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# RELATIONS ON BI-PERIODIC JACOBSTHAL SEQUENCE

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Abstract. Following the new generalization of the Jacobsthal sequence defined by Uygun and Owusu [10] as $\hat{j}_n = a\hat{j}_{n-1}+2\hat{j}_{n-2}$, when $n$ is even and $\hat{j}_n = b\hat{j}_{n-1}+2\hat{j}_{n-2}$, when $n$ is odd, with initial conditions $\hat{j}_0 = 0, \hat{j}_1 = a$ for all values of $n \geq 2$. In this paper, through a much broader study on this new generalization, particularly considering its Binet formula, some numerous new identities and properties of this sequence are investigated.

## 1. Introduction

There are so many articles about the special integer sequences in the literaure specially about the Fibonacci sequences and generalizations of this sequence [1, 2, 3]. But in the recent years many studies have been seen about the other sequences such as Jacobsthal sequence. The classical Jacobsthal sequence is defined as $j_n = j_{n-1} + 2j_{n-2}$ with initial condition $j_0 = 0,\ j_1 = 1$. Many authors studied about generalization of Jacobsthal numbers in [4].

Bi-periodic Fibonacci sequence was first introduced in literature on 2009 by Edson and Yayenie in [5]. They gave the generating function, Cassini, Catalan and D'Ocagne properties for the bi periodic Fibonacci sequence etc. And then Yayenie found interesting properties of this sequence in [6]. And also Jun and Choi in [11] gave some properties of this sequence by defining a matrix related to bi periodic Fibonacci sequence. Just like the bi-periodic Fibonacci sequence, Bilgici [7] introduced into literature the bi-periodic Lucas sequence and gave some properties of bi-periodic Lucas sequence and relationship between bi-periodic Fibonacci sequence. In [8] the authors defined bi-periodic Fibonacci matrix sequence and found nth general term and Binet formula of this matrix sequence. Similarly in [9] the authors investigated properties of bi-periodic Lucas matrix sequence. In [10], the authors defined bi-periodic Jacobsthal sequences similar to the bi-periodic Fibonacci and Lucas sequences and then proceed to find the basic properties such as Binet formula, generating function, D' Ocagne. In [11], the authors defined bi-periodic Jacobsthal matrix sequences.

In this paper there will be a much broader study on the properties of bi-periodic Jacobsthal sequences. The bi-periodic Jacobsthal sequence $\{\hat{j}_n\}_{n=0}^{\infty}$ is defined as

$$\hat{j}_0 = 0,\ \hat{j}_1 = 1,\ \hat{j}_n = \begin{cases} a\hat{j}_{n-1} + 2\hat{j}_{n-2}, & \text{if } n \text{ is even} \\ b\hat{j}_{n-1} + 2\hat{j}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{1}$$

where $\lfloor a \rfloor$ is the floor function of $a$ and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function in [10, 11]. Similarly we can define the recurrence relation by

$$\hat{j}_{n+2} = a^{1-\xi(n)} b^{\xi(n)} \hat{j}_{n+1} + 2\hat{j}_n. \tag{2}$$

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From the above definition we have the nonlinear quadratic equation for the bi-periodic Jacobsthal sequence by

$$x^2 - abx - 2ab = 0.$$

with roots $\alpha$ and $\beta$ defined by

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 8ab}}{2}, \quad \beta = \frac{ab - \sqrt{a^2b^2 + 8ab}}{2}. \tag{3}$$

Some important properties of bi-periodic Jacobsthal sequence are given in the following proposition. By using the results we have a great chance to get different properties of bi-periodic Jacobsthal sequence.

**Proposition 1.** *The bi-periodic Jacobsthal sequence and the roots are satisfied the following relations*

- *The extended Binet formula*

$$\hat{j}_m = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\right) \frac{\alpha^m - \beta^m}{\alpha - \beta} \tag{4}$$

- *The generating function*

$$\hat{j}(x) = \frac{x(1 + ax - 2x^2)}{1 - (ab + 4)x^2 + 4x^4}.$$

- $$\hat{j}_{n+4} = (ab + 4)\,\hat{j}_{n+2} - 4\hat{j}_n \tag{5}$$

- $$(\alpha + 2)(\beta + 2) = 4 \tag{6}$$

$$\alpha + \beta = ab, \qquad \alpha\beta = -2ab$$

$$\beta + 2 = \frac{\beta^2}{ab}, \qquad \alpha + 2 = \frac{\alpha^2}{ab}$$

$$-(\alpha + 2)\beta = 2\alpha, \qquad -(\beta + 2)\alpha = 2\beta.$$

## 2. New Identities of Bi-periodic Jacobsthal Sequence

**Theorem 1.** *For any nonnegative integer $n$, it is obtained that*

$$\alpha^n = a^{\lfloor \frac{n}{2} \rfloor - \xi(n-1)} b^{\lfloor \frac{n}{2} \rfloor} \hat{j}_n \alpha + 2\,(a)^{\lfloor \frac{n}{2} \rfloor}\, b^{\lfloor \frac{n}{2} \rfloor + \xi(n)} \hat{j}_{n-1}$$

$$\beta^n = a^{\lfloor \frac{n}{2} \rfloor - \xi(n-1)} b^{\lfloor \frac{n}{2} \rfloor} \hat{j}_n \beta + 2\,(a)^{\lfloor \frac{n}{2} \rfloor}\, b^{\lfloor \frac{n}{2} \rfloor + \xi(n)} \hat{j}_{n-1}$$

*Proof.* The identity holds for $n = 1$. Let $n \geq 2$ any integer. Using the extended Binet's formula, we have

$$\begin{aligned}
\hat{j}_n - \frac{\beta^2}{ab}\hat{j}_{n-2} &= \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right) \frac{\alpha^n - \beta^n}{\alpha - \beta} - \frac{\beta^2}{ab}\left(\frac{a^{1-\xi(n-2)}}{(ab)^{\lfloor \frac{n-2}{2} \rfloor}}\right) \frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta} \\
&= \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} - \frac{\beta^2\left(\alpha^{n-2} - \beta^{n-2}\right)}{\alpha - \beta}\right) \\
&= \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right) \left(\frac{\alpha^{n-2}\left(\alpha^2 - \beta^2\right)}{\alpha - \beta}\right) \\
&= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor - 1}} \alpha^{n-2}.
\end{aligned}$$

Since $\alpha^2 - ab\alpha - 2ab = 0$, multiplying by $\frac{\alpha^2}{ab} = \alpha + 2$ and using $\alpha\beta = -2ab$, we have

$$\left(\hat{\jmath}_n - \frac{\beta^2}{ab}\hat{\jmath}_{n-2}\right)(\alpha+2) = \frac{\alpha^2}{ab}.\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor - 1}}\alpha^{n-2}$$

$$\frac{(ab)^{\lfloor \frac{n}{2} \rfloor}}{a^{1-\xi(n)}}\left(\hat{\jmath}_n\alpha + 2\hat{\jmath}_n - 4\hat{\jmath}_{n-2}\right) = \alpha^n$$

$$\alpha^n = a^{\lfloor \frac{n}{2} \rfloor - \xi(n-1)} b^{\lfloor \frac{n}{2} \rfloor}\hat{\jmath}_n\alpha + 2\,(a)^{\lfloor \frac{n}{2} \rfloor}\, b^{\lfloor \frac{n}{2} \rfloor + \xi(n)}\hat{\jmath}_{n-1}.$$

□

**Theorem 2.** *For any nonnegative integer $n$, we have*

$$\hat{\jmath}_{n+6} = (ab+6)\,a^{1-\xi(n)}b^{\xi(n)}\hat{\jmath}_{n+3} + 8\hat{\jmath}_n$$

*Proof.* The theorem can be proved by using of the Binet's formula or the recurrence relation as follows. For any positive integer $n \geq 2$, by (2) and (5) we have

$$\hat{\jmath}_{n+6} = (ab+4)\,\hat{\jmath}_{n+4} - 4\hat{\jmath}_{n+2}.$$

Hence

$$\begin{aligned}
\hat{\jmath}_{n+6} &= (ab+4)\,[a^{1-\xi(n)}b^{\xi(n)}\hat{\jmath}_{n+3} + 2\hat{\jmath}_{n+2}] - 4\hat{\jmath}_{n+2} \\
&= (ab+4)\,a^{1-\xi(n)}b^{\xi(n)}\hat{\jmath}_{n+3} + (2ab+4)\,\hat{\jmath}_{n+2} \\
&= (ab+6)\,a^{1-\xi(n)}b^{\xi(n)}\hat{\jmath}_{n+3} + (2ab+4)\,\hat{\jmath}_{n+2} - 2a^{1-\xi(n)}b^{\xi(n)}\hat{\jmath}_{n+3} \\
&= (ab+6)\,a^{1-\xi(n)}b^{\xi(n)}\hat{\jmath}_{n+3} + (2ab+4)\,\hat{\jmath}_{n+2} \\
&\quad - 2a^{1-\xi(n)}b^{\xi(n)}[a^{1-\xi(n)}b^{\xi(n)}\hat{\jmath}_{n+2} + 2\hat{\jmath}_{n+1}] \\
&= (ab+6)\,a^{1-\xi(n)}b^{\xi(n)}\hat{\jmath}_{n+3} + 8\hat{\jmath}_n
\end{aligned}$$

When $a = b = 1$ the above result reduces to a known identity of Jacobsthal numbers

$$\hat{\jmath}_{n+6} = 7\hat{\jmath}_{n+3} + 8\hat{\jmath}_n$$

□

**Theorem 3.** *For any positive integer $m > 1$, we have*

$$a\hat{\jmath}_{2m-1} = \hat{\jmath}_{m+1}\hat{\jmath}_m + \hat{\jmath}_{m-1}\hat{\jmath}_{m-2}$$

*Proof.* Using Theorem 3 we get

$$\hat{\jmath}_{m+n-1} = a^{\xi(mn+n-m)-1}b^{1-\xi(mn+n-m)}\hat{\jmath}_m\hat{\jmath}_n + 2a^{-\xi(mn)}b^{\xi(mn)}\hat{\jmath}_{m-1}\hat{\jmath}_{n-1}$$

$$\begin{aligned}
a\hat{\jmath}_{2m-1} &= a^{\xi(m+1)}b^{1-\xi(mn+n-m)}\hat{\jmath}_{m+1}\hat{\jmath}_{m-1} \\
&\quad + 2\alpha^{1-\xi((m+1)(m-1))}b^{\xi((m+1)(m-1))}\hat{\jmath}_m\hat{\jmath}_{m-2} \\
&= \alpha^{1-\xi(m)}b^{\xi(m)}\hat{\jmath}_{m+1}\hat{\jmath}_{m-1} + 2\alpha^{1-\xi(m+1)}b^{\xi(m+1)}\hat{\jmath}_m\hat{\jmath}_{m-2} \\
&= \hat{\jmath}_{m+1}\left(\alpha^{1-\xi(m)}b^{\xi(m)}\hat{\jmath}_{m-1}\right) + 2\alpha^{1-\xi(m+1)}b^{\xi(m+1)}\hat{\jmath}_m\hat{\jmath}_{m-2} \\
&= \hat{\jmath}_{m+1}\left(\hat{\jmath}_m - 2\hat{\jmath}_{m-2}\right) + 2\alpha^{1-\xi(m+1)}b^{\xi(m+1)}\hat{\jmath}_m\hat{\jmath}_{m-2} \\
&= \hat{\jmath}_{m+1}\hat{\jmath}_m - 2\hat{\jmath}_{m-2}\left(j_{m+1} - \alpha^{1-\xi(m+1)}b^{\xi(m+1)}\hat{\jmath}_m\right) \\
&= \hat{\jmath}_{m+1}\hat{\jmath}_m - 4\hat{\jmath}_{m-2}\hat{\jmath}_{m-1}
\end{aligned}$$

□

**Theorem 4.** *For any positive integer m, it is obtained that*

$$\hat{\jmath}_m = a^{\xi(m-1)} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k} 2^k$$

*Proof.* We will use the principle of induction to show the validity of the this formula satisfying for defining the bi-periodic Jacobsthal numbers by using sum formula. It is easily seen that the assertion is true for $m = 1$,

$$\hat{\jmath}_1 = 1 = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k} 2^k$$

Consider that it is true for any $n$ such that $1 \leq n \leq m$. Then by the assumption and the property of binomial coefficients, it is obtained that

$$\begin{aligned}
\hat{\jmath}_{m+1} &= a^{\xi(m)} b^{1-\xi(m)} \hat{\jmath}_m + 2\hat{\jmath}_{m-1} \\
&= a^{\xi(m)} b^{1-\xi(m)} a^{\xi(m-1)} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k} 2^k \\
&+ 2a^{\xi(m-2)} \sum_{k=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-k-2}{k} (ab)^{\lfloor \frac{m-2}{2} \rfloor - k} 2^k \\
&= ab^{\xi(m+1)} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} ab^{\lfloor \frac{m-2}{2} \rfloor - k} 2 \\
&+ a^{\xi(m)} \sum_{k=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-k-2}{k} (ab)^{\lfloor \frac{m-2}{2} \rfloor - k} 2^{k+1} \\
&= a^{\xi(m)} (ab)^{\xi(m+1)} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k} 2^k \\
&+ a^{\xi(m)} \sum_{k=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-k-2}{k} (ab)^{\lfloor \frac{m-2}{2} \rfloor - k} 2^{k+1} \\
&= a^{\xi(m)} \left[ \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k + \xi(m-1)} 2^k \right. \\
&+ \left. \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-k-2}{k} (ab)^{\lfloor \frac{m}{2} \rfloor - 1 - k} 2^{k+1} \right] \\
&= a^{\xi(m)} \left[ \binom{m-1}{0} (ab)^{\lfloor \frac{m}{2} \rfloor - 0} + \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m}{2} \rfloor - k} 2^k \right.
\end{aligned}$$

$$+\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k-1}{k-1} (ab)^{\lfloor \frac{m}{2} \rfloor - k} 2^k \Bigg]$$

$$= a^{\xi(m)} \left[ \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k}{k} (ab)^{\lfloor \frac{m}{2} \rfloor - k} 2^k + (ab)^{\lfloor \frac{m}{2} \rfloor} + 2^{\lfloor \frac{m}{2} \rfloor} (1-\xi(m)) \right]$$

$$= a^{\xi(m)} \left[ \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} (ab)^{\lfloor \frac{m}{2} \rfloor - k} 2^k \right]$$

□

**Theorem 5.** *Let $m$ and $n$ be any two positive integers, the following property is satisfied*

$$\hat{\jmath}_{m+n-1} = \left(\frac{b}{a}\right)^{1-\xi(mn+n-m)} \hat{\jmath}_m \hat{\jmath}_n + 2\left(\frac{b}{a}\right)^{\xi(mn)} \hat{\jmath}_{m-1}\hat{\jmath}_{n-1}. \tag{7}$$

*Proof.* We prove the above result using the extended Binet's formula. First, note that $\xi(m+n) = \xi(m) + \xi(n) - 2\xi(m)\xi(n)$. The first part of the sum of the right hand side is denoted by

$$= \left(\tfrac{b}{a}\right)^{1-\xi(mn+n-m)} \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) \frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)$$

$$= \frac{a^{1+\frac{\xi(m+n)}{2}} b^{1-\frac{\xi(m+n)}{2}}}{(ab)^{\frac{m+n}{2}}} \frac{\alpha^{m+n}+\beta^{m+n}-\alpha^n\beta^m-\alpha^m\beta^n}{(\alpha-\beta)^2}.$$

And the second part of the sum of the right hand side is denoted by

$$= 2\frac{b^{\xi(mn)} a^{-\xi(mn)+1-\xi(n-1)+1-\xi(m-1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)\left(\frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}\right)$$

$$= 2\frac{a^{1+\frac{\xi(m+n)}{2}} b^{1-\frac{\xi(m+n)}{2}}}{(ab)^{\frac{m+n}{2}-1}} \frac{\alpha^{m+n-2}+\beta^{m+n-2}-\alpha^{n-1}\beta^{m-1}-\alpha^{m-1}\beta^{n-1}}{(\alpha-\beta)^2}.$$

Now we add the right hand side of part (7)

$$= \frac{a^{1+\frac{\xi(m+n)}{2}} b^{1-\frac{\xi(m+n)}{2}}}{(ab)^{\frac{m+n}{2}}} \frac{\alpha^{m+n-1}\left(\alpha+2\alpha b\alpha^{-1}\right)+\beta^{m+n-1}\left(\beta+2ab\beta^{-1}\right)}{(\alpha-\beta)^2}$$

$$= \frac{a^{1+\frac{\xi(m+n)}{2}} b^{1-\frac{\xi(m+n)}{2}}}{(ab)^{\frac{m+n}{2}}(\alpha-\beta)^2} \alpha^{m+n-1}(\alpha-\beta) - \beta^{m+n-1}(\alpha-\beta)$$

$$= \frac{a^{1+\frac{\xi(m+n)}{2}} b^{1-\frac{\xi(m+n)}{2}}}{(ab)^{\frac{m+n}{2}}} \frac{\alpha^{m+n-1}-\beta^{m+n-1}}{\alpha-\beta} = \hat{\jmath}_{m+n-1}$$

When $a=b=1$ the above result reduces to an identity of Jacobsthal numbers

$$\hat{\jmath}_{m+n-1} = \hat{\jmath}_m\hat{\jmath}_n + 2\hat{\jmath}_{m-1}\hat{\jmath}_{n-1}.$$

If we choose $m=n$ the theorem turns out the identity

$$\hat{\jmath}_{2m-1} = (\hat{\jmath}_m)^2 + 2(\hat{\jmath}_{m-1})^2.$$

If we choose $m \to 2m$, $n \to 2n+1$, the theorem turns out the identity

$$\hat{\jmath}_{2m+2n} = \hat{\jmath}_{2m}\hat{\jmath}_{2n+1} + 2\hat{\jmath}_{2m-1}\hat{\jmath}_{2n}.$$

□

**Theorem 6.** *For any positive integer $m > 2$, we have*

$$\hat{\jmath}_m = \frac{a^{\xi(m+1)}}{2^{m-1}} \sum_{k=0}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \binom{m}{2k+1} (ab)^{\left\lfloor \frac{m-2}{2} \right\rfloor - k} (ab+8)^k$$

*Proof.* Since

$$2\alpha = ab + \sqrt{ab\,(ab+8)}, \qquad 2\beta = ab - \sqrt{ab\,(ab+8)}$$

we get

$$(2\alpha)^m = \left(ab + \sqrt{ab\,(ab+8)}\right)^m = \sum_{k=0}^{m} \binom{m}{k} (ab)^{m-\frac{k}{2}} (ab+8)^{\frac{k}{2}}$$

$$(2\beta)^m = \left(ab - \sqrt{ab\,(ab+8)}\right)^m = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (ab)^{m-\frac{k}{2}} (ab+8)^{\frac{k}{2}}$$

Therefore, we obtain

$$(2\alpha)^m - (2\beta)^m = \sum_{k=0}^{m} \binom{m}{k} \left(1-(-1)^k\right) (ab)^{m-\frac{k}{2}} (ab+8)^{\frac{k}{2}}$$

$$2^m\left(\alpha^m - \beta^m\right) = 2\sqrt{ab\,(ab+8)} \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2j+1} (ab)^{m-j-1} (ab+8)^j$$

$$\alpha^m - \beta^m = \frac{\alpha-\beta}{2^{m-1}} \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2j+1} (ab)^{m-j-1} (ab+8)^j$$

By using the definition of bi-periodic Jacobsthal sequence

$$\begin{aligned}
\hat{\jmath}_m &= \left(\frac{a^{1-\xi(m)}}{(ab)^{\left\lfloor \frac{m}{2} \right\rfloor}}\right) \frac{\alpha^m - \beta^m}{\alpha - \beta} \\
&= \left(\frac{a^{1-\xi(m)}}{(ab)^{\left\lfloor \frac{m}{2} \right\rfloor}}\right) \frac{1}{2^{m-1}} \sum_{j=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2j+1} (ab)^{m-j-1} (ab+8)^j \\
&= \frac{a^{\xi(m+1)}}{2^{m-1}} \sum_{j=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2j+1} (ab)^{m-\left\lfloor \frac{m}{2} \right\rfloor - j-1} (ab+8)^j \\
&= \frac{a^{\xi(m+1)}}{2^{m-1}} \sum_{j=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2j+1} (ab)^{\left\lfloor \frac{m-1}{2} \right\rfloor - j} (ab+8)^j
\end{aligned}$$

When $a = b = 1$ the above result reduces to an identity of Jacobsthal numbers as

$$\hat{\jmath}_m = \frac{1}{2^{m-1}} \sum_{j=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2j+1} 9^j$$

□

**Theorem 7.** *For any positive integer $n$, we have*

$$\sum_{i=1}^{2n} \frac{\hat{\jmath}_i \hat{\jmath}_{i+1}}{2^i} = \frac{1}{b}\left[\left(\frac{\hat{\jmath}_{2n+1}}{2^n}\right)^2 - 1\right] \tag{8}$$

*Proof.*

$$\frac{\hat{\jmath}_i\hat{\jmath}_{i+1}}{2^i} = \frac{a^{1-\xi(i)}}{(ab)^{\lfloor \frac{i}{2} \rfloor}}\left(\frac{\alpha^i-\beta^i}{\alpha-\beta}\right)\left(\frac{\alpha^{i+1}-\beta^{i+1}}{\alpha-\beta}\right)\frac{a^{1-\xi(i+1)}}{(ab)^{\lfloor \frac{i+1}{2} \rfloor}2^i}$$
$$= \frac{a\left[\alpha^{2i+1}+\beta^{2i+1}-(\alpha\beta)^i(\alpha+\beta)\right]}{(\alpha-\beta)^2(ab)^i 2^i}$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\left(\frac{\alpha^2}{2ab}\right)^i\alpha+\left(\frac{\beta^2}{2ab}\right)^i\beta-\frac{(-2ab)^i(ab)}{(2ab)^i}\right]$$

$$\sum_{i=1}^{2n}\frac{\hat{\jmath}_i\hat{\jmath}_{i+1}}{2^i} = \frac{a}{(\alpha-\beta)^2}\left[\alpha\sum_{i=1}^{2n}\left(\frac{\alpha^2}{2ab}\right)^i+\beta\sum_{i=1}^{2n}\left(\frac{\beta^2}{2ab}\right)^i-ab\sum_{i=1}^{2n}(-1)^i\right]$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\alpha\sum_{i=0}^{2n-1}\left(\frac{\alpha^2}{2ab}\right)^{i+1}+\beta\sum_{i=0}^{2n-1}\left(\frac{\beta^2}{2ab}\right)^{i+1}\right]$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\frac{\alpha^3}{2ab}\left(\frac{\frac{\alpha^{4n}}{(2ab)^{2n}}-1}{\frac{\alpha^2}{2ab}-1}\right)+\frac{\beta^3}{2ab}\left(\frac{\frac{\beta^{4n}}{(2ab)^{2n}}-1}{\frac{\beta^2}{2ab}-1}\right)\right]$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\frac{\alpha^2}{ab}\left(\frac{\alpha^{4n}-(2ab)^{2n}}{(2ab)^{2n}}\right)+\frac{\beta^2}{ab}\left(\frac{\beta^{4n}-(2ab)^{2n}}{(2ab)^{2n}}\right)\right]$$
$$= \frac{a}{(\alpha-\beta)^2}\left(\frac{\alpha^{4n+2}+\beta^{4n+2}}{4^n(ab)^{2n+1}}\right)-\frac{a}{ab(ab+8)}\left(\frac{ab(ab+4)}{ab}\right)$$
$$= \frac{a}{(\alpha-\beta)^2}\left(\frac{\alpha^{4n+2}+\beta^{4n+2}}{4^n(ab)^{2n+1}}\right)-\left(\frac{(ab+4)}{b(ab+8)}\right)$$

since

$$\alpha^2-2ab = \alpha^2+\alpha\beta = \alpha(\alpha+\beta) = \alpha ab,$$
$$\beta^2-2ab = \beta^2+\alpha\beta = \beta(\alpha+\beta) = \beta ab,$$
$$\alpha^2+\beta^2 = a^2b^2+4ab = ab(ab+4),$$

therefore

$$= \frac{\alpha}{(\alpha-\beta)^2}\left[\alpha^2\left(\frac{\alpha^{4n}-(2ab)^{2n}}{(2ab)^{2n}(ab)}\right)+\beta^2\left(\frac{\beta^{4n}-(2ab)^{2n}}{(2ab)^{2n}(ab)}\right)\right]$$
$$= \frac{\alpha\left(\alpha^{4n+2}-\beta^{4n+2}\right)}{(\alpha-\beta)^2 4^n(ab)^n}-\frac{(ab+4)}{b(ab+8)}$$

The right hand side of part (8) is given by

$$\frac{1}{b}\left[\left(\frac{\hat{\jmath}_{2n+1}}{2^n}\right)^2-1\right] = \frac{1}{b}\left[\left(\frac{a^{1-\xi(2n+1)}}{(ab)^{\lfloor\frac{2n+1}{2}\rfloor}2^n}\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}\right)^2-1\right]$$
$$= \frac{a}{ab}\left[\left(\frac{a^{4n+2}+\beta^{4n+2}-2(\alpha\beta)^{2n+1}}{(ab)^{2n}(\alpha-\beta)^2 4^n}\right)-\frac{1}{b}\right]$$

$$\begin{aligned}
&= \frac{a\left(a^{4n+2}+\beta^{4n+2}\right)}{(ab)^{2n+1}(\alpha-\beta)^2 4^n} - \frac{2(-2)^{2n+1}(ab)^{2n+1}a}{(ab)^{2n+1}(\alpha-\beta)^2 4^n} - \frac{1}{b} \\
&= \frac{a\left(a^{4n+2}+\beta^{4n+2}\right)}{(ab)^{2n+1}(\alpha-\beta)^2 4^n} + \frac{4a}{ab(ab+8)} - \frac{1}{b} \\
&= \frac{a\left(a^{4n+2}+\beta^{4n+2}\right)}{(ab)^{2n+1}(\alpha-\beta)^2 4^n} - \frac{1}{b}\left(\frac{ab+4}{ab+8}\right)
\end{aligned}$$

□

**Theorem 8.** *For any positive integer $n$, we obtain*

$$\hat{\jmath}_n\hat{\jmath}_{n+2} = \left(\frac{a}{b}\right)^{\xi(n+1)}\left[\left(\frac{b}{a}\right)^{\xi(n)}\frac{a^{(1-\xi(n+1))^2}}{(ab)^{2\left\lfloor\frac{n+1}{2}\right\rfloor}}\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)^2 - (-2)^{n+1}\right]. \tag{9}$$

*Proof.*

$$\begin{aligned}
\hat{\jmath}_n\hat{\jmath}_{n+2} &= \frac{\left(a^{1-\xi(n)}\right)^2}{(ab)^{2\left\lfloor\frac{n}{2}\right\rfloor+1}}\frac{(\alpha^n-\beta^n)\left(\alpha^{n+2}-\beta^{n+2}\right)}{(\alpha-\beta)^2} \\
&= \frac{a^{2\xi(n+1)}}{(ab)^{n-\xi(n)+1}}\frac{\alpha^{2n+2}+\beta^{2n+2}-(\alpha\beta)^n\left(\alpha^2+\beta^2\right)}{(\alpha-\beta)^2} \\
&= \frac{a^{\xi(n+1)}}{b^{\xi(n+1)}(ab)^n}\left[\frac{\alpha^{2n+2}+\beta^{2n+2}-(-2ab)^n\left(a^2b^2+4ab\right)}{(\alpha-\beta)^2}\right]
\end{aligned}$$

For the right hand side of the equality (9)

$$\begin{aligned}
&\left(\frac{a}{b}\right)^{\xi(n+1)}\left[\left(\frac{b}{a}\right)^{\xi(n)}\frac{a^{(1-\xi(n+1))^2}}{(ab)^{2\left\lfloor\frac{n+1}{2}\right\rfloor}}\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)^2-(-2)^{n+1}\right] \\
&= \left(\frac{a}{b}\right)^{\xi(n+1)}\left[\left(\frac{b}{a}\right)^{\xi(n)}\frac{a^{2\xi(n)}}{(ab)^{n+1-\xi(n+1)}}\left(\frac{\alpha^{2n+2}+\beta^{2n+2}-2(\alpha\beta)^{n+1}}{(\alpha-\beta)^2}\right)^2-(-2)^{n+1}\right] \\
&= \left(\frac{a}{b}\right)^{\xi(n+1)}\left[\frac{1}{(ab)^n}\frac{\alpha^{2n+2}+\beta^{2n+2}-2(\alpha\beta)^{n+1}+(-2)^{n+1}(\alpha-\beta)^2(ab)^n}{(\alpha-\beta)^2}\right] \\
&= \left(\frac{a}{b}\right)^{\xi(n+1)}\left[\frac{1}{(ab)^n}\frac{\alpha^{2n+2}+\beta^{2n+2}-2(-2ab)^{n+1}-(-2ab)^n\left(a^2b^2+8ab\right)}{(\alpha-\beta)^2}\right] \\
&= \left(\frac{a}{b}\right)^{\xi(n+1)}\left[\frac{1}{(ab)^n}\frac{\alpha^{2n+2}+\beta^{2n+2}-(-2ab)^n\left(-4ab+a^2b^2+8ab\right)}{(\alpha-\beta)^2}\right]
\end{aligned}$$

□

**Theorem 9.** *For any positive integer $n$, we have*

$$\sum_{i=1}^{2n}\left(\frac{a}{b}\right)^{\xi(i)}\frac{\hat{\jmath}_i\hat{\jmath}_{i+2}}{2^i} = \frac{1}{b}\left(\frac{\hat{\jmath}_{2n+1}\hat{\jmath}_{2n+2}}{2^{2n}} - a\right)$$

*Proof.*

$$\left(\frac{a}{b}\right)^{\xi(i)}\frac{\hat{\jmath}_i\hat{\jmath}_{i+2}}{2^i} = \left(\frac{a}{b}\right)^{\xi(i)}\frac{a^{2(1-\xi(i))}}{(ab)^{2\left\lfloor\frac{i}{2}\right\rfloor+1}}\frac{\left(\alpha^i-\beta^i\right)\left(\alpha^{i+2}-\beta^{i+2}\right)}{(\alpha-\beta)^2 2^i}$$

$$= \left(\frac{a}{b}\right)^{\xi(i)} \frac{a^{2\xi(i+1)}}{(ab)^{i-\xi(i)+1}} \frac{\alpha^{2i+2} + \beta^{2i+2} - (\alpha\beta)^i \left(\alpha^2 + \beta^2\right)}{(\alpha - \beta)^2 2^i}$$

$$= \left(\frac{a}{b}\right) \frac{\alpha^{2i+2} + \beta^{2i+2} - (-2ab)^i \left(a^2b^2 + 4ab\right)}{(\alpha - \beta)^2 (2ab)^i}$$

$$\sum_{i=1}^{2n} \left(\frac{a}{b}\right)^{\xi(i)} \frac{\hat{\jmath}_i \hat{\jmath}_{i+2}}{2^i} = \frac{a}{b(\alpha-\beta)^2} \sum_{i=1}^{2n} \frac{\alpha^{2i+2} + \beta^{2i+2} - (-2ab)^i \left(a^2b^2 + 4ab\right)}{(2ab)^i}$$

$$= \frac{a}{b(\alpha-\beta)^2} \left\{ \sum_{i=1}^{2n} \left[ \alpha^2 (\frac{\alpha^2}{2ab})^i + \beta^2 (\frac{\beta^2}{2ab})^i \right] - \left(a^2b^2 + 4ab\right) \sum_{i=1}^{2n} (-1)^i \right\}$$

$$= \frac{a}{b(\alpha-\beta)^2} \left\{ \sum_{i=0}^{2n-1} \left[ \alpha^2 (\frac{\alpha^2}{2ab})^{i+1} + \beta^2 (\frac{\beta^2}{2ab})^{i+1} \right] - \left(a^2b^2 + 4ab\right) \sum_{i=1}^{2n} (-1)^i \right\}$$

$$= \frac{a}{b(\alpha-\beta)^2} \left\{ \frac{\alpha^3}{ab} \frac{\alpha^{4n} - (\alpha\beta)^{2n}}{(2ab)^{2n}} + \frac{\beta^3}{ab} \frac{\beta^{4n} - (\alpha\beta)^{2n}}{(2ab)^{2n}} \right\}$$

$$= \frac{a}{b(\alpha-\beta)^2} \left\{ \frac{\alpha^{4n+3} + \beta^{4n+3}}{ab(2ab)^{2n}} - (\alpha^3 + \beta^3) \right\}$$

similary

$$\frac{1}{b} \left( \frac{\hat{\jmath}_{2n+1} \hat{\jmath}_{2n+2}}{2^{2n}} - a \right) = \frac{1}{b 2^{2n}} \left[ \frac{1}{(ab)^n} \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \frac{1}{(ab)^{n+1}} \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta} - a \right]$$

$$= \frac{a}{b(\alpha-\beta)^2} \left[ \frac{1}{(2ab)^{2n}} \frac{\alpha^{4n+3} + \beta^{4n+3} - (\alpha\beta)^{2n+1}(\alpha+\beta) - a(ab)(\alpha-\beta)^2}{ab} \right]$$

$$= \frac{a}{b(\alpha-\beta)^2} \left[ \frac{1}{(2ab)^{2n}} \frac{\alpha^{4n+3} + \beta^{4n+3} - (2ab)^{2n}((\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)}{ab} \right]$$

$$= \frac{a}{b(\alpha-\beta)^2} \left[ \frac{\alpha^{4n+3} + \beta^{4n+3}}{ab(2ab)^{2n}} - (\alpha^3 + \beta^3) \right]$$

□

**Theorem 10.** *For any positive integer n, we have*

$$\sum_{i=1}^{2n} \frac{\hat{\jmath}_i \hat{\jmath}_{i+3}}{2^{i+1}} = \frac{1}{b} \left[ \frac{\hat{\jmath}_{2n+1} \hat{\jmath}_{2n+3}}{2^{2n+1}} - (ab+1) \right] \tag{10}$$

*Proof.* Note that

$$\frac{\hat{\jmath}_i \hat{\jmath}_{i+3}}{2^{i+1}} = \frac{a^{1-\xi(i)}}{(ab)^{\lfloor \frac{i}{2} \rfloor}} \frac{\alpha^i - \beta^i}{\alpha - \beta} \frac{a^{1-\xi(i+3)}}{(ab)^{\lfloor \frac{i+3}{2} \rfloor}} \frac{\alpha^{i+3} - \beta^{i+3}}{\alpha - \beta} \frac{1}{2^{i+1}}$$

$$= \frac{a\left(\alpha^{2i+3} + \beta^{2i+3} - (\alpha\beta)^i \left(\alpha^3 + \beta^3\right)\right)}{(ab)^{\lfloor \frac{i}{2} \rfloor + \lfloor \frac{i+1}{2} \rfloor + 1} (\alpha-\beta)^2 2^{i+1}}$$

$$= \frac{a}{(\alpha-\beta)^2}\left(\alpha(\frac{\alpha^2}{2ab})^{i+1}+\beta(\frac{\beta^2}{2ab})^{i+1}-\frac{(-2)^i a^2b^2(ab+6)}{ab2^{i+1}}\right)$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\alpha(\frac{\alpha^2}{2ab})^{i+1}+\beta(\frac{\beta^2}{2ab})^{i+1}-\frac{ab(ab+6)}{2}(-1)^i\right]$$

$$\sum_{i=1}^{2n}\frac{\hat{\jmath}_i\hat{\jmath}_{i+3}}{2^{i+1}}$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\alpha\sum_{i=1}^{2n}(\frac{\alpha^2}{2ab})^{i+1}+\beta\sum_{i=1}^{2n}(\frac{\beta^2}{2ab})^{i+1}-\frac{ab(ab+6)}{2}\sum_{i=1}^{2n}(-1)^i\right]$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\frac{(\frac{\alpha^2}{2ab})^{2n}-1}{\frac{\alpha^2}{2ab}-1}\frac{\alpha^5}{(2ab)^2}+\frac{(\frac{\beta^2}{2ab})^{2n}-1}{\frac{\beta^2}{2ab}-1}\frac{\beta^5}{(2ab)^2}\right]$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\frac{\alpha^{4n}-(2ab)^{2n}}{(2ab)^{2n+1}}\frac{\alpha^4}{ab}+\frac{\beta^{4n}-(2ab)^{2n}}{(2ab)^{2n+1}}\frac{\beta^4}{ab}\right]$$
$$= \frac{a}{(\alpha-\beta)^2}\left[\frac{\alpha^{4n}-(2ab)^{2n}}{(2ab)^{2n+1}}\frac{\alpha^4}{ab}+\frac{\beta^{4n}-(2ab)^{2n}}{(2ab)^{2n+1}}\frac{\beta^4}{ab}\right]$$
$$= \frac{1}{(\alpha-\beta)^2}\left[\frac{\alpha^{4n+4}+\beta^{4n+4}}{(2ab)^{2n+1}b}\right]-\frac{ab(ab+8)+4}{b(ab+8)}$$

It's written for the right hand side of the equation (10)

$$\frac{1}{b}\left[\frac{\hat{\jmath}_{2n+1}\hat{\jmath}_{2n+3}}{2^{2n+1}}-(ab+1)\right]$$
$$= \frac{1}{b}\left[\frac{a^{1-\xi(2n+1)}}{(ab)^{\lfloor\frac{2n+1}{2}\rfloor}}\frac{a^{1-\xi(2n+3)}}{(ab)^{\lfloor\frac{2n+3}{2}\rfloor}}\frac{(\alpha^{2n+1}-\beta^{2n+1})(\alpha^{2n+3}-\beta^{2n+3})}{(\alpha-\beta)^2 2^{2n+1}}-(ab+1)\right]$$
$$= \frac{1}{b}\left[\frac{\alpha^{4n+4}+\beta^{4n+4}-(\alpha\beta)^{2n+1}(\alpha^2+\beta^2)}{(ab)^{2n+1}(\alpha-\beta)^2 2^{2n+1}}-(ab+1)\right]$$
$$= \frac{1}{b}\left[\frac{\alpha^{4n+4}+\beta^{4n+4}-(-2ab)^{2n+1}(a^2b^2+4ab)}{(2ab)^{2n+1}(\alpha-\beta)^2}-(ab+1)\right]$$
$$= \frac{1}{b}\left(\frac{\alpha^{4n+4}+\beta^{4n+4}}{(\alpha-\beta)^2(2ab)^{2n+1}}\right)+\frac{(2ab)^{2n+1}ab(ab+4)}{b(2ab)^{2n+1}ab(ab+8)}-\frac{ab+1}{b}$$
$$= \frac{1}{b}\left[\frac{\alpha^{4n+4}+\beta^{4n+4}}{(\alpha-\beta)^2(2ab)^{2n+1}}+\frac{ab+4-(a^2b^2+9ab+8)}{ab+8}\right]$$
$$= \frac{1}{b}\left(\frac{\alpha^{4n+4}+\beta^{4n+4}}{(\alpha-\beta)^2(2ab)^{2n+1}}\right)-\frac{ab(ab+8)+4}{b(ab+8)}$$

□

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