TJMM
10 (2018), No. 2, 121-129

A THREE-DIMENSIONAL GENERALIZED BVP OF THERMOELASTICITY FOR A LAYER: GREEN'S FUNCTIONS AND INTEGRATION FORMULA

ȘEREMET VICTOR

ABSTRACT. This study devoted to constructing displacements Green's functions for a generalized 3D BVP of uncoupled thermoelasticity within a layer by using harmonic integral representations method, proposed in the previous works. First are derived structural formulas for main thermoelastic Green's functions for displacements, expressed via respective Green's functions for Poisson's equations. These structural formulas satisfy the equations of thermoelasticity with respect to point of application the heat source and the nonhomogeneous Poisson's equation with respect to point of response in which the thermoelastic displacements appeared. In addition, they satisfy boundary conditions for temperature Green's function with respect to point application the displacements and to mechanical boundary conditions with respect to point of application the heat source. The final analytical expressions for displacements Green's functions obtained on the base of mentioned above structural formulas for four 3D BVPs of thermoelasticity for the layer contain Bessel functions of the zero-order and the second type.

1. Introduction

As is known uncoupled thermoelasticity, including theory of thermal stresses is a combination of the theory of heat conduction and theory of elasticity, when the temperature field does not depend on the field of elastic displacements, and, when inertial terms can be ignored is described in classical scientific literature [1, 2, 3, 4, 5, 6, 7]. However, many new developments of thermoelasticity and many references are presented in [8]. In this paper the Green's function (GF) and the integration formula (IF) was proposed and developed for finding solutions in integrals for boundary value problems (BVPs) of uncoupled thermoelasticity by using harmonic integral representations method (HIRM) in the works [9, 10, 11, 12, 13, 14, 15, 16]. Some interesting solutions for Green's functions were given in the papers [17, 18, 19]. In this paper is proposed the development of the HIRM to derivation of thermolactic structural formulas for a generalized BVP, which permitted us to obtain analytical expressions for thermoelastic Green's functions and integration formulas for four BVPs for the layer

2. Objectives

The main objective of this paper is to develop HIRM for constructing for main thermoelastic displacements Green's functions (MTDGFs) for 3D BVPs of thermoelastic layer $V\,(-\infty \leq x_1, x_2 \leq \infty,\ 0 \leq x_3 \leq a_3)$, which have parallel boundary planes and to derive analytical expressions for MTDGFs for four new locally-mixed 3D BVPs of thermoelasticity.

2010 *Mathematics Subject Classification.* .

Key words and phrases. harmonic integral representation method, structural formulas, integration formulas, thermoelastic Green's functions; thermoelastic volume dilatation; thermoelastic layer.

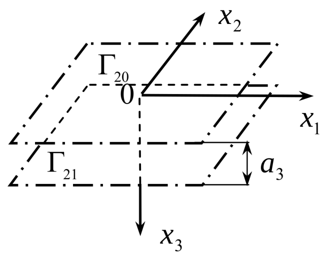

FIGURE 1. The scheme of the layer $V\left(-\infty \leq x_1, x_2 \leq \infty,\ 0 \leq x_3 \leq a_3\right)$ with boundary planes $\Gamma_{30}\left(-\infty \leq x_1, x_2 \leq \infty,\ x_3 = 0\right)$ and $\Gamma_{31}(-\infty \leq x_1, x_2 \leq \infty,\ x_3 = a_3)$.

This paper is organized in such a way that the sections of investigations are given in the following consequence: formulation of the generalized BVP to uncoupled thermoelasticity for the layer; general integral representations for thermoelastic volume dilatation (TVD) and MTDGFs; final simplified structural formulas for MTDGFs and TVD; the checking of the derived final structural formulas for MTDGFs and TVD; the checking of the derived final analytical expressions for MTDGFs and TVD; and integration formulas for thermoelastic displacements and thermoelastic stresses.

3. Formulation of the generalized BVP to uncoupled thermoelasticity for the layer

The generalized BVP to uncoupled thermo-elasticity for determining structural formulas for MTDGFs for displacements $U_i\left(x, \xi\right)$ within the layer consist from Lame's and Poisson's equations

$$\begin{array}{c} \mu \nabla_{\xi}^{2} U_i\left(x, \xi\right) + \left(\lambda + \mu\right) \Theta,_{\ \xi_i}\left(x, \xi\right) - \gamma G_{T,\xi_i}\left(x, \xi\right) = 0 \\ \nabla_{\xi}^{2} G_T\left(x; \xi\right) = -\delta\left(x - \xi\right); x \equiv \left(x_1, x_2, x_3\right),\ \xi \equiv \left(\xi_1, \xi_2, \xi_3\right) \end{array} \quad (1)$$

and the following possible combinations of the boundary conditions for $U_i\left(x, y\right)$, $i = 1, 2, 3$, thermal stresses $\sigma_{3i}^{*}\left(x, y\right)$, and Green's function for temperature $G_T\left(y, \xi\right)$ or its derivative on external normal $\partial G_T\left(y, \xi\right)/\partial n_{30}$:

$$\sigma_{31}^{*}\left(x, y\right) = U_3\left(x, y\right) = \sigma_{32}^{*}\left(x, y\right) = 0, \partial G_T\left(y, \xi\right)/\partial n_{30} = 0; x, \xi \in V; y \equiv \left(y_1, y_2, 0\right) \in \Gamma_{30} \quad (2)$$

or

$$U_1\left(x, y\right) = \sigma_{33}^{*}\left(x, y\right) = U_2\left(x, y\right) = 0,\ \ G_T\left(y, \xi\right) = 0;\ x, \xi \in V; y \equiv \left(y_1, y_2, 0\right) \in \Gamma_{30} \quad (3)$$

on the boundary plane $\Gamma_{30}\left(-\infty \leq x_1, x_2 \leq \infty,\ x_3 = 0\right)$, and

$$\begin{array}{c} \sigma_{31}^{*}\left(x, y\right) = U_3\left(x, y\right) = \sigma_{32}^{*}\left(x, y\right) = 0, \\ \partial G_T\left(y, \xi\right)/\partial n_{31} = 0; x, \xi \in V; y \equiv \left(y_1, y_2, a_3\right) \in \Gamma_{31} \end{array} \quad (4)$$

or

$$U_1\left(x, y\right) = \sigma_{33}^{*}\left(x, y\right) = U_2\left(x, y\right) = 0,\ \ G_T\left(y, \xi\right) = 0;\ x, \xi \in V; y \equiv \left(y_1, y_2, a_3\right) \in \Gamma_{31} \quad (5)$$

on the boundary plane $\Gamma_{31}\left(-\infty \leq x_1, x_2 \leq \infty,\ x_3 = a_3\right)$.

4. General integral representations for thermoelastic volume dilatation (TVD) and main thermoelastic displacements Green's functions (MTDGFs)

To derive the constructive formulas for thermoelastic volume dilatation and displacements we use the following representations of harmonic integral representations method (HIRM) [10, 11, 12, 13]

$$\Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu} G_\Theta(x,\xi) + \int_\Gamma \left[\frac{\partial \Theta(x,y)}{\partial n_\Gamma} - \Theta(x,y)\frac{\partial}{\partial n_\Gamma}\right] G_\Theta(y,\xi)\, d\Gamma(y) \tag{6}$$

for TVD, and

$$\begin{gathered} U_i(x,\xi) = \tfrac{\gamma\xi_i}{2\mu} G_T(x,\xi) - \tfrac{\lambda+\mu}{2\mu}\xi_i\Theta(x,\xi) - \tfrac{\gamma}{2(\lambda+2\mu)} x_i G_i(x,\xi) + \\ \int_\Gamma \left(G_i(y,\xi)\tfrac{\partial}{\partial n_\Gamma} - \tfrac{G_i(y,\xi)}{\partial n_\Gamma}\right)\left[U_i(x,y) + \tfrac{y_i}{2\mu}\left((\lambda+\mu)\Theta(x,y) - \gamma G_T(x,y)\right)\right] d\Gamma(y); \\ i = 1,2,3 \end{gathered} \tag{7}$$

for MTDGFs.

The representations (6) and (7) for layer V can be rewritten as following:

$$\begin{gathered} \Theta(x,\xi) = \tfrac{\gamma}{\lambda+2\mu} G_\Theta(x,\xi) + \int_{\Gamma_{30}} \left(\tfrac{\partial\Theta(x,y)}{\partial n_{30}} - \Theta(x,y)\tfrac{\partial}{\partial n_{30}}\right) G_\Theta(y,\xi)\, d\Gamma_{30}(y) + \\ \int_{\Gamma_{31}} \left(\tfrac{\partial\Theta(x,y)}{\partial n_{31}} - \Theta(x,y)\tfrac{\partial}{\partial n_{31}}\right) G_\Theta(y,\xi)\, d\Gamma_{31}(y) \end{gathered} \tag{8}$$

– for TVD Θ, and

$$\begin{gathered} U_i(x,\xi) = \tfrac{\gamma\xi_i}{2\mu} G_T(x,\xi) - \tfrac{\lambda+\mu}{2\mu}\xi_i\Theta(x,\xi) - \tfrac{\gamma}{2(\lambda+2\mu)} x_i G_i(x,\xi) - \\ \int_{\Gamma_{30}} \left(\tfrac{\partial G_i(y,\xi)}{\partial n_{\Gamma_{30}}} - G_i(y,\xi)\tfrac{\partial}{\partial n_{\Gamma_{30}}}\right)\left[U_i(x,y) + \tfrac{y_i}{2\mu}\left((\lambda+\mu)\Theta(x,y) - \gamma G_T(x,y)\right)\right] d\Gamma_{30}(y) - \\ \int_{\Gamma_{31}} \left(\tfrac{\partial G_i(y,\xi)}{\partial n_{\Gamma_{31}}} - G_i(y,\xi)\tfrac{\partial}{\partial n_{\Gamma_{31}}}\right)\left[U_i(x,y) + \tfrac{y_i}{2\mu}\left((\lambda+\mu)\Theta(x,y) - \gamma G_T(x,y)\right)\right] d\Gamma_{30}(y); \\ i = 1,2,3 \end{gathered} \tag{9}$$

for MTDGFs U_i.

In addition, the boundary conditions (2) - (5) can be substantially simplified if it is taking into account the following links between displacements U_i, stresses σ^*_{ij}, TVD Θ and respective GFPE G_i, G_Θ on boundary planes $\Gamma_{3l} = 0$; $l = 0,1$ of the layer V. So, if $U_i = 0$, then $G_i = 0$; if $U_{i,n} = 0$, then $G_{i,n} = 0$. If zero normal displacements, zero tangential stresses and $G_{T,n} = 0$ are given, then $\Theta_{,n} = 0$ and $G_{\Theta,n} = 0$. Finally, if zero normal stresses, zero tangential displacements and $G_T = 0$, then $\Theta = 0$ and $G_\Theta = 0$ [9, 11, 12, 14, 15]. So, the simplified boundary conditions (2) - (5) looks as following:

$$\begin{gathered} \sigma^*_{31}(x,y) = U_3(x,y) = \sigma^*_{32}(x,y) = 0;\ G_{T,y_3}(x,y) = 0 \Rightarrow \\ U_3(x,y) = U_{1,y_3}(x,y) = U_{3,y_1}(x,y) = U_{3,y_3}(x,y) = U_{3,y_2}(x,y) = U_{2,y_3}(x,y) = 0 \Rightarrow \\ \Theta_{,y_3}(x,y) = G_{1,y_3}(x,y) = G_{2,y_3}(x,y) = G_3(x,y) = G_{\Theta,y_3}(x,y) = \\ G_{T,y_3}(x,y) = 0;\ x,\xi \in V,\ y \equiv (y_1,y_2,0) \in \Gamma_{30} \end{gathered} \tag{10}$$

or

$$\begin{gathered} U_1(x,y) = \sigma^*_{33}(x,y) = U_2(x,y) = 0;\ G_T(y,\xi) = 0 \Rightarrow \\ U_1(x,y) = U_{1,y_1}(x,y) = U_{1,y_3}(x,y) = U_{3,y_3}(x,y) = U_{2,y_1}(x,y) = U_{2,y_3}(x,y) = 0 \Rightarrow \\ \Theta(x,y) = G_1(x,y) = G_{3,y_3}(x,y) = G_2(x,y) = G_\Theta(x,y) = G_T(x,y) = 0; \\ x \in V,\ y \equiv (y_1,y_2,0) \in \Gamma_{30} \end{gathered} \tag{11}$$

– on the boundary plane $\Gamma_{30} \equiv (-\infty \leq y_1, y_2 \leq \infty,\ y_3 = 0)$,

$$\begin{gathered}\sigma^*_{31}(x,y) = U_3(x,y) = \sigma^*_{32}(x,y) = 0;\ G_{T,y_3}(x,y)\ = 0 \Rightarrow U_{1,y_3}(x,y) = U_3(x,y) = \\ U_{3,y_1}(x,y) = U_{3,y_3}(x,y) = U_{2,y_3}(x,y) = 0 \Rightarrow \\ \Theta_{,y_3}(x,y) =\ G_{1,y_3}(x,y) =\ G_3(x,y) = G_{2,y_3}(x,y) = \\ G_{\Theta,y_3}(x,y) =\ G_{T,y_3}(x,y) = 0;\ x \in V,\ y \equiv (y_1, y_2, a_3) \in \Gamma_{31}\end{gathered} \tag{12}$$

or

$$\begin{gathered}U_1(x,y) = \sigma^*_{33}(x,y) = U_2(x,y) = 0;\ G_T(y,\xi) = 0 \Rightarrow U_1(x,y) = U_{1,y_1}(x,y) = \\ U_{1,y_3}(x,y) = U_{3,y_3}(x,y) = U_{2,y_1}(x,y) = U_{2,y_3}(x,y) = 0 \Rightarrow \\ \Theta(x,y) = G_1(x,y) = G_{3,y_3}(x,y) = G_2(x,y) = G_\Theta(x,y) = \\ G_T(x,y) = 0;\ x,\xi \in V,\ y \equiv (y_1, y_3, a_3) \in \Gamma_{31}\end{gathered} \tag{13}$$

on the boundary plane $\Gamma_{31} \equiv (-\infty \leq y_1, y_2 \leq \infty,\ y_3 = a_3)$.

Taken into account the boundary conditions (10) - (13) for Θ and G_Θ in the representation (8), TVD $\Theta(x,\xi)$ can be rewritten in the form:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda + 2\mu} G_\Theta(x,\xi) \tag{14}$$

because for each boundary planes Γ_{3l} of the layer V, due the mentioned above four links for any combinations of the boundary conditions (11) - (13), the integrals in Eq. (8) vanish. Therefore, the final formula for TVD looks as in Eq. (14). Also, from boundary conditions (11) - (13) for G_Θ and G_T follows $G_\Theta(x,\xi) = G_T(x,\xi)$.

5. Final simplified structural formulas for MTDGFs and TVD

Substituting, the proved constructive formula (14) and boundary conditions (11) - (13) into (9), the integral representations will be simplified and look as follows:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda + 2\mu} G_T(x,\xi) \tag{15}$$

– for TVD, and

$$U_1(x,\xi) = \frac{\gamma}{2(\lambda + 2\mu)}\left[\xi_1 G_T(x,\xi) - x_1 G_1(x,\xi)\right] \tag{16}$$

$$U_2(x,\xi) = \frac{\gamma}{2(\lambda + 2\mu)}\left[\xi_2 G_T(x,\xi) - x_2 G_2(x,\xi)\right] \tag{17}$$

$$\begin{gathered}U_3(x,\xi) = \frac{\gamma}{2(\lambda + 2\mu)}\left[\xi_3 G_T(x,\xi) - x_3 G_3(x,\xi) + \right. \\ \left. a_3 \int_{\Gamma_{31}} \left(\frac{\partial G_3(y,\xi)}{\partial n_{31}} - G_3(y,\xi)\frac{\partial}{\partial n_{31}}\right) G_T(x,y)\, d\Gamma_{31}(y)\right]\end{gathered} \tag{18}$$

– for structural formulas for MTDGFs.

The integral in Eq. (18) can be calculated as follows:

$$\begin{gathered}I_3 = a_3 \int_{\Gamma_{31}} \left(\frac{\partial G_3(y,\xi)}{\partial n_{31}} - G_3(y,\xi)\frac{\partial}{\partial n_{31}}\right) G_T(x,y)\, d\Gamma_{31}(y) = x_3 G_3(x,\xi) - \xi_3 G_T(x,y) - \\ - \int [\xi_1 G_{T,\xi_1}(x,\xi) - x_1 G_{1,\xi_1}(x,\xi)] d\xi_3 - \int \left(\xi_2 G_{T,\xi_2}(x,\xi) - x_2 G_{2,\xi_2}(x,\xi)\right) d\xi_3\end{gathered} \tag{19}$$

Thus, substituting (19) into (18) we obtain the following structural formula for MTGFs $U_3(x,\xi)$:

$$\begin{gathered}U_3(x,\xi) = -\frac{\gamma}{2(\lambda + 2\mu)}\left[\int [\xi_1 G_{T,\xi_1}(x,\xi) - x_1 G_{1,\xi_1}(x,\xi)] d\xi_3 \right. \\ \left. + \int \left(\xi_2 G_{T,\xi_2}(x,\xi) - x_2 G_{2,\xi_2}(x,\xi)\right) d\xi_3\right]\end{gathered} \tag{20}$$

6. The checking of the derived final MTDGFs and TVD

The displacements $U_i(x,\xi)$ in the Eqs (16), (17) and (20) have to satisfy with respect to variables $x \equiv (x_1, x_2, x_3)$ the following proved before equations [9, 10, 11]:

$$\nabla_x^2 U_i(x,\xi) = -\gamma \Theta^{(i)}(x,\xi) \tag{21}$$

where $\gamma = \alpha(2\lambda + 3\mu)$ is the thermoelastic constant, α is coefficient of linear temperature dilatation and λ, μ are Lame's constants of elasticity. According to the boundary conditions (2) and (4) and handbook [20] the influence functions for mechanical volume dilatation $\Theta^{(i)}(x,\xi)$, created by a unit concentrated force, is determining as follows:

$$\Theta^{(i)}(x,\xi) = -\frac{1}{\lambda + 2\mu}\frac{\partial}{\partial \xi_i} G_\Theta \tag{22}$$

Thus, according to the given above mechanical boundary conditions (2) and (4) and to the Problem 18.L.2 of the handbook [20] at boundary conditions

$$U_3^{(k)} = \sigma_{31}^{(k)} = \sigma_{32}^{(k)} = 0;\ x_3 = 0,\ a_3,\ -\infty \le x_1,\ x_2 \le \infty;\ k = 1,2,3 \tag{23}$$

we have

$$\begin{gathered}\Theta^{(i)}(x,\xi) = -\frac{1}{\lambda + 2\mu}\frac{\partial}{\partial \xi_i} G^{(2)}(x,\xi) = \\ \frac{1}{\pi a_3}\sum_{n=1}^{\infty} k_0(\mu_1 r)\cos\mu_1 x_3 \cos\mu_1 \xi_3 - \frac{1}{\pi a_3}\ln\ r; \mu_1 = \frac{n\pi}{a_3}\end{gathered} \tag{24}$$

The checking the equation (21) – (24)

$$\begin{gathered}\nabla_x^2 U_1(x,\xi) = \frac{\gamma}{2(\lambda + 2\mu)}\nabla_x^2\left[\xi_1 G_T(x,\xi) - x_1 G_1(x,\xi)\right] = \\ \frac{\gamma}{2(\lambda + 2\mu)}\left[\xi_1 \nabla_x^2 G_T(x,\xi) - \nabla_x^2\left(x_1 G_1(x,\xi)\right)\right] = \frac{\gamma}{2(\lambda + 2\mu)}\left(2\frac{\partial}{\partial x_1} G_1(x,\xi)\right) = \\ \frac{\gamma}{(\lambda + 2\mu)}\left(\frac{\partial}{\partial x_1} G^{(2)}(x,\xi)\right) = -\frac{\gamma}{(\lambda + 2\mu)}\frac{\partial}{\partial \xi_1} G^{(2)}(x,\xi)\end{gathered} \tag{25}$$

$$\begin{gathered}\nabla_x^2 U_2(x,\xi) = \tfrac{\gamma}{2(\lambda+2\mu)}\left[\nabla_x^2 \xi_2 G_T(x,\xi) - \nabla_x^2 x_2 G_2(x,\xi)\right] = \\ -\tfrac{\gamma}{(\lambda+2\mu)}\tfrac{\partial}{\partial x_2} G_2(x,\xi) = -\tfrac{\gamma}{(\lambda+2\mu)}\tfrac{\partial}{\partial \xi_2} G^{(2)}(x,\xi)\end{gathered} \tag{26}$$

$$\begin{gathered}\nabla_x^2 U_2(x,\xi) = \tfrac{\gamma}{2(\lambda+2\mu)}\left[\nabla_x^2 \xi_2 G_T(x,\xi) - \nabla_x^2 x_2 G_2(x,\xi)\right] = \\ -\tfrac{\gamma}{(\lambda+2\mu)}\tfrac{\partial}{\partial x_2} G_2(x,\xi) = -\tfrac{\gamma}{(\lambda+2\mu)}\tfrac{\partial}{\partial \xi_2} G^{(2)}(x,\xi)\end{gathered} \tag{27}$$

In addition with respect to variable $\xi \equiv (\xi_1, \xi_2, \xi_3)$ the displacements $U_i(x,\xi)$ have satisfy the equations [9, 10, 11]:

$$\mu \nabla_\xi^2 U_i(x,\xi) + (\lambda + \mu)\Theta,_{\xi_i}(x,\xi) - \gamma G_{T,\xi_i}(x,\xi) = 0 \tag{28}$$

The checking the equation (28)

$$\begin{gathered}\mu \nabla_\xi^2 U_1(x,\xi) = \tfrac{\gamma\mu}{2(\lambda+2\mu)}\left[\xi_1 G_T(x,\xi) - x_1 G_1(x,\xi)\right] = \tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial \xi_1} G_T(x,\xi); \\ (\lambda + \mu)\Theta,_{\xi_1}(x,\xi) - \gamma G_{T,\xi_1}(x,\xi) = \tfrac{(\lambda+\mu)\gamma}{\lambda+2\mu}\tfrac{\partial}{\partial \xi_1} G_T(x,\xi) - \tfrac{\gamma\partial}{\partial \xi_1} G_T(x,\xi) = \\ = -\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial \xi_1} G_T(x,\xi); \\ \mu \nabla_\xi^2 U_1(x,\xi) + (\lambda + \mu)\Theta,_{\xi_1}(x,\xi) - \gamma G_{T,\xi_1}(x,\xi) = \\ = \tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial \xi_1} G_T(x,\xi) - \tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial \xi_1} G_T(x,\xi) = 0\end{gathered} \tag{29}$$

$$\begin{gathered}\mu\nabla_\xi^2 U_2(x,\xi)=\tfrac{\gamma\mu}{2(\lambda+2\mu)}\left[\xi_2 G_T(x,\xi)-x_2 G_1(x,\xi)\right]=\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_2}G_T(x,\xi);\\ (\lambda+\mu)\,\Theta_{,\xi_2}(x,\xi)-\gamma G_{T,\xi_2}(x,\xi)=\\ =\tfrac{(\lambda+\mu)\gamma}{\lambda+2\mu}\tfrac{\partial}{\partial\xi_2}G_T(x,\xi)-\tfrac{\gamma\partial}{\partial\xi_2}G_T(x,\xi)=-\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_2}G_T(x,\xi);\\ \mu\nabla_\xi^2 U_2(x,\xi)+(\lambda+\mu)\,\Theta_{,\xi_2}(x,\xi)-\gamma G_{T,\xi_2}(x,\xi)=\\ =\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_2}G_T(x,\xi)-\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_2}G_T(x,\xi)=0\end{gathered}\tag{30}$$

$$\begin{gathered}\mu\nabla_\xi^2 U_3(x,\xi)=\tfrac{\gamma\mu}{2(\lambda+2\mu)}\left[-\int[\xi_1 G_{T,\xi_1}(x,\xi)-x_1 G_{1,\xi_1}(x,\xi)]d\xi_3-\right.\\ \left.-\int\left(\xi_2 G_{T,\xi_2}(x,\xi)-x_2 G_{2,\xi_2}(x,\xi)\right)d\xi_3\right]=\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_3}G_T;\\ (\lambda+\mu)\,\Theta_{,\xi_3}(x,\xi)-\gamma G_{T,\xi_3}(x,\xi)=\\ =\tfrac{(\lambda+\mu)\gamma}{\lambda+2\mu}\tfrac{\partial}{\partial\xi_3}G_T(x,\xi)-\tfrac{\gamma\partial}{\partial\xi_3}G_T(x,\xi)=-\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_3}G_T(x,\xi);\\ \mu\nabla_\xi^2 U_3(x,\xi)+(\lambda+\mu)\,\Theta_{,\xi_3}(x,\xi)-\gamma G_{T,\xi_3}(x,\xi)=\\ =\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_3}G_T(x,\xi)-\tfrac{\gamma\mu}{(\lambda+2\mu)}\tfrac{\partial}{\partial\xi_3}G_T(x,\xi)=0\end{gathered}\tag{31}$$

Finally, the TVD calculated on the basis of structural formulas (16), (17) and (20) have to be equal to the respective thermoelastic volume dilatation given by equation (15)

$$U_{i,i}(x,\xi)=\Theta(x,\xi)=\frac{\gamma}{\lambda+2\mu}G_T(x,\xi)\tag{32}$$

The checking the equation (32)

$$U_{1,1}(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)}\left[G_T(x,\xi)+\xi_1 G_{T,\xi_1}(x,\xi)-x_1 G_{1,\xi_1}(x,\xi)\right]\tag{33}$$

$$U_{2,2}(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)}\left[G_T(x,\xi)+\xi_2 G_{T,\xi_2}(x,\xi)-x_2 G_{2,\xi_2}(x,\xi)\right]\tag{34}$$

$$\begin{gathered}U_{3,3}(x,\xi)=\\ =-\tfrac{\gamma}{2(\lambda+2\mu)}\left[\xi_1 G_{T,\xi_1}(x,\xi)-x_1 G_{1,\xi_1}(x,\xi)+\left(\xi_2 G_{T,\xi_2}(x,\xi)-x_2 G_{2,\xi_2}(x,\xi)\right)\right]\end{gathered}\tag{35}$$

Thus, from Eqs (33) - (35) follows that Eq. (32) is satisfied.

7. Analytical expressions for MTDGFs

Finally, to obtain analytical expressions for MTDGFs $U_i(x,\xi)$ we have to use the respective analytical expressions for Green's functions for Poisson's equation: $G_1, G_2, G_3$and G_T in according to respective boundary conditions (2) and (4). So, using the handbook [20] (see the Problems 18.P.1–18.P.2 and answers to them) we obtain the following analytical expressions:

$$\begin{gathered}G_3=G^{(1)}=(\pi a_3)^{-1}\textstyle\sum_{n=1}^{\infty}k_0(\mu_1 r)\sin\mu_1 x_3\sin\mu_1\xi_3;\ \mu_1=n\pi a_3^{-1};\\ G_1=G_2=G_T=G^{(2)}=\tfrac{1}{\pi a_3}\textstyle\sum_{n=1}^{\infty}k_0(\mu_1 r)\cos\mu_1 x_3\cos\mu_1\xi_3-\tfrac{1}{\pi a_3}\ln\ r+b;\ \mu_1=\tfrac{n\pi}{a_3}\end{gathered}\tag{36}$$

Here, and hereafter in this section $k_0(\mu_1 r)$ are Bessel functions (or cylindrical functions) of the zero-order of the second type, respectively; $r=\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}$. Thus, substituting the functions (36) into (16), (17) and (20) we obtain:

$$\begin{gathered}U_1(x,\xi)=\tfrac{\gamma}{2(\lambda+2\mu)\pi a_3}(\xi_1-x_1)\left(\textstyle\sum_{n=1}^{\infty}k_0(\mu_1 r)\cos\mu_1 x_3\cos\mu_1\xi_3-\tfrac{1}{\pi a_3}\ln r+\tfrac{b}{\pi a_3}\right);\\ \mu_1=\tfrac{n\pi}{a_3}\end{gathered}\tag{37}$$

$$U_2(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)\pi a_3}(\xi_2-x_2)\left(\sum_{n=1}^{\infty}k_0(\mu_1 r)\cos\mu_1 x_3\cos\mu_1\xi_3-\ln\ r+\frac{b}{\pi a_3}\right)\tag{38}$$

$$U_3(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)\pi a_3}\left[\xi_3 r-\sum_{n=1}^{\infty}rk_1(\mu_1 r)\cos\mu_1 x_3\sin\mu_1\xi_3\right]\tag{39}$$

where the relation $\frac{dk_0(\mu_1 r)}{d(\mu_1 r)} = -k_1(\mu_1 r)$ was taken into account.

As could be seen the MTDGFs (37) - (39) satisfy boundary conditions for temperature Green's function $G_T(x, \xi)$ (see boundary conditions (2) and (4) with respect to point of application $x \equiv (x_1, x_2, x_3)$ of the displacements and to mechanical boundary conditions (2) and (4) with respect to point of application the heat source $\xi \equiv (\xi_1, \xi_2, \xi_3)$).

In analogical way, to obtain analytical expressions for MTDGFs $U_i(x, \xi)$ in Eqs (16), (17) and (20) for remained three BVPs we have to use the respective analytical expressions for Green's functions for Poisson's equation: G_1, G_2, G_3 and G_T in according to the following three combinations of boundary conditions: (2), (5); (3), (4) and (3), (5). Thus, using the handbook [20] (see the Problems 18.P.3–18.P.4 and answers to them) we obtain the following analytical expressions for Green's functions:

$$\begin{gathered} G_1 = G_2 = G_T = G^{(4)} = \frac{1}{\pi a_3}\sum_{n=1}^{\infty} k_0(\mu_1 r)\cos\mu_1 x_3 \cos\mu_1\xi_3;\ \mu_1 = (2n-1)\frac{\pi}{2a_3} \\ G_3 = G^{(3)} = \frac{1}{\pi a_3}\sum_{n=1}^{\infty} k_0(\mu_1 r)\sin\mu_1 x_3 \sin\mu_1\xi_3;\ \mu_1 = (2n-1)\frac{\pi}{2a_3} \end{gathered} \tag{40}$$

– for the combination (2), (5) of boundary conditions;

$$\begin{gathered} G_1 = G_2 = G_T = G^{(3)} = \frac{1}{\pi a_3}\sum_{n=1}^{\infty} k_0(\mu_1 r)\sin\mu_1 x_3 \sin\mu_1\xi_3;\ \mu_1 = (2n-1)\frac{\pi}{2a_3}, \\ G_3 = G^{(4)} = \frac{1}{\pi a_3}\sum_{n=1}^{\infty} k_0(\mu_1 r)\cos\mu_1 x_3 \cos\mu_1\xi_3;\ \ \mu_1 = (2n-1)\frac{\pi}{2a_3} \end{gathered} \tag{41}$$

– for the combination (3), (4) of boundary conditions;

$$\begin{gathered} G_3 = G^{(2)} = \frac{1}{\pi a_3}\sum_{n=1}^{\infty} k_0(\mu_1 r)\cos\mu_1 x_3 \cos\mu_1\xi_3 - \frac{1}{\pi a_3}\ln r;\ \mu_1 = \frac{n\pi}{a_3}; \\ G_1 = G_2 = G_T = G^{(1)} = (\pi a_3)^{-1}\sum_{n=1}^{\infty} k_0(\mu_1 r)\sin\mu_1 x_3 \sin\mu_1\xi_3;\ \mu_1 = n\pi a_3^{-1} \end{gathered} \tag{42}$$

– for the combination (3), (5) of boundary conditions.

Next, by using the Duhamel-Newman law, on the base of expressions (37) - (39), we can obtain the analytical expressions for thermal stresses Green's functions:

$$\sigma_{ij}^* = \mu(U_{i,j} + U_{j,i}) + \lambda(\Theta - \gamma G_T). \tag{43}$$

8. Integration formulas for thermoelastic displacements and stresses

At the final steep by using general integration formulas for displacements $u_i(x)$ and stresses

$$\sigma_{ij}(x)\,[9-14]:$$

$$\begin{gathered} u_i(\xi) = a^{-1}\int_V F(x)\,U_i(x,\xi)\,dV(x) - \int_{\Gamma_D} T(y)\frac{\partial U_i(y,\xi)}{\partial n_y}d\Gamma_D(y) \\ + \int_{\Gamma_N}\frac{\partial T(y)}{\partial n_y}U_i(y,\xi)\,d\Gamma_N(y) + a^{-1}\int_{\Gamma_M}\left[\alpha T(y) + a\frac{\partial T(y)}{\partial n_y}\right]U_i(y,\xi)\,d\Gamma_M(y); \\ i = 1,2,3 \end{gathered} \tag{44}$$

$$\begin{gathered} \sigma_{ij}(\xi) = a^{-1}\int_V F(x)\,\sigma_{ij}^*(x,\xi)\,dV(x) - \int_{\Gamma_D} T(y)\frac{\partial \sigma_{ij}^*(y,\xi)}{\partial n_y}d\Gamma_D(y) \\ + \int_{\Gamma_N}\frac{\partial T(y)}{\partial n_y}\sigma_{ij}^*(y,\xi)\,d\Gamma_N(y) + a^{-1}\int_{\Gamma_M}\left[\alpha T(y) + a\frac{\partial T(y)}{\partial n_y}\right]\sigma_{ij}^*(y,\xi)\,d\Gamma_M(y) \end{gathered} \tag{45}$$

we can to obtain the respective integration formulas for displacements $u_i(x)$ and for thermal stresses $\sigma_{ij}(x)$ within layer, created by the inner heat source $F(x)$, temperature $T(y)$ and thermal flux $\partial T(y)/\partial n_y$ given on the boundaries plane Γ_{30} and Γ_{31} in the forms:

$$\begin{gathered} u_i(x) = a^{-1}\int_V F(x)\,U_i(x,\xi)\,dV(x) - \\ -\int_{\Gamma_{30}}\left[\frac{\partial U_i(y,x)}{\partial n_y} - U_i(y,x)\frac{\partial}{\partial n_y}\right]T_{30}(y)\,d\Gamma_{30}(y) - \\ -\int_{\Gamma_{30}}\left[\frac{\partial U_i(y,x)}{\partial n_y} - U_i(y,x)\frac{\partial}{\partial n_y}\right]T_{31}(y)\,d\Gamma_{31}(y) \end{gathered} \tag{46}$$

$$\begin{aligned}\sigma_{ij}(x) = a^{-1}\int_V F(x)\,\sigma_{ij}^{*}(y,x)\,dV(x) - \\ - \int_{\Gamma_{30}}\left[\frac{\partial\sigma_{ij}^{*}(y,x)}{\partial n_y} - \sigma_{ij}^{*}(y,x)\frac{\partial}{\partial n_y}\right] T_{30}(y)\,d\Gamma_{30}(y) - \\ - \int_{\Gamma_{30}}\left[\frac{\partial\sigma_{ij}^{*}(y,x)}{\partial n_y} - \sigma_{ij}^{*}(y,x)\frac{\partial}{\partial n_y}\right] T_{31}(y)\,d\Gamma_{31}(y)\end{aligned} \qquad (47)$$

Thus on the base of the formulas (46) and (47) the readers will be able to obtain many particular solutions of 3D BVPs for the thermoelastic layer.

9. CONCLUSION

The structural formulas and analytical expressions for MTDGFs and TVD to four new BVPs for a themoelastic layer are obtained for the first time by using HIRM.

References

[1] Boley, B.A. and Weiner, J.F., *Theory of Thermal Stresses*, Wiley, New York, 1960.

[2] Kovalenko, A.D., *Fundamentals of Thermoelasticity*, Naukova Dumka, Kiev, Russian, 1970.

[3] Mayzel, V.M., *The Temperature Problem of the Theory of Elasticity*, AN SSSR Publisher, Kiev, 1951.

[4] Melan, E. and Parkus, H., *Thermoelastic Stresses Caused by the Stationary Heat Fields*, Fizmatgiz, Moscow, 1958.

[5] Nowacki, W., *The Theory of Elasticity*. Mir, Moscow, 1975.

[6] Nowacki, W., *Thermoelasticity*, Pergamon Press and Polish Science Publishers, Oxford, Warszawa, 1962.

[7] Nowinski, J.L., *Theory of Thermoelasticity with Applications*, Sijthoff and Noordhoff International Publishers, Alphen Aan Den Rijn, 1978.

[8] Hetnarski, R.B. and Eslami, M.R., *Thermal Stresses—Advanced Theory and Applications*, Springer, Dordrecht, 2009.

[9] Seremet, V., *A New Approach to Constructing Green's Functions and Integral Solutions in Thermoelasticity*, Acta Mechanica **225** (3), 737–755, 2014.

[10] Seremet, V., *Recent Integral Representations for Thermoelastic Green's Functions and Many Examples of Their Exact Analytical Expressions*, Journal of Thermal Stresses **37** (5), 561–584, 2014.

[11] Seremet, V., *A New Efficient Unified Method to Derive New Constructive Formulas and Explicit Expressions for Plane and Spatial Thermoelastic Green's Functions*, Acta Mechanica **226** (1), 211–230, 2015.

[12] Seremet, V. and Carrera, E., *Solution in Elementary Functions to a BVP of Thermoelasticity: Green's Functions and Green's-Type Integral Formula for Thermal Stresses Within a Half-strip*, Journal of Thermal Stresses **37** (8), 947–968, 2014.

[13] Seremet, V., Wang, H., *Thermoelastic Equilibrium of Some Semi-Infinite Domains Subjected to the Action of a Heat Source*, Journal of Thermal Stresses **38** (5), 509–525, 2015.

[14] Seremet, V., Wang, H., *Two-Dimensional Green's function for thermal stresses in a semi-layer under a point heat source*, Journal of Thermal Stresses **38** (7), 756–774, 2015.

[15] Şeremet, V., *Steady-state Green's functions for thermal stresses within rectangular region under point heat source*, Journal of Thermal Stresses **39** (8), 906–927, 2016.

[16] Şeremet, V., *A method to derive thermoelastic Green's functions for bounded domains (on examples of two-dimensional problems for parallelepipeds)*, Acta Mechanica, pp. 1–18, 2016, doi:10.1007/s00707-016-1680-8.

[17] Peng-Fei Hou, Qiu-Hua, Hai-Yang Jiang, *Three-Dimensional steady-state general solution for isotropic thermoelastic materials with applications II. Green's Functions for two-phase infinite body*, Journal of Thermal Stresses **36** (8), 851–867, 2015.

[18] Qing-Hua Qin, *Thermoelectroelastic Green's function for a piezoelectric plate containing an elliptic hole*, Mechanics of Materials **30** (1), 21–29, 1998.

[19] Ernian Pan, Weiqiu Chen, *Static Green's Functions in Anisotropic Media*, Cambridge University Press, 2015.

[20] Şeremet, V.D., *Handbook on Green's Functions and Matrices*, WIT Press, Southampton, Boston, 2003.

Agrarian State University of Moldova,
Laboratory of Green's functions,
Mircești street 44, Chisinau MD 2049, Moldova.

Institute of Mathematics and Computer Sciences,
Academy of Science of Moldova, Laboratory of Mathematical Modeling,
Academiei 5 street, Chisinau MD 2028.
E-mail address: v.seremet@uasm.md