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SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE OPERATOR (α, m)-PREINVEX

ERDAL ÜNLÜYOL AND HÜMEYRA KARBUZ

Abstract. In this paper, it is introduced operator (α, m)-preinvex function in Hilbert spaces. And then it is obtained some new Hermite-Hadamard type inequalities for functions whose absolute value of derivatives are operator (α, m)-preinvex.

1. Introduction

The following inequality holds for any convex function $f$define $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

Both inequalities hold in the reversed direction if f is concave.
The inequality (1) is known in the literature as the Hermite-Hadamard's inequality.The Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. Now we give information about the operators that are necessary for this work.

Firstly, we review the operator order in $B(H)$, it is set of all bounded operators from H to H, and the continuous functional calculus for a bounded selfadjoint operator.For $A, B \in B(H)$ selfadjoint operators we can write for every $x \in H$

$$A \leq B \text{ if } < Ax, x > \leq < Bx, x >$$

or

$$B \geq A \text{ if } < Bx, x > \geq < Ax, x >$$

in the operator order. Let A be a selfadjoint linear operator on a complex Hilbert space $(H, < .,. >)$ and $C(Sp(A))$ the C^*-algebra of all continuous complex-valued functions on the spectrum A.The Gelfand map establishes a *-isometrical isomorphism Φ between $C(Sp(A))$ and the C^* generated by A and the identity operator 1_H on H as follows [1]
For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have
1.$\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
2.$\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f)^*$;
3.$\|\Phi(f)\| = \|f\| : sup_{t \in Sp(A)}|f(t)|$;
4.$\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$
with this notation, we define for all $f \in C(Sp(A))$

$$f(A) := \Phi(f)$$

and we call it the continuous functional calculus for a selfadjoint operator A.

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If f is a continuous complex-valued functions on $C(Sp(A))$,the element $\phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A.

If A is bounded selfadjoint operator and f is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e $f(A)$ is a positive operator on H. Moreover if both f and g are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$ then $f(A) \leq g(A)$ in the operator order $B(H)$.

Definition 1. [2]*: Let F be a nonempty closed set in $\mathbb{R}^n$,$f : F \to \mathbb{R}$ be a continuous function and $\eta(.,.) : F \times F \to \mathbb{R}^n$ be a continuous bi-function.Then a set F is said to be invex set with respect to $\eta(.,.)$, if for every $x, y \in F$ and $t \in [0,1]$*

$$y + t\eta(x, y) \in F.$$

Remark 1. [3]*: It is obvious that every convex set in invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex.*

Let X be a real vector space and $F \subseteq X$ be an invex set with respect to $\eta : F \times F \to X$.For every $x, y \in F$ the η-path P_{xv} joining the points X and $v := x + \eta(y, x)$ is defined as follows

$$P_{xv} := \left\{ z : z = x + t\eta(y, x), t \in [0, 1] \right\}$$

You can find some properties about η-function in [4] and [5] for details. e.g. (C) condition and etc.

Definition 2. [6]*:Let $F \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : F \times F \to B(H)_{sa}$, where $B(H)_{sa}$ is denoted by all bounded self adjoint operators set from H to H.Then the continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be operator preinvex with respect to η on F, if for every $A, B \in F$ and $t \in [0, 1]$.*

$$f(A + t\eta(B, A)) \leq (1 - t)f(A) + tf(B)$$

in the operator order in $B(H)$.

Remark 2. [6]*: Every operator convex function is on operator preinvex with respect to the map $\eta(A, B) = A - B$ but converse does not holds.*

Example 1. [6]*: The function $f(x) = -|x|$ is not a convex function, but it is a preinvex function with respect to $\eta(A, B)$, where*

$$\eta(A, B) := \begin{cases} A - B, & A.B < 0 \\ B - A, & A.B > 0 \end{cases}$$

S.-H. Wang and X.-M. Liu [7] introduced the concept of operator s-preinvex function. They established some new Hermite-Hadamard type inequalities for operator s-preinvex functions and provided the estimates of both sides of Hermite-Hadamard type inequality in which some operator s-preinvex functions of positive selfadjoint operators in Hilbert space was involved.

S.-H. Wang and X.-W. Sun [8] similarly introduced the concept of operator α-preinvex function and some inequalities.

Also operator h-preinvex function is defined by E. Ünlüyol and E. Başköy [9]. And They obtained some algebraic properties of this class. Moreover they established some new inequalities via Hermite-Hadamard type inequality for operator h-preinvex function.

In this paper, we defined operator (α, m)-preinvex function in Hilbert spaces. Then, we obtained some new Hermite-Hadamard type inequalities for functions whose absolute value of derivatives are operator (α, m)-preinvex.

2. Main Results

Definition 3. *: The function f on the invex set $K \subseteq [0, b^*], b^* > 0$ is said to be operator (α, m)-preinvex with respect to η if*

$$f(A + t\eta(B, A)) \leq (1 - t^\alpha)f(A) + mt^\alpha f(\frac{B}{m})$$

holds for all $A, B \in S$, $t \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$.
The function f is said to be operator (α, m)-preconcave if and only if $-f$ is operator (α, m)-preinvex.

Theorem 1. *Let $K \subseteq [0, b^*], b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on K, then for every $A, B \in K$ the following inequality holds*

$$\left| \frac{1}{\eta(A, B)} \int_A^{A+\eta(B,A)} f(x)dx - f(\frac{2A + \eta(B, A)}{2}) \right| \leq \frac{\eta(B, A)}{8} \Big[\mid f'(A) \mid + \mid f'(B) \mid \Big].$$

Proof. Since K is invex with respect to η and A is belongs to K, we get $A+\eta(B, A) \in K$. For every $t \in [0, 1]$ integrating by parts implies that

$$\begin{aligned} \int_0^{\frac{1}{2}} tf'(A + t\eta(B, A))dt \quad &+ \quad \int_{\frac{1}{2}}^1 (t - 1)f'(A + t\eta(B, A))dt \\ &= \left[\frac{tf(A + t\eta(B, A))}{\eta(B, A)}\right]_0^{\frac{1}{2}} + \left[\frac{(t - 1)f(B + t\eta(B, A))}{\eta(B, A)}\right]_{\frac{1}{2}}^1 \\ &\quad - \frac{1}{\eta(B, A)} \int_0^1 f(A + t\eta(B, A))dt \\ &= \frac{1}{\eta(B, A)} f(\frac{2A + \eta(B, A)}{2}) \\ &\quad - \frac{1}{[\eta(B, A)]^2} \int_A^{A+\eta(B,A)} f(x)dx. \end{aligned}$$

By preinvex function of $|f'|$, we have

$$\begin{aligned} \Bigg| \frac{1}{\eta(A, B)} \quad & \int_A^{A+\eta(B,A)} f(x)dx - f\left(\frac{2A + \eta(B, A)}{2}\right) \Bigg| \\ \leq \quad & \eta(B, A) \Bigg[\int_0^{\frac{1}{2}} t \mid f'(A + t\eta(B, A)) \mid dt \\ & + \int_{\frac{1}{2}}^1 (1 - t) \mid f'(A + t\eta(B, A)) \mid dt \Bigg] \\ \leq \quad & \eta(B, A) \Bigg[\int_0^{\frac{1}{2}} t\big[(1 - t) \mid f'(A) \mid \\ & + t|f'(B)|\big]dt + \int_{\frac{1}{2}}^1 (1 - t)\big[(1 - t) \mid f'(A) \mid + t \mid f'(B) \mid \big]dt \Bigg] \\ \leq \quad & \eta(B, A) \left[\frac{|f'(A)| + |f(B)|}{8} \right] \end{aligned}$$

The proof is completed. □

Lemma 1. *Let $K \subseteq [0,b^*]$ be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}, (\alpha,m) \in (0,1]\times(0,1]$ and $A,B \in K, V = A+\eta(B,A)$ with $A < V$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([A, A+\eta(B,A)])$, then the following equality holds:*

$$\frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx - \frac{f(A)+f(A+\eta(B,A))}{2} = \frac{\eta(B,A)}{2}\int_0^1 (1-2t)f'(A+t\eta(B,A))dt.$$

Proof. According to lemma's claim, we have following equalities

$$\frac{\eta(B,A)}{2}\int_0^1 (1-2t)f'(A+t\eta(B,A))dt$$

$$\begin{aligned}
&= \frac{\eta(B,A)}{2}\int_A^{A+\eta(B,A)} \frac{1}{\eta(B,A)}\left(1-\frac{2(x-A)}{\eta(B,A)}\right)f'(x)dx \\
&= \frac{1}{2}\int_A^{A+\eta(B,A)} f'(x)dx - \frac{1}{2}\int_A^{A+\eta(B,A)} \frac{2xf'(x)}{\eta(B,A)}dx \\
&\quad +\frac{1}{2}\int_A^{A+\eta(B,A)} \frac{2Af'(x)}{\eta(B,A)}dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{f(A+\eta(B,A))-f(A)}{2} - \frac{1}{\eta(B,A)}\Big[xf(x)\Big]_A^{A+\eta(B,A)} \\
&\quad +\frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx + \frac{A(f(A+\eta(B,A))-f(A))}{\eta(B,A)} \\
&= \frac{f(A+\eta(B,A))-f(A)}{2} - \frac{(A+\eta(B,A))f(A+\eta(B,A))-Af(A)}{\eta(B,A)} \\
&\quad +\frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx + \frac{A(f(A+\eta(B,A))-f(A))}{\eta(B,A)} \\
&= \frac{f(A+\eta(B,A))-f(A)}{2} \\
&\quad +\frac{f(A+\eta(B,A))(A-(A+\eta(B,A)))+f(A)(-A+A)}{\eta(B,A)} \\
&\quad +\frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx \\
&= \frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx - \frac{f(A)+f(A+\eta(B,A))}{2}
\end{aligned}$$

Consequently the proof is completed. □

Theorem 2. *Let $K \subseteq [0,b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}$, $(\alpha,m) \in (0,1]\times(0,1]$ and $A,B \in K$, $V = A+\eta(B,A)$ with $A < V$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([A, A+\eta(B,A)])$. If $|f'|^q$ is operator (α,m)-preinvex with respect to η on η-path P_{AV} with spectra of A and spectra of V on*

K, $q > 1$, then we have the following inequality

$$\begin{aligned} \Big|\frac{f(A)+f(A+\eta(B,A))}{2} &- \frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx\Big| \\ &\leq \frac{\eta(B,A)}{2.(p+1)^{\frac{1}{p}}}\Big[\frac{\alpha \mid f'(\alpha)\mid^q + m \mid f'(\frac{B}{m})\mid^q}{1+\alpha}\Big]^{\frac{1}{q}} \end{aligned}$$

Proof. Using Lemma 1 and the Hölder's integral inequality, we have

$$\Big|\frac{f(A)+f(A+\eta(B,A))}{2} - \frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx\Big|$$

$$\leq \frac{\eta(B,A)}{2}\Big(\int_0^1 \mid 1-2t\mid^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 \mid f'(A+t\eta(B,A))\mid^q dt\Big)^{\frac{1}{q}} \tag{2}$$

By the operator (α, m)-preinvexity of $\mid f' \mid^q$, we have for every $t \in [0,1]$

$$\Big|f'(A+t\eta(B,A))\Big|^q \leq (1-t^\alpha)\Big|f'(A)\Big|^q + mt^\alpha\Big|f'(\frac{B}{m})\Big|^q$$

for $(\alpha, m) \in (0,1] \times (0,1]$. Hence

$$\begin{aligned} \int_0^1 \Big|f'(A+t\eta(B,A))\Big|^q dt &\leq \mid f'(A)\mid^q \int_0^1 (1-t^\alpha) + m \mid f'(\frac{B}{m})\mid^q \int_0^1 t^\alpha dt \\ &= \frac{\alpha}{1+\alpha}\mid f'(A)\mid^q + \frac{m}{1+\alpha}\mid f'(\frac{B}{m})\mid^q \end{aligned}$$

An application of the above inequality in (2) and the following fact

$$\int_0^1 \mid 1-2t\mid^p dt = \frac{1}{p+1}$$

gives the desired inequality. □

Corollary 1. *If we take $\eta(B,A) = B - A$ in the Theorem 2, we get the following inequality:*

$$\begin{aligned} \Big|\frac{f(A)+f(B)}{2} &- \frac{1}{B-A}\int_A^B f(x)dx\Big| \\ &\leq \frac{B-A}{2(p+1)^{\frac{1}{p}}}\Big[\frac{\alpha \mid f'(A)\mid^q + m \mid f'(\frac{B}{m})\mid^q}{1+\alpha}\Big]^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3. *Let $K \subseteq [0, b^*], b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$, $(\alpha, m) \in (0,1] \times (0,1]$ and $A, B \in K$, $V = A + \eta(B,A)$ with $A < V$. Suppose that $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L[A, A+\eta(B,A)]$. If $\mid f' \mid^q$ is an operator (α, m) preinvex with respect to η on η-path P_{AV} with spectra of A and spectra of V on K, $q \geq 1$, then we have the following inequality:*

$$\Big|\frac{f(A)+f(A+\eta(B,A))}{2} - \frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx\Big|$$

$$\leq \frac{\eta(B,A)}{2}(\frac{1}{2})^{1-\frac{1}{q}}\Big[v_2|f'(A)|^q + mv_1|f'(\frac{B}{m})|^q\Big]^{\frac{1}{q}} \tag{3}$$

where

$$v_1 = \frac{1+\alpha.2^{\alpha}}{2^{\alpha}(1+\alpha)(2+\alpha)}, \quad v_2 = \frac{1}{2} - v_1$$

Proof. Using Lemma 1, then we have following inequality:

$$\begin{aligned} \Big|\frac{f(A)+f(A+\eta(B,A))}{2} &- \frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx\Big| \\ &\leq \frac{\eta(B,A)}{2}\Big(\int_0^1 \mid 1-2t \mid dt\Big)^{1-\frac{1}{q}} \\ &\quad \Big(\int_0^1 \mid 1-2t \mid\mid f'(A+t\eta(B,A)) \mid^q dt\Big)^{\frac{1}{q}} \qquad (4)\end{aligned}$$

By the operator (α, m)-preinvexity of $\mid f' \mid^q$ on K, for every $t \in [0,1]$ and $(\alpha, m) \in (0,1] \times (0,1]$ we have

$$\begin{aligned} \int_0^1 \mid 1-2t \mid\mid f'(A+t\eta(B,A)) \mid^q dt &\leq \int_0^1 \mid 1-2t \mid \Big[(1-t^{\alpha}) \mid f'(A) \mid^q \\ &\quad + mt^{\alpha} \mid f'(B) \mid^q \Big]dt \\ &= \mid f'(A) \mid^q \int_0^1 \mid 1-2t \mid (1-t)^{\alpha} dt \\ &\quad + m \mid f'(B) \mid^q \int_0^1 \mid 1-2t \mid t^{\alpha} dt \\ &= v_2 \mid f'(A) \mid^q + mv_1 \mid f'(B) \mid^q \qquad (5)\end{aligned}$$

Putting (5) in (4), we get the required inequality (3). This completes the proof of the theorem. □

Corollary 2. *: If we take $\eta(B,A) = B - A$ in Theorem 3, then one has the inequality:*

$$\begin{aligned} \Big|\frac{f(A)+f(B)}{2} &- \frac{1}{B-A}\int_A^B f(x)dx\Big| \\ &\leq \frac{B-A}{2}(\frac{1}{2})^{1-\frac{1}{q}}\Big[v_2 \mid f'(A) \mid^q + mv_1 \mid f'(B) \mid^q \Big]^{\frac{1}{q}}\end{aligned}$$

where

$$v_1 = \frac{1+\alpha.2^{\alpha}}{2^{\alpha}(1+\alpha)(2+\alpha)} \quad v_2 = \frac{1}{2} - v_1$$

Theorem 4. *Let $K \subseteq [0, b^*]$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$, $(\alpha, m) \in (0,1] \times (0,1]$ and $A, B \in K$, $V = A + \eta(B,A)$ with $A < V$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\mid f' \mid$ is operator (α, m) preinvex with respect with η on η-path P_{AV} with spectra of A and spectra of V on K, then the following inequality holds:*

$$\Big|\frac{1}{\eta(B,A)}\int_A^{A+\eta(B,A)} f(x)dx - f\Big(\frac{2A+\eta(B,A)}{2}\Big)\Big|$$

$$\leq \eta(B,A)\Big[\frac{f'(A)}{4} + \frac{\big((\frac{1}{2})^{\alpha+1}-1\big)\big(f'(A) - mf'(\frac{B}{m})\big)}{(\alpha+1)(\alpha+2)}\Big] \qquad (6)$$

Proof. Since K is invex with respect to η and $A \in K$, we have $A + \eta(B, A) \in K$. For every $t \in [0, 1]$ integrating by parts implies that

$$\begin{aligned}
\int_0^{\frac{1}{2}} tf'(A + t\eta(B, A))dt &+ \int_{\frac{1}{2}}^1 (t-1)f'(A + t\eta(B, A))dt \\
&= \left[\frac{tf(A + t\eta(B, A))}{\eta(B, A)}\right]_0^{\frac{1}{2}} + \left[\frac{(t-1)f(A + t\eta(B, A))}{\eta(B, A)}\right]_{\frac{1}{2}}^1 \\
&\quad - \frac{1}{\eta(B, A)} \int_0^1 f(A + t\eta(B, A))dt \\
&= \frac{1}{\eta(B, A)} f\left(\frac{2A + \eta(B, A)}{2}\right) \\
&\quad - \frac{1}{[\eta(B, A)]^2} \int_A^{A+\eta(B,A)} f(x)dx. \qquad (7)
\end{aligned}$$

By operator (α, m)-preinvex function of $|f'|$ and (7), we get the followings,

$$\left|\frac{1}{\eta(A, B)} \int_A^{A+\eta(B,A)} f(x)dx - f\left(\frac{2A + \eta(B, A)}{2}\right)\right|$$

$$\begin{aligned}
&\leq \eta(B, A)\left[\int_0^{\frac{1}{2}} t \mid f'(A + t\eta(B, A)) \mid dt + \int_{\frac{1}{2}}^1 (1-t) \mid f'(A + t\eta(B, A)) \mid dt\right] \\
&\leq \eta(B, A)\left[\int_0^{\frac{1}{2}} t[(1 - t^\alpha)|f'(A)| + mt^\alpha \mid f'(\frac{B}{m}) \mid]dt\right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 (1-t)[(1 - t^\alpha)|f'(A)| + mt^\alpha \mid f'(\frac{B}{m}) \mid]dt\right] \\
&= \eta(B, A)\left[\int_0^{\frac{1}{2}} t|f'(A)| - t^{\alpha+1}|f'(A)| + mt^{\alpha+1}|f'(\frac{B}{m})|dt\right. \\
&\quad + \int_{\frac{1}{2}}^1 f'(A) - t^\alpha \mid f'(A) \mid + mt^\alpha |f'(\frac{B}{m})| \\
&\quad \left. - tf'(A) + t^{\alpha+1} \mid f'(A) \mid - mt^{\alpha+1} \mid f'(\frac{B}{m}) \mid dt\right] \\
&= \eta(B, A)\left[\frac{\mid f'(A) \mid t^2}{2} - \frac{t^{\alpha+2} \mid f'(A) \mid}{\alpha+2} + \frac{mt^{\alpha+2} \mid f'\frac{B}{m} \mid}{\alpha+2}\right]_0^{\frac{1}{2}} \\
&\quad + \left[t \mid f'(A) \mid - \frac{\alpha + 1 \mid f'(A) \mid}{\alpha+1} + \frac{mt^{\alpha+1} \mid f'\frac{B}{m} \mid}{\alpha+1}\right. \\
&\quad \left. - \frac{t^2 \mid f'(A) \mid}{\alpha+2} - \frac{mt^{\alpha+2} \mid f'\frac{B}{m} \mid}{\alpha+2}\right]_{\frac{1}{2}}^1 \\
&= \eta(B, A)\left[\frac{f'(A)}{8} - \frac{(\frac{1}{2})^{\alpha+2} \mid f'(A) \mid}{\alpha+2} + \frac{m(\frac{1}{2})^{\alpha+2} \mid f'(\frac{B}{m}) \mid}{\alpha+2}\right. \\
&\quad + \mid f'(A) \mid - \frac{\mid f'(A) \mid}{\alpha+1} + \frac{m \mid f'(\frac{B}{m}) \mid}{\alpha+1} - \frac{\mid f'(A) \mid}{2} + \frac{\mid f'(A) \mid}{\alpha+2} - \frac{m \mid f'(\frac{B}{m}) \mid}{\alpha+2} \\
&\quad - \frac{\mid f'(A) \mid}{2} + \frac{(\frac{1}{2})^{\alpha+1} \mid f'(A) \mid}{\alpha+1} - \frac{m(\frac{1}{2})^{\alpha+1} \mid f'(\frac{B}{m}) \mid}{\alpha+1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\,f'(A)\,|}{8}-\frac{(\frac{1}{2})^{\alpha+2}\,|\,f'(A)\,|}{\alpha+2}+\frac{m(\frac{1}{2})^{\alpha+2}\,|\,f'(\frac{B}{m})\,|}{\alpha+2}\Bigg]\\
=\;&\eta(B,A)\Bigg[\frac{|\,f'(A)\,|}{4}+\frac{\big((\frac{1}{2})^{\alpha+1}-1\big)\,|\,f'(A)\,|}{\alpha+1}\\
&+\frac{(1-(\frac{1}{2})^{\alpha+1})m\,|\,f'(\frac{B}{m})\,|}{\alpha+1}+\frac{\big(1-2(\frac{1}{2})^{\alpha+2}\big)\,|\,f'(A)\,|}{\alpha+2}\\
&+\frac{\big(-1+2(\frac{1}{2})^{\alpha+2}\big)\,|\,f'(\frac{B}{m})\,|}{\alpha+2}\Bigg]\\
=\;&\eta(B,A)\times\\
&\Bigg[\frac{f'(A)}{4}+\frac{(1-(\frac{1}{2})^{\alpha+1})(mf'(\frac{B}{m})-f'(A))}{\alpha+1}+\frac{(1-2(\frac{1}{2})^{\alpha+2})(mf'(f'(A)-\frac{B}{m}))}{\alpha+2}\Bigg]\\
=\;&\eta(B,A)\times\\
&\Bigg[\frac{f'(A)}{4}+\frac{((\frac{1}{2})^{\alpha+1}-1)(f'(A)-mf'(\frac{B}{m}))}{\alpha+1}+\frac{(1-(\frac{1}{2})^{\alpha+1})(f'(A)-mf'(\frac{B}{m}))}{\alpha+2}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
=\;&\eta(B,A)\Bigg[\frac{|\,f'(A)\,|}{4}+\frac{((\frac{1}{2})^{\alpha+1}-1)(\alpha f'(A)-\alpha mf'(\frac{B}{m})+2f'(A))}{(\alpha+1)(\alpha+2)}\\
&-\frac{2mf'(\frac{B}{m})-\alpha f'(A)+\alpha mf'(\frac{B}{m})-f'(A)+mf'(\frac{B}{m})}{(\alpha+1)(\alpha+2)}\Bigg]\\
=\;&\eta(B,A)\Bigg[\frac{f'(A)}{4}+\frac{((\frac{1}{2})^{\alpha+1}-1)(f'(A)-mf'(\frac{B}{m}))}{(\alpha+1)(\alpha+2)}\Bigg]
\end{aligned}
$$

The proof is completed. □

Corollary 3. *If we take $\alpha=1$ in Theorem 4, we get the following inequality:*

$$
\begin{aligned}
\eta(B,A)\Bigg[\frac{f'(A)}{4}+\frac{-f'(A)+mf'(\frac{B}{m})}{8}\Bigg]\;&=\;\eta(B,A)\Bigg[\frac{2f'(A)-f'(A)+mf'(\frac{B}{m})}{8}\Bigg]\\
&\le\;\eta(B,A)\Bigg[\frac{f'(A)+f'(B)}{8}\Bigg]
\end{aligned}
$$

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Ordu University
Department of Mathematics, Faculty of Arts and Science
52200, Ordu, Turkey
E-mail address: eunluyol@yahoo.com

Ordu University
Department of Mathematics, Institute of Science
52200, Ordu, Turkey
E-mail address: humeyrakarbuz15@hotmail.com