SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR
FUNCTIONS WHOSE DERIVATIVES ARE OPERATOR
($\alpha, m$)-PREINVEX

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Abstract. In this paper, it is introduced operator ($\alpha, m$)-preinvex function in Hilbert spaces. And then it is obtained some new Hermite-Hadamard type inequalities for functions whose absolute value of derivatives are operator ($\alpha, m$)-preinvex.

1. Introduction

The following inequality holds for any convex function $f$ define $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

Both inequalities hold in the reversed direction if $f$ is concave. The inequality (1) is known in the literature as the Hermite-Hadamard’s inequality. The Hermite-Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality.

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. Now we give information about the operators that are necessary for this work.

Firstly, we review the operator order in $B(H)$, it is set of all bounded operators from $H$ to $H$, and the continuous functional calculus for a bounded selfadjoint operator. For $A, B \in B(H)$ selfadjoint operators we can write for every $x \in H$

$$A \leq B \text{ if } \langle Ax, x \rangle \leq \langle Bx, x \rangle$$

or

$$B \geq A \text{ if } \langle Bx, x \rangle \geq \langle Ax, x \rangle$$

in the operator order. Let $A$ be a selfadjoint linear operator on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and $C(Sp(A))$ the $C^*$-algebra of all continuous complex-valued functions on the spectrum $A$. The Gelfand map establishes a *-isometrical isomorphism $\Phi$ between $C(Sp(A))$ and the $C^*$ generated by $A$ and the identity operator $1_H$ on $H$ as follows [1]

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

1. $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
2. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f)^*$;
3. $\|\Phi(f)\| = \|f\| : \sup_{t \in Sp(A)}|f(t)|$;
4. $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$

with this notation, we define for all $f \in C(Sp(A))$

$$f(A) := \Phi(f)$$

and we call it the continuous functional calculus for a selfadjoint operator $A$. 

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If $f$ is a continuous complex-valued functions on $C(Sp(A))$, the element $\phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator $A$.

If $A$ is bounded selfadjoint operator and $f$ is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on $H$. Moreover if both $f$ and $g$ are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$ then $f(A) \leq g(A)$ in the operator order $B(H)$.

**Definition 1.** [2]: Let $F$ be a nonempty closed set in $\mathbb{R}^n$, $f : F \to \mathbb{R}$ be a continuous function and $\eta(.,.) : F \times F \to \mathbb{R}^n$ be a continuous bi-function. Then a set $F$ is said to be invex set with respect to $\eta(.,.)$, if for every $x, y \in F$ and $t \in [0,1]$ 
\[ y + t\eta(x,y) \in F. \]

**Remark 1.** [3]: It is obvious that every convex set in invex with respect to the map $\eta(x,y) = x - y$, but there exist invex sets which are not convex.

Let $X$ be a real vector space and $F \subseteq X$ be an invex set with respect to $\eta : F \times F \to X$. For every $x, y \in F$ the $\eta$-path $P_{xy}$ joining the points $X$ and $v := x + \eta(y, x)$ is defined as follows 
\[ P_{xy} := \{ z : z = x + t\eta(y, x), t \in [0,1] \} \]
You can find some properties about $\eta$-function in [4] and [5] for details. e.g. (C) condition and etc.

**Definition 2.** [6]: Let $F \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : F \times F \to B(H)_{sa}$, where $B(H)_{sa}$ is denoted by all bounded self adjoint operators set from $H$ to $H$. Then the continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be operator preinvex with respect to $\eta$ on $F$, if for every $A,B \in F$ and $t \in [0,1]$ 
\[ f(A + t\eta(B, A)) \leq (1 - t)f(A) + tf(B) \]
in the operator order in $B(H)$.

**Remark 2.** [6]: Every operator convex function is on operator preinvex with respect to the map $\eta(A,B) = A - B$ but converse does not holds.

**Example 1.** [6]: The function $f(x) = -|x|$ is not a convex function, but it is a preinvex function with respect to $\eta(A,B)$, where 
\[ \eta(A,B) := \begin{cases} A - B, & A.B < 0 \\ B - A, & A.B > 0 \end{cases} \]

S.-H. Wang and X.-M. Liu [7] introduced the concept of operator $s$-preinvex function. They established some new Hermite-Hadamard type inequalities for operator $s$-preinvex functions and provided the estimates of both sides of Hermite-Hadamard type inequality in which some operator $s$-preinvex functions of positive selfadjoint operators in Hilbert space was involved.


Also operator $h$-preinvex function is defined by E. Ünlüyol and E. Başköy [9]. And they obtained some algebraic properties of this class. Moreover they established some new inequalities via Hermite-Hadamard type inequality for operator $h$-preinvex function.

In this paper, we defined operator $(\alpha, m)$-preinvex function in Hilbert spaces. Then, we obtained some new Hermite-Hadamard type inequalities for functions whose absolute value of derivatives are operator $(\alpha, m)$-preinvex.
2. Main Results

Definition 3. The function \( f \) on the invex set \( K \subseteq [0, b^*], b^* > 0 \) is said to be operator \((\alpha, m)\)-preinvex with respect to \( \eta \) if

\[
f(A + t\eta(B, A)) \leq (1 - t^\alpha)f(A) + mt^\alpha f(B/m)
\]

holds for all \( A, B \in S, t \in [0, 1] \) and \( (\alpha, m) \in (0, 1) \times (0, 1) \).

The function \( f \) is said to be operator \((\alpha, m)\)-preconcave if and only if \(-f\) is operator \((\alpha, m)\)-preinvex.

Theorem 1. Let \( K \subseteq [0, b^*], b^* > 0 \) be an open invex subset with respect to \( \eta : K \times K \to \mathbb{R} \). Suppose that \( f : K \to \mathbb{R} \) is a differentiable function. If \(|f'|\) is preinvex on \( K \), then for every \( A, B \in K \) the following inequality holds

\[
\left| \frac{1}{\eta(A, B)} \int_A^{A+\eta(B, A)} f(x)dx - f\left( \frac{2A + \eta(B, A)}{2} \right) \right| \leq \frac{\eta(B, A)}{8} \left[ |f'(A)| + |f'(B)| \right].
\]

Proof. Since \( K \) is invex with respect to \( \eta \) and \( A \) belongs to \( K \), we get \( A + \eta(B, A) \in K \). For every \( t \in [0, 1] \) integrating by parts implies that

\[
\int_0^\frac{1}{2} t f'(A + t\eta(B, A))dt + \int_\frac{1}{2}^1 (1 - t) f'(A + t\eta(B, A))dt \\
= \left[ \frac{t f(A + t\eta(B, A))}{\eta(B, A)} \right]_0^\frac{1}{2} + \left[ \frac{(1 - t) f(B + t\eta(B, A))}{\eta(B, A)} \right]_\frac{1}{2}^1 \\
- \frac{1}{\eta(B, A)} \int_0^1 f(A + t\eta(B, A))dt \\
= \frac{1}{\eta(B, A)} \left( \frac{2A + \eta(B, A)}{2} \right) \\
- \frac{1}{[\eta(B, A)]^2} \int_A^{A+\eta(B, A)} f(x)dx.
\]

By preinvex function of \(|f'|\), we have

\[
\left| \frac{1}{\eta(A, B)} \int_A^{A+\eta(B, A)} f(x)dx - f\left( \frac{2A + \eta(B, A)}{2} \right) \right| \\
\leq \eta(B, A) \left[ \int_0^\frac{1}{2} t \left| f'(A + t\eta(B, A)) \right| dt \\
+ \int_\frac{1}{2}^1 (1 - t) \left| f'(A + t\eta(B, A)) \right| dt \right] \\
\leq \eta(B, A) \left[ \int_0^\frac{1}{2} t [(1 - t) \left| f'(A) \right| \\
+ t |f'(B)|] dt + \int_\frac{1}{2}^1 (1 - t) [(1 - t) \left| f'(A) \right| + t \left| f'(B) \right|] dt \right] \\
\leq \eta(B, A) \left[ |f'(A)| + |f'(B)| \right] \frac{1}{8}
\]

The proof is completed. \( \square \)
Lemma 1. Let $K \subseteq [0, b^*]$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$, $(\alpha, m) \in (0,1] \times (0,1]$ and $A, B \in K, V = A + \eta(B, A)$ with $A < V$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on $K$ such that $f' \in L([A, A + \eta(B, A)])$, then the following equality holds:

$$
\frac{1}{\eta(B, A)} \int_A^{A+\eta(B, A)} f(x)dx - \frac{f(A) + f(A + \eta(B, A))}{2} = \frac{\eta(B, A)}{2} \int_0^1 (1 - 2t)f'(A + t\eta(B, A))dt.
$$

Proof. According to lemma’s claim, we have following equalities:

$$
\frac{\eta(B, A)}{2} \int_0^1 (1 - 2t)f'(A + t\eta(B, A))dt
$$

$$
= \frac{\eta(B, A)}{2} \int_A^{A+\eta(B, A)} \frac{1}{\eta(B, A)} \left( 1 - \frac{2(x - A)}{\eta(B, A)} \right) f'(x)dx
$$

$$
= \frac{1}{2} \int_A^{A+\eta(B, A)} f'(x)dx - \frac{1}{2} \int_A^{A+\eta(B, A)} \frac{2xf'(x)}{\eta(B, A)} dx + \frac{1}{2} \int_A^{A+\eta(B, A)} \frac{2Af'(x)}{\eta(B, A)} dx
$$

$$
= \frac{f(A + \eta(B, A)) - f(A)}{2} - \frac{1}{\eta(B, A)} \left[ xf(x) \right]_A^{A+\eta(B, A)} + \frac{1}{\eta(B, A)} \int_A^{A+\eta(B, A)} f(x)dx + \frac{A(f(A + \eta(B, A)) - f(A))}{\eta(B, A)}
$$

$$
= \frac{f(A + \eta(B, A)) - f(A)}{2} - \frac{(A + \eta(B, A))f(A + \eta(B, A)) - Af(A)}{\eta(B, A)} + \frac{1}{\eta(B, A)} \int_A^{A+\eta(B, A)} f(x)dx + \frac{A(f(A + \eta(B, A)) - f(A))}{\eta(B, A)}
$$

$$
= \frac{f(A + \eta(B, A)) - f(A)}{2} + \frac{f(A + \eta(B, A))(A - (A + \eta(B, A))) + f(A)(-A + A)}{\eta(B, A)}
$$

$$
+ \frac{1}{\eta(B, A)} \int_A^{A+\eta(B, A)} f(x)dx
$$

$$
= \frac{1}{\eta(B, A)} \int_A^{A+\eta(B, A)} f(x)dx - \frac{f(A) + f(A + \eta(B, A))}{2}
$$

Consequently the proof is completed. □

Theorem 2. Let $K \subseteq [0, b^*], b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}, (\alpha, m) \in (0,1] \times (0,1]$ and $A, B \in K, V = A + \eta(B, A)$ with $A < V$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on $K$ such that $f' \in L([A, A + \eta(B, A)])$. If $|f'|^q$ is operator $(\alpha, m)$-preinvex with respect to $\eta$ on $\eta$-path $P_{AV}$ with spectra of $A$ and spectra of $V$ on
Let \( K, q > 1 \), then we have the following inequality
\[
\left| \frac{f(A) + f(A + \eta(B, A))}{2} \right| - \frac{1}{\eta(B, A)} \int_A^{A+\eta(B, A)} f(x)dx \leq \frac{\eta(B, A)}{2(p+1)^{\frac{1}{q}}} \left\{ \frac{\alpha}{1 + \alpha} \left| f'(A) \right|^q + m \left| f'(\frac{B}{m}) \right|^q \right\}^{\frac{1}{q}}
\]

**Proof.** Using Lemma [1] and the Hölder’s integral inequality, we have
\[
\left| \frac{f(A) + f(A + \eta(B, A))}{2} \right| - \frac{1}{\eta(B, A)} \int_A^{A+\eta(B, A)} f(x)dx \leq \frac{\eta(B, A)}{2(p+1)^{\frac{1}{q}}} \left( \int_0^1 (1 - 2t)^{\frac{1}{q}} dt \right)^{\frac{1}{q}} \left( \int_0^1 (1 - t^\alpha)^{\frac{1}{q}} dt \right)^{\frac{1}{q}}
\]
By the operator \((\alpha, m)\)-preinvexity of \( f' \), we have for every \( t \in [0, 1] \)
\[
\left| f'(A + t\eta(B, A)) \right|^q \leq (1 - t^\alpha) \left| f'(A) \right|^q + mt^\alpha \left| f'(\frac{B}{m}) \right|^q
\]
for \((\alpha, m) \in (0, 1) \times (0, 1)\). Hence
\[
\int_0^1 \left| f'(A + t\eta(B, A)) \right|^q dt \leq \left| f'(A) \right|^q \frac{1}{\alpha} \int_0^1 (1 - t^\alpha) dt + m \left| f'(\frac{B}{m}) \right|^q \frac{1}{\alpha} \int_0^1 t^\alpha dt
\]
An application of the above inequality in [2] and the following fact
\[
\int_0^1 (1 - 2t)^{\frac{1}{q}} dt = \frac{1}{p+1}
\]
gives the desired inequality. \(\square\)

**Corollary 1.** If we take \( \eta(B, A) = B - A \) in the Theorem [3] we get the following inequality:
\[
\left| \frac{f(A) + f(B)}{2} \right| - \frac{1}{B-A} \int_A^B f(x)dx \leq \frac{B-A}{2(p+1)^{\frac{1}{q}}} \left\{ \frac{\alpha}{1 + \alpha} \left| f'(A) \right|^q + m \left| f'(\frac{B}{m}) \right|^q \right\}^{\frac{1}{q}}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Theorem 3.** Let \( K \subseteq [0, b^*], b^* > 0 \) be an open invex subset with respect to \( \eta : K \times K \to \mathbb{R}, (\alpha, m) \in (0, 1) \times (0, 1) \) and \( A, B \in K \), \( V = A + \eta(B, A) \) with \( A < V \). Suppose that \( f : K \to \mathbb{R} \) is a differentiable mapping on \( K \) such that \( f' \in L[A, A + \eta(B, A)] \). If \( f' \) is an operator \((\alpha, m)\) preinvex with respect to \( \eta \) on \( \eta \)-path \( P_{AV} \) with spectra of \( A \) and spectra of \( V \) on \( K, q \geq 1 \), then we have the following inequality:
\[
\left| \frac{f(A) + f(A + \eta(B, A))}{2} \right| - \frac{1}{\eta(B, A)} \int_A^{A+\eta(B, A)} f(x)dx \leq \frac{\eta(B, A)}{2} \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left[v_2 \left| f'(A) \right|^q + mv_1 \left| f'(\frac{B}{m}) \right|^q \right]^{\frac{1}{q}}
\]
(3)
where
\[ v_1 = \frac{1 + \alpha \, 2^\alpha}{2\alpha(1 + \alpha)(2 + \alpha)}, \quad v_2 = \frac{1}{2} - v_1 \]

Proof. Using Lemma 1 then we have following inequality:
\[
\left| \frac{f(A) + f(A + \eta(B, A))}{2} \right| - \frac{1}{\eta(B, A)} \int_A^{A + \eta(B, A)} f(x)dx \\
\leq \frac{\eta(B, A)}{2} \left( \int_0^1 |1 - 2t| \, dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 |1 - 2t| \|f'(A + t\eta(B, A))\|^q \, dt \right)^{\frac{1}{q}}
\]

(4)

By the operator \((\alpha, m)\)-preinvexity of \(|f'|^q\) on \(K\), for every \(t \in [0, 1]\) and \((\alpha, m) \in (0, 1] \times (0, 1]\) we have
\[
\int_0^1 |1 - 2t| \|f'(A + t\eta(B, A))\|^q \, dt \leq \int_0^1 |1 - 2t| \left[ (1 - t^\alpha) |f'(A)|^q + m t^\alpha |f'(B)|^q \right] \, dt \\
= |f'(A)|^q \int_0^1 |1 - 2t| (1 - t^\alpha) \, dt + m |f'(B)|^q \int_0^1 |1 - 2t| t^\alpha \, dt \\
= v_2 |f'(A)|^q + mv_1 |f'(B)|^q
\]

(5)

Putting (5) in (4), we get the required inequality (3). This completes the proof of the theorem. \(\square\)

Corollary 2. If we take \(\eta(B, A) = B - A\) in Theorem 3, then one has the inequality:
\[
\left| \frac{f(A) + f(B)}{2} \right| - \frac{1}{B - A} \int_A^B f(x)dx \\
\leq \frac{B - A}{2} \left( \frac{1}{2} \right)^{1 - \frac{1}{q}} \left[ v_2 |f'(A)|^q + mv_1 |f'(B)|^q \right]^{\frac{1}{q}}
\]

where
\[ v_1 = \frac{1 + \alpha \, 2^\alpha}{2\alpha(1 + \alpha)(2 + \alpha)}, \quad v_2 = \frac{1}{2} - v_1 \]

Theorem 4. Let \(K \subseteq [0, b^*]\) be an open invex subset with respect to \(\eta : K \times K \to \mathbb{R}\), \((\alpha, m) \in (0, 1] \times (0, 1]\) and \(A, B \in K\), \(V = A + \eta(B, A)\) with \(A < V\). Suppose that \(f : K \to \mathbb{R}\) is a differentiable function. If \(|f'|^q\) is operator \((\alpha, m)\)-preinvex with respect with \(\eta\) on \(\eta\)-path \(P_{AV}\) with spectra of \(A\) and spectra of \(V\) on \(K\), then the following inequality holds:
\[
\left| \frac{1}{\eta(B, A)} \int_A^{A + \eta(B, A)} f(x)dx - f \left( \frac{2A + \eta(B, A)}{2} \right) \right| \\
\leq \eta(B, A) \left[ \frac{f'(A)}{4} + \left( \frac{((1/2)^{\alpha + 1} - 1)(f'(A) - m f'(B/m))}{(\alpha + 1)(\alpha + 2)} \right) \right]
\]

(6)
Proof. Since $K$ is invex with respect to $\eta$ and $A \in K$, we have $A + \eta(B, A) \in K$. For every $t \in [0, 1]$ integrating by parts implies that

$$
\int_{0}^{1} t f'(A + t \eta(B, A))dt + \int_{0}^{1} (t - 1) f'(A + t \eta(B, A))dt
= \left[ \frac{t f(A + t \eta(B, A))}{\eta(B, A)} \right]_{0}^{1} + \left[ \frac{(t - 1) f(A + t \eta(B, A))}{\eta(B, A)} \right]_{0}^{1}
- \frac{1}{\eta(B,A)} \int_{0}^{1} f(A + t \eta(B, A))dt
= \frac{1}{\eta(B,A)} \int_{a}^{A+B} f(B,A) \ \frac{2A+\eta(B,A)}{2} dt
- \frac{1}{[\eta(B,A)]^2} \int_{a}^{A+B} f(x)dx.
$$

By operator $(\alpha, m)$-preinvex function of $|f'|$ and (7), we get the followings,

$$
\left| \frac{1}{\eta(A,B)} \int_{A}^{A+\eta(B,A)} f(x)dx - f \left( \frac{2A+\eta(B,A)}{2} \right) \right|
\leq \eta(B,A) \left[ \int_{0}^{1} t |f'(A + t \eta(B, A))| \ \ \ \ \ \ dt + \int_{0}^{1} (1 - t) |f'(A + \eta(B, A))| \ \ \ \ \ \ dt \right]
\leq \eta(B,A) \left[ \int_{0}^{1} t |(1 - t^\alpha)| f'(A) + mt^\alpha |f'(B_m)| \ \ \ \ \ \ dt + \int_{0}^{1} (1 - t)(1 - t^\alpha)|f'(A) + mt^\alpha| f'(B_m)| \ \ \ \ \ \ dt \right]
= \eta(B,A) \left[ \int_{0}^{1} f'(A) - t^\alpha |f'(A)| + m t^\alpha |f'(B_m)| \ \ \ \ \ \ dt \right]
+ \int_{0}^{1} t^2 |f'(A)| - t^\alpha |f'(A)| + m t^\alpha |f'(B_m)| \ \ \ \ \ \ dt
- \int_{0}^{1} t^2 |f'(A)| - t^\alpha |f'(A)| + m t^\alpha |f'(B_m)| \ \ \ \ \ \ dt
= \eta(B,A) \left[ \frac{f'(A)}{8} - \frac{\frac{1}{2} t^\alpha |f'(A)|}{\alpha + 2} + \frac{m \frac{1}{2} t^\alpha |f'(B_m)|}{\alpha + 2} \right]
+ \frac{1}{\alpha + 1} \left[ \frac{f'(A)}{2} - \frac{\frac{1}{2} t^\alpha |f'(A)|}{\alpha + 1} + \frac{m \frac{1}{2} t^\alpha |f'(B_m)|}{\alpha + 2} \right]
- \frac{1}{\alpha + 1} \left[ \frac{f'(A)}{2} - \frac{\frac{1}{2} t^\alpha |f'(A)|}{\alpha + 1} + \frac{m \frac{1}{2} t^\alpha |f'(B_m)|}{\alpha + 2} \right]$. 

\textbf{Theorem 2.2} Let $A$ be a real number and $f : \mathbb{R} \to \mathbb{R}$ be a function.
Corollary 3. If we take $\alpha = 1$ in Theorem 4, we get the following inequality:

$$\eta(B, A) \left[ \frac{f'(A)}{4} + \frac{1}{2} \right] = \eta(B, A) \left[ \frac{2f'(A) - f'(A) + mf' \left( \frac{B}{m} \right)}{8} \right] \leq \eta(B, A) \left[ \frac{f'(A) + f'(B)}{8} \right]$$

The proof is completed. $\square$

References
