = 113$.
(2) Find the triangles that have their sides, height and area expressed by natural numbers. This problem was formulated by T.R. Running and many mathematicians might say it is too pretentious. In the book I have read, the most general case is not analyzed, but the solutions to the problem are given only if the sides of the triangles are integers in arithmetic progression with the ratio 1.

Reflecting on these surprises, which delighted many mathematicians over the years, we have asked ourselves the question: ***are there triangles that have all the sides, the area and all the heights expressed by natural numbers?***

It must be recognized that there are too many requirements, which is why, in this article, we have focused our attention on the class of right triangles. We mention that solving this particular problem is a **pleasant surprise** because we have succeeded in demonstrating that starting from two natural numbers n and m, where $n < m$ one can

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generate an infinity of rectangular triangles that have their sides, their area and their heights expressed by natural numbers.

We have to say that the provocative problems formulated by many mathematicians passionate about this discipline, considered the **queen of science**, but solved by another army of mathematicians, inevitably lead to the discovery of new scientific truths of unequivocal utility.

2. The theoretical foundations underlying the algorithm

Theorem 1. *If x, y and z are the solutions of the Diophantine equation* [1] $x^2 + y^2 = z^2$ *then area of the right triangle ABC (fig. 1) in which $AC = x$, $AB = y$ is a natural number.*

Proof. The natural numbers x, y and z, that satisfy the Diophantine equation $x^2+y^2 = z^2$ are called a Pythagorean triple, and the set of solutions to this equations is:

$$\begin{cases} x = m^2 - n^2 \\ y = 2mn \\ z = m^2 + n^2 \end{cases}, \text{ where } m, n \ (n < m) \text{ are natural numbers.}$$

The are of triangle ABC is

$$S = \frac{AB \cdot AC}{2} = \frac{x \cdot y}{2} = \frac{(m^2 - n^2) \cdot 2mn}{2} = mn(m^2 - n^2),$$

thus $S \in \mathbb{N}$. □

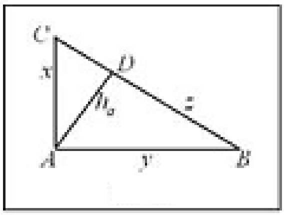

Figure 1.

Consequence 1. ***All rectangular triangles whose sides form a Pythagorean triple have the area and two heights (catheti) expressed by natural numbers.***

Problem: ***Are there any rectangular triangles that have all the sides, the area and all the heights expressed by natural numbers?***
The answer to this question is deduced from the consequences of the following theorems.

[1] Diophantine equations are algebraic equations with integer coefficients that have only natural roots. Among the Diophantine equations, over time, $x^n + y^n = z^n$ has played the most important role as the mathematician Pierre de Fermat postulated that this equation has no roots for $n \geq 3$. The problem is known in literature as **Fermat's Last Theorem**. Regarding this problem, in 1637, Fermat wrote on the edge of a page from an edition of Diophantus' "Arithmetica": ***I have discovered a truly wonderful proof, but I do not have enough space here to write it***. For 358 years, the greatest mathematicians of the world have tried in vain to find the proof to Fermat's last theorem, which in time became a symbol of mathematical mystery. Their efforts led to the opening of new fields of research and in 1995 the British mathematician Andrew Wiles succeeded in proving **Fermat's last theorem**.

Theorem 2. *If x, y, z form a Pythagorean triple and $h_a = \frac{x \cdot y}{z} = \frac{p}{q}$, where p and q are coprime numbers (i.e., relatively prime), then the sides, area and heights of the right triangle which have the catheti $x_q = x \cdot q$, $y_q = y \cdot q$ are natural numbers.*

Proof. Since x, y, $q \in \mathbb{N}$, we have that: $x_q = x \cdot q \in \mathbb{N}$, $y_q = y \cdot q \in \mathbb{N}$, and from the right triangle that has as catheti $x_q = x \cdot q$, $y_q = y \cdot q$ we deduce that:

$$z_q = \sqrt{x^2 \cdot q^2 + y^2 \cdot q^2} = q \cdot \sqrt{x^2 + y^2} = q \cdot \sqrt{(m^2 - n^2)^2 + 4m^2n^2} = q \cdot (m^2 + n^2) = z \cdot q \in \mathbb{N};$$

$$S_q = \frac{x_q \cdot y_q}{2} = \frac{x \cdot y \cdot q^2}{2} = q^2 \cdot \frac{x \cdot y}{2} = q^2 \cdot mn(m^2 - n^2) = q^2 \cdot S \in \mathbb{N};$$

$$h_a^{(q)} = \frac{x_q \cdot y_q}{z_q} = \frac{x \cdot q \cdot y \cdot q}{z \cdot q} = \frac{x \cdot y}{z} \cdot q = h_a \cdot q = \frac{p}{q} \cdot q = p \in \mathbb{N}.$$

□

Example 1. *For $n = 1$ and $m = 2$, one obtains the Pythagorean triple$x = 3$, $y = 4$, $z = 5$. The area of this right triangle which has these side lengths is $S = \frac{x \cdot y}{2} = \frac{3 \cdot 4}{2} = 6$ and the third height is $h_a = \frac{x \cdot y}{z} = \frac{3 \cdot 4}{5} = \frac{12}{5} \in \mathbb{Q}$. Because $p = 12$ and $q = 5$ are coprime numbers (greatest common divisor of these numbers is 1), the triangle with catheti $x_q = x \cdot q = 3 \cdot 5 = 15$ and $y_q = y \cdot q = 4 \cdot 5 = 20$, has the hypotenuse $z_q = z \cdot q = 5 \cdot 5 = 25$, the area $S_q = q^2 \cdot S = 25 \cdot 5 = 125$ and the height $h_a^{(q)} = p = 12$. Thus, the sides, the area and the heights of this triangle are all natural numbers.*

Example 2. *For $n = 9$ and $m = 12$ one obtains the Pythagorean triple $x = 63$, $y = 216$, $z = 225$. The area of this right triangle which has these side lengths is $S = \frac{x \cdot y}{2} = \frac{63 \cdot 216}{2} = 6804$ and the third height is $h_a = \frac{x \cdot y}{z} = \frac{63 \cdot 216}{225} = \frac{13608}{225} = \frac{2^3 \cdot 3^5 \cdot 7}{3^2 \cdot 5^2} = \frac{2^3 \cdot 3^3 \cdot 7}{5^2} = \frac{1512}{25} \in \mathbb{Q}$, (greatest common divisor of $x \cdot y$ and z is 3^2). Thus, the numbers $p = 1512$ and $q = 25$ are coprime and the right triangle that has catheti of length $x_q = x \cdot q = 1575$, $y_q = y \cdot q = 5400$, will have the hypotenuse of length $z_q = z \cdot q = 5625$ and the area $S_q = q^2 \cdot S = 625 \cdot 6804 = 4252500$. The third height of this triangle is $h_a^{(q)} = p = 1512$. Thus, the sides, the area and the heights of this triangle are all natural numbers.*

Consequence 2. For any pair of natural numbers (n, m), where $n < m$, one can generate a right triangle that has its sides x_q, y_q, z_q, area S_q and height x_q, y_q, $h_a^{(q)}$ expressed by natural numbers. Thus, ***there is an infinite number of rectangular triangles that have their sides, their area and their heights expressed by natural numbers.***

Theorem 3. *If x_q, y_q, z_q form a Pythagorean triple and denote the sides lengths of a right triangle that has its sides, area and heights expressed by natural numbers, then for any $k \in \mathbb{N}$, $k \geq 1$, the sides, area and heights of the right triangle that has catheti $x_k = x_q \cdot k$, $y_k = y_q \cdot k$ are natural numbers.*

Proof. From the right triangle with catheti $x_k = x_q \cdot k$, $y_k = y_q \cdot k$, we deduce that:

(1) The hypotenuse is : $z_k = \sqrt{x_q^2 \cdot k^2 + y_q^2 \cdot k^2} = k \cdot \sqrt{x_q^2 + y_q^2} = k \cdot z_q \in \mathbb{N}$;

(2) The area of triangle is: $S_k = \frac{x_k \cdot y_k}{2} = \frac{x_q \cdot y_q \cdot k^2}{2} = k^2 \cdot \frac{x_q \cdot y_q}{2} = k^2 \cdot S_q \in \mathbb{N}$;

(3) The third height is: $h_a^{(k)} = \frac{x_k \cdot y_k}{z_k} = \frac{x_q \cdot y_q \cdot k^2}{z_q \cdot k} = k \cdot \frac{x_q \cdot y_q}{z_q} = k \cdot h_a^{(q)} \in \mathbb{N}$.

□

Consequence 3. For any pair of natural numbers (n, m), where $n < m$, one can generate an initial right triangle with sides x_q, y_q, z_q, area S_q and heights x_q, y_q, $h_a^{(q)}$ expressed by natural numbers, and starting from this initial triangle, it is possible to generate an infinite number of rectangular triangles having the sides, the area and the heights expressed by natural numbers. Therefore, we can conclude that there is a double infinity of rectangular triangles having the sides, the area and the heights expressed by natural numbers.

3. Narrative and pseudocode description of the algorithm

3.1. **Narrative description of the algorithm.** From the issues outlined in §2, we have come to the conclusion that the set of rectangular triangles having the sides, the area and the heights expressed by integers is infinite.

The algorithm we present in this article and the program that encodes it will only generate a subset of this set. This restriction is due to the fact that the current high-level programming languages have certain limits on memorizing and operating with very large numbers. Starting from these observation, we introduce the following notations:

- $nmax$ - the maximum value the natural number n can take;
- $mmax$ - the maximum value the natural number m can take;
- $kmax$ - the maximum value the natural number $k$can take;

Taking into account the above mentioned explanations, the algorithm contains the following steps:
S1. We read the values of the variables $nmax$, $mmax$, $kmax$.
S2. We initialize the variable nt with the value 0 (nt- number of generated triangles).
S3. For any $n = \overline{1, nmax}$ and any $m = \overline{n+1, mmax}$ we perform the following operations:

(1) we determine the values of the variables x, y, z (the Pythagorean triple the expresses the lengths of the catheti and of the hypotenuse of a right triangle).
(2) we store in u and in $u1$ the product of catheti and in v and $v1$ the value z of the hypotenuse.
(3) we determine the greatest common divisor of numbers $u$and v - i.e., the value $d = (u, v)$.
(4) we assign to variable p the value $[u/d]$ and to variable q we assign the value $[v/d]$;
(5) we increment the value of the variable nt with one unit;
(6) we write the values obtained for the variables nt, n, m, x, y, z, p, q;
(7) we determine and write values of variables xq, yq, zq, haq, Sq (the sides, the heights and the area of the right triangle with the order number nt);
(8) for any $k = \overline{1, kmax}$ we determine and write the values of the variables xk, yk, zk, hak, Sk (the sides, the heights and the area of the right triangle with the order number k generated by starting from the triangle with the order number nt).

3.2. **Pseudocode description of the algorithm fig. 2.**

4. Conclusions

(1) The mathematical model designed to solve the **surprise problem** formulated in this article is rigorously substantiated from a scientific point of view.
(2) We do not know the mathematical models by which Charles Lutwidge Dodgson and T.R. Running solved the two problems mentioned in §1, but we specify that their solutions were obtained "with relative ease" by a Borland Pascal program encoding a slightly modified version of the algorithm presented in §3.

```
/ Generate rectangular triangles/
integer n,m,nmax,mmax,kmax,nt
integer x,y,z,xq,yq,zq,Sq,haq,xk,yk,zk,hak,Sk
integer u,v,u1,v1,p,q,r,d
read  nmax,mmax,k
nt ← 0
  for n = 1,nmax
    for m = n+1,mmax
      x ← m*m − n*n ; y ← 2*m*n ; z ← m*m+n*n
      u ← x*y; u1 ← u ; v ← z ; v1 ← v
      do
        r ← u1 − [u1/v1]
        u1 ← v1; v1 ← r
      until r = 0
      d ← u1; p ← [u/d] ; q ← [v/d]
      nt ← nt +1
      write nt,n,m,x,y,z,p,q
      xq ← x*q ; yq ← y*q ; zq ← z*q ;haq ← p
      Sq ← m*n*x*q*q
      write xq,yq,zq,haq,Sq
      for k = 1,kmax
        xk ← xq*k ; yk ← yq*k ; zk ← yq*k ; hak ← haq*k ; Sk ← Sq*k*k
        write nt,k,xk,yk,zk,hak,sk
      repeat
    repeat
  repeat
stop
end
```

Figure 2.

(3) By slightly modifying the general algorithm in §3, we can determine the rectangular triangles with integer side lengths for which the difference of the catheti lengths equals 1 (problem formulated by Waclaw Sierpinscki), or we can solve Diophantus' problem: "Determine right triangles with integer side lengths so that the difference between the length of the hypotenuse and the length of any of the two catheti is the cube of a natural number".

(4) The presented algorithm can be easily modified so as to determine the set of ***fundamental triangles***, meaning those Pythagorean triangles in which the side lengths are coprime.

(5) We present some sequences provided by the Borland Pascal program encoding the algorithm in §3 for some concrete parameter values of n and m.

1.	n= 1	m= 2	x =	3	y =	4	z =	5	p =	12	q =	5
			xq=	15	yq=	20	zq=	25	haq=	12	sq=	150
1. 1.			x1=	15	y1=	20	z1=	25	ha1=	12	s1=	150
1. 2.			x2=	30	y2=	40	z2=	50	ha2=	24	s2=	600
1. 3.			x3=	45	y3=	60	z3=	75	ha3=	36	s3=	1350
1. 4.			x4=	60	y4=	80	z4=	100	ha4=	48	s4=	2400
1. 5.			x5=	75	y5=	100	z5=	125	ha5=	60	s5=	3750

87.	n= 9	m=12	x = 63	y = 216	z = 225	p = 1512	q = 25
			xq= 1575	yq= 5400	zq= 5625	haq= 1512	sq= 4252500
87. 1.			x1= 1575	y1= 5400	z1= 5625	ha1= 1512	s1= 4252500
87. 2.			x2= 3150	y2= 10800	z2= 11250	ha2= 3024	s2= 17010000
87. 3.			x3= 4725	y3= 16200	z3= 16875	ha3= 4536	s3= 38272500
87. 4.			x4= 6300	y4= 21600	z4= 22500	ha4= 6048	s4= 68040000
87. 5.			x5= 7875	y5= 27000	z5= 28125	ha5= 7560	s5= 106312500

105.	n=14	m=15	x = 29	y = 420	z = 421	p = 12180	q = 421
			xq= 12209	yq=176820	zq=177241	haq= 12180	sq=1079397690
105. 1.			x1= 12209	y1=176820	z1=177241	ha1= 12180	s1=1079397690
105. 2.			x2= 24418	y2=353640	z2=354482	ha2= 24360	s2= 22623464
105. 3.			x3= 36627	y3=530460	z3=531723	ha3= 36540	s3=1124644618
105. 4.			x4= 48836	y4=707280	z4=708964	ha4= 48720	s4= 90493856
105. 5.			x5= 61045	y5=884100	z5=886205	ha5= 60900	s5=1215138474

(6) From the examples presented above, it is easy to deduce that for any $n = \overline{1, 14}$ and any $m = \overline{n + 1, 15}$, **105 "initial right triangles"** can be generated. These initial right triangles have their sides, their area and their heights expressed by natural numbers, and starting from any such initial triangle, one can generate an infinity of secondary triangles with sides, area and heights expressed by natural numbers; In the presented examples, 5 secondary triangles are generated.

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