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GENERALIZED BLASIUS EQUATION: EXISTENCE, UNIQUENESS AND ANALYTICAL APPROXIMATION SOLUTION

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ABSTRACT. In this paper, we consider the following nonlinear boundary value problem with varying coefficients for the generalized Blasius equation proposed in [1] as an open problem:

$$\begin{cases} (a(\eta)f''(\eta))' + f(\eta)f''(\eta) &= 0,\ 0 < \eta < \infty, \\ f(0) = 0,\ f'(0) = 0,\ f'(\infty) &= 1, \end{cases}$$

where $a(\eta) > 0$ is a positive function of η. This problem contains the classical Blasius equation as a special case when $a(\eta)$ is a constant, that is, $a(\eta) = a$.

Our first attempt is to discuss, using the classical fixed point theorem, the existence and uniqueness of the solution of this nonlinear problem. Some properties for $a(\eta)$ constant are also exhibited. Then, a highly accurate analytic approximation of the solution of this problem is provided by using the Adomian decomposition method.

1. Introduction

In [1], the authors proposed the following nonlinear boundary value problem:

$$\begin{cases} (a(\eta)f''(\eta))' + f(\eta)f''(\eta) &= 0,\ 0 < \eta < \infty, \\ f(0) = 0,\ f'(0) = 0,\ f'(\infty) &= 1, \end{cases} \tag{1}$$

where $a(\eta) > 0$ is a continuous function of η, as an open problem, which can be found in boundary layer flows with temperature-dependent viscosity for engineering applications. This problem contains the well-known nonlinear third-order differential Blasius equation as a special case [2]:

$$\begin{cases} af'''(\eta) + f(\eta)f''(\eta) &= 0,\ 0 < \eta < \infty, \\ f(0) = 0,\ f'(0) = 0,\ f'(\infty) &= 1, \end{cases} \tag{2}$$

where a is a constant, which can be described as the non-dimensional velocity distribution in the laminar boundary layer over a flat plate. The Blasius equation is one of the basic equations of fluid dynamics which describes the velocity profile of the fluid in the boundary layer theory on a half-infinite interval. Different analytical approximation methods have been proposed to deal with the nonlinear Blasius equation (2) [3]-[7].

It is often very difficult, if not impossible, to find exact solutions of such differential equations. Moreover, at our knowledge, the existence and uniqueness theorem of Pr.(1) has not been yet proved.

The aim of this paper is double. Our first attempt is to discuss, by using the classical fixed point theorem [8]-[9], the existence and uniqueness of the solution of our problem. Some properties for $a(\eta) = a$ (a constant) will also be exhibited. Secondly, a highly accurate analytic approximation of the solution of this problem is provided by using the Adomian decomposition method [10]-[26].

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2. Fundamental lemmas

The following nonlinear differential equation

$$(a(\eta)f''(\eta))' + f(\eta)f''(\eta) = 0,\ 0 < \eta < \infty \tag{3}$$

can be written in an equivalent form as

$$\frac{(a(\eta)f''(\eta))'}{a(\eta)f''(\eta)} = -\frac{1}{a(\eta)}f(\eta),\ 0 < \eta < \infty\ , \tag{4}$$

since functions such $f''(\eta) = 0$ cannot be considered because they don't satisfy the boundary conditions.

Integrating Eq. (4) from 0 to η, we obtain

$$a(\eta)f''(\eta) = C_1 \exp\left(-\int_0^\eta \frac{f(\xi)}{a(\xi)}d\xi\right),\ 0 < \eta < \infty\ , \tag{5}$$

where $C_1 = a(0)f''(0)$ is an unknown constant.
Hence,

$$f'(\eta) = C_1 \int_0^\eta \frac{1}{a(\xi)} \exp\left(-\int_0^\xi \frac{f(t)}{a(t)}dt\right) d\xi,\ 0 < \eta < \infty\ , \tag{6}$$

and by integration by parts,

$$f(\eta) = C_1 \int_0^\eta (\eta - \zeta)\frac{1}{a(\zeta)} \exp\left(-\int_0^\zeta \frac{f(t)}{a(t)}dt\right) d\zeta,\ 0 < \eta < \infty\ , \tag{7}$$

which is a nonlinear Volterra integral equation of the first kind that will be used in the sequel, for calculating the analytical approximate solution by the Adomian decomposition method. The constant $C_1 = a(0)f''(0)$ can also be determined by using the third boundary condition $f'(\infty) = 1$ and Eq. (6) to get

$$C_1 = \frac{1}{\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{f(\xi)}{a(\xi)}d\xi\right) d\eta}\ . \tag{8}$$

We can also mix both conditions by defining

$$C_1 = (1-\alpha)\ a(0)f''(0) + \alpha\ \frac{1}{\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{f(\xi)}{a(\xi)}d\xi\right) d\eta},\ (0 \le \alpha \le 1)\ . \tag{9}$$

For $0 \le \eta \le \infty$, we have the following inequalities.

Lemma 1.

$$0 \le f''(\eta) \le \frac{C_1}{a(\eta)}\ ;\ 0 \le f'(\eta) \le C_1 \int_0^\eta \frac{1}{a(t)}\ dt\ ;\ 0 \le f(\eta) \le C_1 \int_0^\eta \frac{\eta - t}{a(t)}\ dt\ , \tag{10}$$

$$\frac{1}{\int_0^\infty \frac{1}{a(t)}\ dt} \le\ C_1 = a(0)f''(0) \le \frac{1}{\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{\xi}{a(\xi)}d\xi\right) d\eta}\ . \tag{11}$$

Proof. Inequalities (10) are easy to obtain from Equations (5) to (7) since $\exp(-\eta) \le \exp(0)$ for $\eta \ge 0$. As $a(\eta) > 0$, from Eq. (5) $\text{sign}(f''(\eta)) = \text{sign}(f''(0))$. The boundary conditions $f'(0) = 0$ and $f'(\infty) = 1$ imply that $f''(\eta) > 0$. Thus, $f'(\eta)$ is increasing and positive on $[0, \infty)$. Consequently, $f(\eta)$ is increasing and positive since $f(\eta) > f(0) = 0$. Using Eq. (4) we deduce, since $a(\eta)f''(\eta) > 0$, that $(a(\eta)f''(\eta))' < 0$ for $\eta \ge 0$.

Futhermore, as $f'(\eta)$ increases for $\eta \geq 0$ and $0 \leq f'(\eta) \leq 1$ we have that $0 \leq f(\eta) \leq \eta$ for $\eta \geq 0$. Thus

$$\int_0^\infty \frac{1}{a(\eta)} d\eta \geq \int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{f(\xi)}{a(\xi)} d\xi\right) d\eta \geq \int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{\xi}{a(\xi)} d\xi\right) d\eta. \quad (12)$$

In view of this we obtain the inequality (11). □

Lemma 2. *If* $0 < a_{\min} \leq a(x) \leq a_{\max}$ *is bounded on* $[0, \infty)$*, then*

$$0 \leq C_1 = a(0) f''(0) \leq \frac{2a_{\max}}{\sqrt{2\pi a_{\min}}}. \quad (13)$$

Proof. Let $\theta(\eta) = -\int_0^\eta \frac{f(t)}{a(t)}\, dt$.
Since $0 \leq f(\eta) \leq \eta$ for $\eta \geq 0$, we have $\theta(\eta) \geq \frac{-\eta^2}{2a_{\min}}$. We obtain that

$$\frac{1}{a(\eta)} \; \exp\left(\theta(\eta)\right) \; \geq \frac{\exp\left(-\frac{\eta^2}{2a_{\min}}\right)}{a_{\max}}. \quad (14)$$

From Lemma 1 and using that $\int_0^\infty \exp\left(-\frac{\eta^2}{2a}\right) d\eta = \frac{1}{2}\sqrt{2a\pi}$, we complete the proof. □

In particular, if $a(\eta) = a$, Eq. (4) leads to

$$f''(\eta) = \widetilde{C}_1 \exp\left(-\frac{1}{a}\int_0^\eta f(\xi) d\xi\right), \;\; 0 < \eta < \infty, \quad (15)$$

and we have the following:

Lemma 3. *For* $a(\eta) = a$ *constant* [1]*, we have*

$$0 < \widetilde{C}_1 = f''(0) \leq \sqrt{\frac{2}{a\pi}}; \quad (16)$$

$$\textit{If} \;\; a = 2, \widetilde{C}_1 \leq \sqrt{\frac{1}{\pi}} \simeq 0.56419 \;. \quad (17)$$

Proof. Establishing a similar estimation as Eq. (11) in Lemma 1 for the case $a(\eta) = a$ and using that $\int_0^\infty \exp\left(-\frac{\eta^2}{2a}\right) d\eta = \frac{1}{2}\sqrt{2a\pi}$, we obtain that

$$\int_0^\infty \exp\left(-\frac{1}{a}\int_0^\eta f(\xi) d\xi\right) d\eta \geq \frac{1}{2}\sqrt{2a\pi}. \quad (18)$$

This completes the proof. □

A better estimate for $\widetilde{C}_1$ is given by the following Lemma :

Lemma 4. *For* $a(\eta) = a$ *(constant) , we have*

$$0 < \widetilde{C}_1 = f''(0) \leq \frac{3^{\frac{2}{3}}}{(2\pi a^3)^{\frac{1}{6}}\, \Gamma(\frac{1}{3})}; \quad (19)$$

$$\textit{If} \;\; a = 2, \widetilde{C}_1 \leq \frac{\left(\frac{3}{2}\right)^{\frac{2}{3}}}{\sqrt[6]{\pi}\, \Gamma(\frac{1}{3})} \simeq 0.404178 \;. \quad (20)$$

Proof. First, we have $f(\eta) \leq \frac{\eta^2}{\sqrt{2a\pi}}$ deduced from Eq. (16) and the fact that $f(\eta) \leq \widetilde{C}_1 \int_o^\eta \left(\int_0^\xi \, d\zeta\right) d\xi$.

Then, from $f'(\infty) = 1 \geq \widetilde{C}_1 \int_0^\infty \exp\left(-\frac{1}{a} \int_0^\xi \frac{\eta^2}{\sqrt{2a\pi}}\, dt\right) d\xi$ and noticing that $\int_0^\infty \exp\left(-\frac{x^3}{\sqrt{18a^3\pi}}\right)\, dx = \frac{\Gamma(\frac{1}{3})\left(2\pi a^3\right)^{\frac{1}{6}}}{3^{\frac{2}{3}}}$, we complete the proof. □

Remark 1. *Using as initial iterate Eq. (16), we can construct a recursive procedure to get better estimates for the decreasing sequence* $\widetilde{C}_1$ *given by*

$$\widetilde{C}_{1,new} \leq \frac{1}{\int_0^\infty \exp\left(-\frac{\widetilde{C}_{1,old}}{6a}\, x^3\right)\, dx} \tag{21}$$

Consider the case $a = 2$*, simple numerical computations show that* $\widetilde{C}_1$ *is bounded by an asymptotic value* 0.3420953217*. Moreover, the convergence to this value does not depend on the initial value of* $\widetilde{C}_1$ *since the sequence of* $\widetilde{C}_1$ *is decreasing and bounded by 0.*

3. Existence and uniqueness of the problem

Pr.(1) can be reformulated as an integral equation and may be written as a fixed point equation

$$f = T_a(f), \tag{22}$$

where $T_a : \Omega \to \Omega$ is defined by

$$T_a\left(f(t)\right) = C_1 \int_0^\eta (\eta - \zeta)\frac{1}{a(\zeta)} \exp\left(-\int_0^\zeta \frac{f(t)}{a(t)} dt\right) d\zeta \tag{23}$$

and Ω is the subset of all real-valued continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$\Omega = \left\{ f \in \mathcal{C}\left([0, \infty[\right), \|f\|_{a,\infty} < \infty \right\} . \tag{24}$$

Here, $\|f\|_{a,\infty} = \left\|\frac{f}{a}\right\|_\infty = \sup_{x\in\mathbb{R}^+} \left|\frac{f(x)}{a(x)}\right|$ is a weighted supremum norm.
Thus, a solution of Pr. (1) is equivalent to a solution of Eq. (22).
Let f defined by equation (7) and g defined as

$$g(\eta) = C_2 \int_0^\eta (\eta - \zeta)\frac{1}{a(\zeta)} \exp\left(-\int_0^\zeta \frac{g(t)}{a(t)} dt\right) d\zeta,\ 0 < \eta < \infty\ , \tag{25}$$

with $C_2 = a(0)g''(0)$. We need the following lemma:

Lemma 5. *Let* f *and* g *be two solutions to Problem (1). Then,*

$$\begin{aligned} |C_2 - C_1| &= a(0)\left|g''(0) - f''(0)\right| \\ &\leq \frac{\int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 d\eta}{\left(\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{\xi}{a(\xi)} d\xi\right) d\eta\right)^2}\ \|g - f\|_{a,\infty}\ . \end{aligned} \tag{26}$$

Proof. Let 's define the mapping $f \mapsto \theta_f(\eta) = -\int_0^\eta \frac{f(t)}{a(t)}\, dt$. We notice that

$$|\theta_f(\eta)| \leq \eta\, \|f\|_{a,\infty}\ . \tag{27}$$

Applying the Mean Value Theorem to the function $\Phi_\eta(f) = \exp\left(\theta_f(\eta)\right)$, we get

$$\begin{aligned}|\Phi_\eta(g) - \Phi_\eta(f)| &= \left|\exp\left(\theta_g(\eta)\right) - \exp\left(\theta_f(\eta)\right)\right| \\ &\leq \frac{h(\eta)}{a(\eta)} \exp\left(\theta_h(\eta)\right) \ |\theta_g(\eta) - \theta_f(\eta)| \ , \\ &\leq \frac{\eta^2}{a(\eta)} \, \|g - f\|_{a,\infty} \end{aligned} \tag{28}$$

where $f(\eta) \leq h(\eta) \leq g(\eta) \leq \eta$. Thus, we have

$$\begin{aligned}|C_1 - C_2| &= \left| \frac{1}{\int_0^\infty \frac{1}{a(\eta)} \ \Phi_\eta(f) \ d\eta} - \frac{1}{\int_0^\infty \frac{1}{a(\eta)} \ \Phi_\eta(g) \ d\eta} \right| \ , \\ &\leq C_1 C_2 \int_0^\infty \frac{1}{a(\eta)} \ \ |\Phi_\eta(g) - \Phi_\eta(f)| \ \ d\eta \ , \\ &\leq C_1 C_2 \ \ \|f - g\|_{a,\infty} \ \int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 d\eta. \end{aligned} \tag{29}$$

The result is straightforward. □

Consequently, for $0 \leq \xi \leq \eta$, we have

$$\begin{aligned}|T_a(g) - T_a(f)| \leq C_1 \int_0^\eta \frac{(\eta - \zeta)}{a(\zeta)} \, |\Phi_\zeta(g) - \Phi_\zeta(f)| \ \ d\zeta \quad + \\ |C_2 - C_1| \ \int_0^\eta \frac{(\eta - \zeta)}{a(\zeta)} \, |\Phi_\zeta(g)| \ \ d\zeta \ . \end{aligned} \tag{30}$$

Using Lemma 6, Eq. (28) and Eq. (29) and since $\Phi_\zeta(g) \leq 1$, we obtain

$$\begin{aligned}|T_a(g) - T_a(g)| \leq C_1 \ \ \|g - f\|_{a,\infty} \ \left(\int_0^\eta (\eta - \zeta) \left(\frac{\zeta}{a(\zeta)}\right)^2 \ d\zeta\right) \quad + \\ C_1 C_2 \, \|g - f\|_{a,\infty} \left(\int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 \ d\eta\right) \left(\int_0^\eta \frac{(\eta - \zeta)}{a(\zeta)} d\zeta\right) \ . \end{aligned} \tag{31}$$

and, since $(\eta - \xi) \leq \eta$ on $[0, \eta]$,

$$\begin{aligned}|T_a(g) - T_a(f)| \leq C_1 \ \eta \ \ \|g - f\|_{a,\infty} \ \left(\int_0^\eta \left(\frac{\zeta}{a(\zeta)}\right)^2 \ d\zeta\right) \quad + \\ C_1 C_2 \ \eta \, \|g - f\|_{a,\infty} \left(\int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 \ d\eta\right) \left(\int_0^\eta \frac{1}{a(\zeta)} d\zeta\right) \ . \end{aligned} \tag{32}$$

Let

$$L_1(\eta) = \left[C_1 \left(\int_0^\eta \left(\frac{\zeta}{a(\zeta)}\right)^2 \ d\zeta\right) + C_1 C_2 \left(\int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 \ d\eta\right) \left(\int_0^\eta \frac{1}{a(\zeta)} d\zeta\right)\right] \ \left(\frac{\eta}{a(\eta)}\right) \ . \tag{33}$$

We observe first that $L_1(\eta) < L_2(\eta)$, where

$$L_2(\eta) = \left(\frac{\eta}{a(\eta)}\right) \ \left(\int_0^\infty \left(\frac{\zeta}{a(\zeta)}\right)^2 \ d\zeta\right) \left(C_1 + C_1 C_2 \ \int_0^\infty \frac{1}{a(\zeta)} d\zeta\right). \tag{34}$$

The function $a(\eta)$ which guarantees the convergence of all integrals (see section 4), will be chosen such that $\frac{\eta}{a(\eta)}$ bounded on $[0,\infty)$. Let M be the supremum of $\frac{\eta}{a(\eta)}$, then we obtain the following inequality

$$\|T_a(g)-T_a(f)\|_{a,\infty} \leq L \ \|g-f\|_{a,\infty} \ . \tag{35}$$

with

$$L = M \left(\int_0^\infty \left(\frac{\zeta}{a(\zeta)} \right)^2 d\zeta \right) \left(C_1 + C_1 C_2 \int_0^\infty \frac{1}{a(\zeta)} d\zeta \right). \tag{36}$$

In order to show that T_a is a contractive mapping, the expression in Eq. (36) must be less than 1. It is obvious that the choice of the function $a(x)$ have to guarantee this condition.

4. Choice of the fonction $a(\eta)$

Eq. (36) contains two integrals whose convergence depends on the parameter $a(\eta) > 0$ for $0 \leq \eta < \infty$. Let us consider a positive-term expansion of the function $a(\eta)$ about $\eta = 0$. In this case we have the following lemma:

Lemma 6. *The series expansion of $a(\eta)$ must have the form*

$$a(\eta) = a_0 + \sum_{k>1}^{\infty} \frac{a_k}{\eta^k} + \sum_{l>0}^{\infty} b_l \eta^l, \quad (k,l) \in \mathbb{Q}^2 \ , \tag{37}$$

and must satisfy both conditions:

(i): *There is at least one rational $l > \frac{3}{2}$ such $b_l \neq 0$,*
(ii): *Either $a_0 \neq 0$ or there is at least one rational $k > 0$ such that $a_k \neq 0$.*

Proof. One may observe that the form of the integrand in the expression of L given by Eq. (36) infers that the positive rational functions $\frac{1}{a(t)}$ and $\left(\frac{t}{a(t)}\right)^2$ must be integrable on $[0,\infty)$. That means that we have to deal with the behavior of these functions around the origin and for large values of η too, to satisfy the convergence of the generalized integrals. Let $M_p t^p$ be a monomial of degree p in the series expansion of $a(t)$. In the neighbourhood of $\eta = 0$ we must have that $p < 1$ and in the neighborhood of infinity we must have $p > \frac{3}{2}$ to ensure the convergence of the generalized integrals. Thus, $a(t)$ must be at most of order $(1-\epsilon)$ at the origin and at least of order $(\frac{3}{2}+\epsilon)$ at infinity $(\epsilon > 0)$. This justifies the expression of $a(\eta)$ in Eq. (37). □

Remark 2.

- *If we assume that k and l are positive integer, Eq. (37) can be rewritten as $a(\eta) = a_0 + \sum_{k=1}^{\infty} \frac{a_k}{\eta^k} + \sum_{l=1}^{\infty} b_l \eta^l$.*
- *In the evaluation of the function f, the generalized integral of the function $\frac{t}{a(t)}$ is needed. In this case we must assume that $a(\eta)$ must be at least of order $(2+\epsilon)$ at infinity $(\epsilon > 0)$.*

Consequently, according to Eq. (37) we have the following results

	$\lim_{\eta\to 0}\frac{1}{a(\eta)}$	$\lim_{\eta\to 0}\frac{\eta}{a(\eta)}$	$\lim_{\eta\to\infty}\frac{1}{a(\eta)}$	$\lim_{\eta\to\infty}\frac{\eta}{a(\eta)}$
$a_0 = 0$	0	0	0	0
$a_0 \neq 0$	$\frac{1}{a_0}$	0	0	0

Moreover, since $\eta > 0$, the function $\frac{\eta}{a(\eta)}$, which is present in equation (36), is positive, continuous and from the above table we deduce that this function has a global maximum on $[0, \infty)$.

Thus, we have the following result

Theorem 1. *If the two following conditions are satisfied*

(i):

$$a(\eta) = a_0 + \sum_{l>0}^{\infty} b_l \eta^l, \quad l \in \mathbb{Q}\ , \quad a_0 \neq 0\ , \tag{38}$$

with at least one rational $l > \frac{3}{2}$ *such* $b_l \neq 0$,

(ii):

$$L_{max} = 2M \left(\int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 d\eta \right) \left(\frac{\int_0^\infty \frac{1}{a(\eta)} d\eta}{\left(\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{t}{a(t)}\, dt\right) d\eta \right)^2} \right) < 1\ , \tag{39}$$

then the mapping T_a *is a contraction.*

Proof. From equation (36) we obtain

$$\begin{aligned} L &< M \left(\int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 d\eta \right) \\ &\times \left(\frac{1}{\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{t}{a(t)}\, dt\right) d\eta} + \frac{\int_0^\infty \frac{1}{a(\eta)} d\eta}{\left(\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{t}{a(t)}\, dt\right) d\eta \right)^2} \right)\ , \\ &< M \left(\int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 d\eta \right) \left(\frac{\int_0^\infty \frac{1}{a(\eta)} d\eta}{\left(\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{t}{a(t)}\, dt\right) d\eta \right)^2} \right) \\ &\times \left(1 + \frac{\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{t}{a(t)}\, dt\right) d\eta}{\int_0^\infty \frac{1}{a(\eta)} d\eta} \right), \end{aligned}$$

From inequality (11) we deduce that

$$L < 2M \left(\int_0^\infty \left(\frac{\eta}{a(\eta)}\right)^2 d\eta \right) \left(\frac{\int_0^\infty \frac{1}{a(\eta)} d\eta}{\left(\int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{t}{a(t)}\, dt\right) d\eta \right)^2} \right) = L_{max}\ .$$

□

In general, inequality (39) is not easy to satisfy analytically, but numerically it is possible. Moreover, if we assume $a(x)$ under the form $a(x) = a_0 + x^s$ ($a_0 > 0$, $s > \frac{3}{2}$), we can easily evaluate the integrals and find many functions that satisfy condition 39 with an appropriate value of a_0. In the case of $s = 2$, the expression of the integrals can be obtained analytically and we observe, using MATLAB software, that the condition 39 is satisfied for $a_0 > 1.6$.

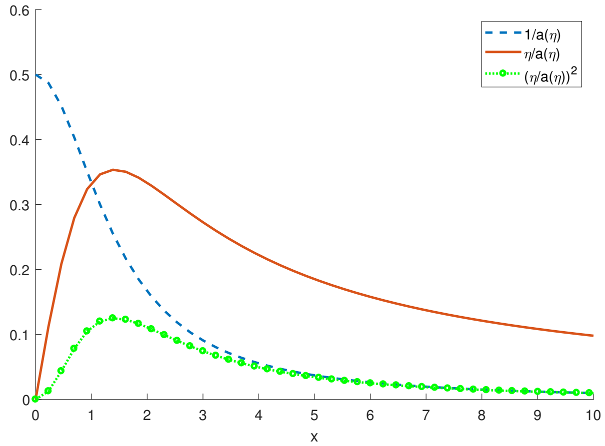

FIGURE 1. functions $\frac{1}{a(\eta)}$, $\frac{\eta}{a(\eta)}$ and $\left(\frac{\eta}{a(\eta)}\right)^2$ for $a(\eta) = 2 + \eta^2$

Figure 1 shows the functions $\frac{1}{a(\eta)}$, $\frac{\eta}{a(\eta)}$ and $\left(\frac{\eta}{a(\eta)}\right)^2$ for $a(\eta) = 2+\eta^2$. We can observe that all these functions are bounded and tend to zero for large values of η.

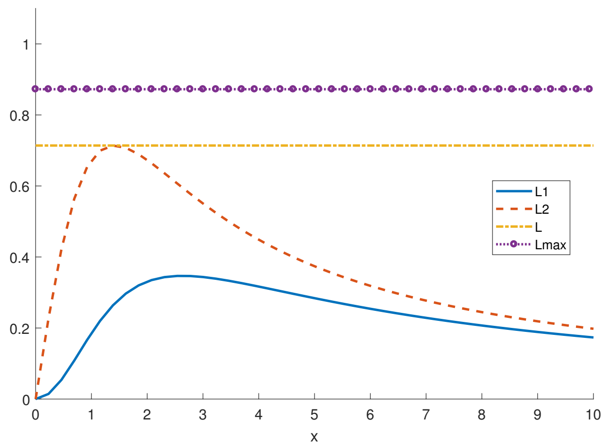

FIGURE 2. functions $L_1(\eta)$, $L_2(\eta)$, L and L_{max} for $a(\eta) = 2 + \eta^2$

Figure 2 shows the functions $L_1(\eta) = \frac{\eta\pi \arctan\left(\frac{\sqrt{2}\eta}{2}\right)}{4(\eta^2+2)} - \frac{\sqrt{2}\eta^2}{2(\eta^4+4\eta^2+4)} + \frac{\eta \arctan\left(\frac{\sqrt{2}\eta}{2}\right)}{2(\eta^2+2)}$, $L_2(\eta) = \frac{\pi\eta(\pi+2)}{8(\eta^2+2)}$, $L = \frac{\pi\sqrt{2}\left(\frac{\pi}{2}+1\right)}{16}$ and $L_{max} = \frac{\sqrt{2}\pi^2}{16}$, respectively, for $a(\eta) = 2 + \eta^2$. We can observe that all these functions are bounded and less than 1.

Table 1 gives some weight functions that satisfy condition 39.

$a(\eta)$	$2+\eta^2$	$2+\eta^4$	$2+\eta^6$	$2+\eta+\eta^3$
L_{max}	0.87	0.16	0.14	0.12

TABLE 1. The value of L_{max} for some weight functions.

5. Approximate solution by the Adomian decomposition method

We propose here to solve the third-order nonlinear boundary value problem (Eq. 1) by the Adomian decomposition method (ADM). Adomian's asymptotic decomposition method (AADM)[10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21] will be used for evaluating integrals of compound integrands which occur in the solution process.

5.1. **A new scheme of Adomian.** In this subsection, two recursive schemes are presented for the behavior of the solution $f(\eta)$ near the origin and at infinity, respectively. For both cases, the solution is decomposed into solution components which will be determined by recursion. The nonlinearity is expanded in terms of Adomian polynomials, as follows

$$f(\eta)=\sum_{n=0}^{\infty} f_n(\eta),\ (Nf)(\eta)=\exp\left(-\int_0^{\eta}\frac{f(t)}{a(t)}dt\right)=\sum_{n=0}^{\infty}A_n(\eta). \tag{40}$$

Let

$$u(\zeta)=\int_0^{\zeta}\frac{f(t)}{a(t)}dt=\int_0^{\zeta}\frac{f_0(t)}{a(t)}dt+\int_0^{\zeta}\frac{f_1(t)}{a(t)}dt+\int_0^{\zeta}\frac{f_2(t)}{a(t)}dt+....=\sum_{n=0}^{\infty}u_n(\zeta). \tag{41}$$

We have

$$u_0=\int_0^{\zeta}\frac{f_0(t)}{a(t)}dt,\ u_1=\int_0^{\zeta}\frac{f_1(t)}{a(t)}dt,\ u_2=\int_0^{\zeta}\frac{f_2(t)}{a(t)}dt,\dots . \tag{42}$$

Thus the Adomian polynomials are, in this case, given by

$$\begin{cases} A_0 &= e^{-u_0},\\ A_1 &= -e^{-u_0}u_1,\\ A_2 &= e^{-u_0}(-u_2+\frac{1}{2!}u_1^2),\\ A_3 &= e^{-u_0}(-u_3+u_1u_2-\frac{1}{3!}u_1^3),\\ A_4 &= e^{-u_0}(-u_4+\frac{1}{2!}u_2^2+u_1u_3-\frac{1}{2!}u_1^2u_2+\frac{1}{4!}u_1^4),\\ & \dots \end{cases} \tag{43}$$

5.2. **For small values of η.** Upon substitution into Eq. (7), we obtain

$$\sum_{n=0}^{\infty} f_n(\eta)=C_1\int_0^{\eta}(\eta-\zeta)\frac{1}{a(\zeta)}\sum_{n=0}^{\infty}A_n(\zeta)d\zeta,\ 0<\eta<\infty. \tag{44}$$

Then, for small values of η, we establish the modified Adomian recursion scheme for the function $f(\eta)$ as follows

$$\begin{cases} f_0(\eta) &= 0, \\ f_{n+1}(\eta) &= C_1 \int_0^\eta (\eta-\zeta)\frac{1}{a(\zeta)}A_n(\zeta)d\zeta,\ n\geq 0. \end{cases} \tag{45}$$

5.3. **For large values of η.** Dividing Eq. (5) by $a(\eta)$ and integrating from η to ∞, we obtain with the use of the boundary condition $f'(\infty)=1$,

$$f'(\eta) = 1 - C_1 \int_\eta^\infty \frac{1}{a(\xi)} \exp\left(-\int_0^\xi \frac{f(t)}{a(t)}dt\right) d\xi,\ 0<\eta<\infty, \tag{46}$$

that leads to

$$f(\eta) = \eta - C_1 \int_0^\eta \int_s^\infty \frac{1}{a(\xi)} \exp\left(-\int_0^\xi \frac{f(t)}{a(t)}dt\right) d\xi ds,\ 0<\eta<\infty, \tag{47}$$

which is a nonlinear Volterra integral equation of the second kind for calculating the solution.
Upon substitution into Eq. (47), we obtain the following recursive scheme

$$\begin{cases} f_0(\eta) &= \eta, \\ f_{n+1}(\eta) &= -C_1 \int_0^\eta \int_s^\infty \frac{1}{a(\zeta)}A_n(\zeta)d\zeta ds,\ n\geq 0. \end{cases} \tag{48}$$

Rewriting the matching equation (Eq. (8)) as

$$C_1 \int_0^\infty \frac{1}{a(\eta)} \exp\left(-\int_0^\eta \frac{f(\xi)}{a(\xi)}d\xi\right) d\eta = 1, \tag{49}$$

and repeating the steps from (Eq. 40) to (Eq. 43), we obtain an approximation of C_1 by solving

$$C_1 \left(\sum_{n=0}^{\infty} \int_0^\infty \frac{1}{a(\eta)} A_n(\eta, C_1)\, d\eta\right) = 1. \tag{50}$$

Since the Adomian operators depends on C_1 too, and considering the Adomian expression of the function Nf (Eq. (40)), we observe that an approximate value of the constant $C_1 = a(0)f''(0)$ can be determined by solving the nonlinear algebraic equation (Eq. 50) in terms of the undetermined coefficient C_1.

Remark 3. *For the calculation of the constant C_1 in Eq. (50) and in order to avoid a problem of integral divergence linked to A_0 , we alter steps (Eq. 40) to (Eq. 43) by substituting $u(\eta) = \sum_{n=0}^{\infty} v_n(\eta) = \sum_{n=0}^{\infty} \left(\int_0^\eta \frac{f_{n+1}(t)}{a(t)}\, dt\right)$ into Eq. (41). Accordingly a slight modification of the Adomian polynomials can be deduced in a straightforward manner.*

6. Applications

6.1. **Case 1:** $a(\eta) = a$**.** In this subsection, we consider the constant case $a(\eta) = a > 0$. In this case, the first components of the solution $f(\eta)$ for small values of η are

$$\begin{cases} f_0(\eta) &= 0, \\ \\ f_1(\eta) &= \frac{C_1}{a}\int_0^\eta(\eta-\zeta)A_0(\zeta)d\zeta,\ A_0(\zeta) = e^{-u_0} = 1 \\ f_1(\eta) &= \frac{C_1}{a}\int_0^\eta(\eta-\zeta)d\zeta = \frac{C_1}{a}\frac{\eta^2}{2!}, \\ \\ f_2(\eta) &= \frac{C_1}{a}\int_0^\eta(\eta-\zeta)A_1(\zeta)d\zeta,\ A_1(\zeta) = -e^{-u_0}u_1 = -\frac{C_1}{a^2}\frac{\zeta^3}{3!} \\ f_2(\eta) &= -\frac{C_1^2}{a^3}\int_0^\eta(\eta-\zeta)\zeta^3 d\zeta = -\frac{C_1^2}{a^3}\frac{\eta^5}{5!}, \\ \\ f_3(\eta) &= \frac{C_1}{a}\int_0^\eta(\eta-\zeta)A_2(\zeta)d\zeta,\ A_2(\zeta) = \frac{11C_1^2}{a^4}\frac{\zeta^6}{6!} \\ f_3(\eta) &= \frac{11C_1^3}{a^5}\frac{\eta^8}{8!}, \\ \\ f_4(\eta) &= \frac{C_1}{a}\int_0^\eta(\eta-\zeta)A_3(\zeta)d\zeta,\ A_3(\zeta) = -\frac{375C_1^3}{a^6}\frac{\zeta^9}{9!} \\ f_4(\eta) &= -\frac{375C_1^4}{a^7}\frac{\eta^{11}}{11!}, \\ \\ & \dots\ . \end{cases} \tag{51}$$

Thus,

$$f_s(\eta;\ a) = \frac{C_1}{a}\frac{\eta^2}{2!} - \frac{C_1^2}{a^3}\frac{\eta^5}{5!} + \frac{11C_1^3}{a^5}\frac{\eta^8}{8!} - \frac{375C_1^4}{a^7}\frac{\eta^{11}}{11!} + \dots\ , \tag{52}$$

where $f_s(\eta;\ a)$ represents $f(\eta;\ a)$ for small values of η.

The calculation of C_1 is done in the following manner : A simple calculation leads to

$$\begin{cases} v_0 &= \int_0^\eta \frac{f_1(t)}{a(t)}dt = \frac{C_1}{a^2}\frac{\eta^3}{3!}, \\ \\ v_1 &= \int_0^\eta \frac{f_2(t)}{a(t)}dt = \frac{-C_1^2}{a^4}\frac{\eta^6}{6!}, \\ \\ v_2 &= \int_0^\eta \frac{f_3(t)}{a(t)}dt = \frac{11C_1^3}{a^6}\frac{\eta^9}{9!}, \\ \\ & \dots \end{cases} \tag{53}$$

and

$$\begin{cases} A_0 &= e^{-v_0} = e^{-\frac{C_1}{3!a^2}\eta^3}, \\ \\ A_1 &= -e^{-v_0}v_1 = \frac{C_1^2}{a^4 6!}\eta^6 e^{-\frac{C_1}{3!a^2}\eta^3}, \\ \\ A_2 &= e^{-v_0}(-v_2 + \frac{1}{2!}v_1^2) = e^{-\frac{C_1}{3!a^2}\eta^3}\left(\frac{-11C_1^3}{a^6}\frac{\eta^9}{6!} + \frac{1}{2}\frac{C_1^4}{a^8}\frac{\eta^{12}}{(6!)^2}\right), \\ \\ & \dots \end{cases} \tag{54}$$

Thus Eq. (50) becomes

$$C_1\left[\frac{1}{a}\int_0^\infty e^{-\frac{C_1}{3!a^2}\eta^3}d\eta + \frac{C_1^2}{a^5 6!}\int_0^\infty \eta^6 e^{-\frac{C_1}{3!a^2}\eta^3}d\eta + \left(\frac{-11C_1^3}{9!a^7}\int_0^\infty \eta^9 e^{-\frac{C_1}{3!a^2}\eta^3}d\eta + \frac{C_1^4}{2a^9(6!)^2}\int_0^\infty \eta^{12}e^{-\frac{C_1}{3!a^2}\eta^3}d\eta\right) + \dots\right] = 1. \tag{55}$$

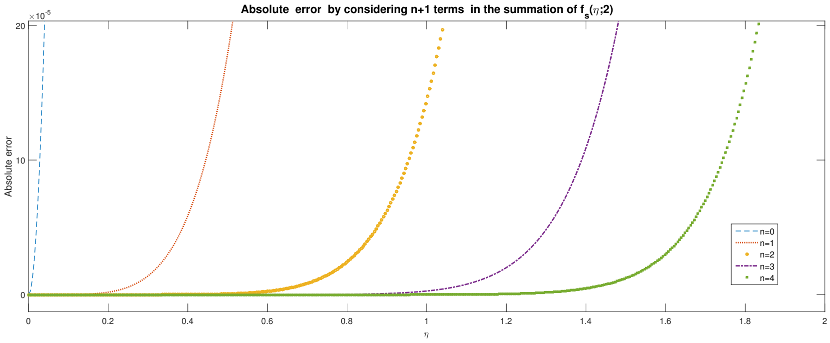

FIGURE 3. Absolute error by considering (n+1) terms in the summation of $f(\eta;2)$

number of terms in the expansion of $f_s(\eta)$	Error
1	0
2	$(C_1^2 * t^2)/8$
3	$(-(C_1^2 * t^5)/960 + (C_1 * t^2)/4) * (C_1/2 - (C_1^2 * t^3)/48) - (C_1^2 * t^2)/8$
4	$((11 * C_1^3 * t^8)/1290240 - (C_1^2 * t^5)/960 + (C_1 * t^2)/4) * ((11 * C_1^3 * t^6)/23040$ $-(C_1^2 * t^3)/48 + C_1/2) - (C_1^2 * t^2)/8 + (11 * C_1^3 * t^5)/1920$
5	$(-(5 * C_1^4 * t^{11})/68124672 + (11 * C_1^3 * t^8)/1290240$ $-(C_1^2 * t^5)/960 + (C_1 * t^2)/4) * (-(25 * C_1^4 * t^9)/3096576 + (11 * C_1^3 * t^6)/23040$ $-(C_1^2 * t^3)/48 + C_1/2) - (C_1^2 * t^2)/8 + (11 * C_1^3 * t^5)/1920 - (25 * C_1^4 * t^8)/172032$

TABLE 2. Absolute error by considering n terms in the summation of $f_s(\eta;2)$

Using the following integral formula

$$I\left[b;\ s,\ n\right] = \int_0^{\infty} \eta^{s-1} e^{-b\eta^n} d\eta = \frac{1}{nb^{\frac{s}{n}}} \Gamma\left(\frac{s}{n}\right),\ n \neq 0, \tag{56}$$

we obtain the nonlinear equation in C_1

$$C_1 \left(\frac{1}{3a} \frac{(3!a^2)^{\frac{1}{3}}}{C_1^{\frac{1}{3}}} \Gamma\left(\frac{1}{3}\right) + \frac{C_1^{-\frac{1}{3}} a^{-\frac{1}{3}}}{3 \times 6! \times 6^{\frac{7}{3}}} \Gamma\left(\frac{7}{3}\right) + ... \right) = 1. \tag{57}$$

If we consider only the first component of the integral $\int_0^{\infty} \frac{1}{a(\eta)} \exp\left(-\int_0^{\eta} \frac{f(\xi)}{a(\xi)} d\xi\right) d\eta$, we obtain

$$C_1 = \left(\frac{\sqrt[3]{6}}{3\sqrt[3]{a}} \Gamma\left(\frac{1}{3}\right) \right)^{-\frac{3}{2}} = \frac{0.484050}{\sqrt{a}}. \tag{58}$$

A improvement of the value of C_1 can be readily achieved by retaining the first two terms in (57). This gives the second approximate equation in C_1,

$$C_1 \left[\frac{1}{3a} \frac{(3!a^2)^{\frac{1}{3}}}{C_1^{\frac{1}{3}}} \Gamma\left(\frac{1}{3}\right) + \frac{C_1^{-\frac{1}{3}} a^{-\frac{1}{3}}}{3 \times 6! \times 6^{\frac{7}{3}}} \Gamma\left(\frac{7}{3}\right) \right] = 1. \tag{59}$$

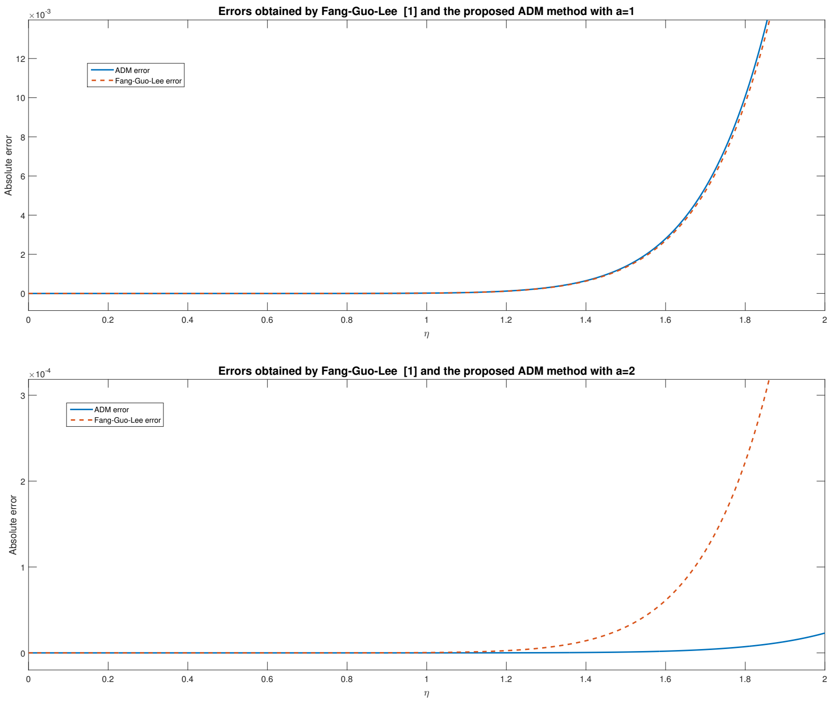

FIGURE 4. Errors obtained by Fang-Guo-Lee [1] and the proposed ADM method with a=1 (above) and a=2 (below)

Thus

$$C_1 = \left(\frac{\sqrt[3]{6}}{3\sqrt[3]{a}} \Gamma\left(\frac{1}{3}\right) + \frac{(\sqrt[3]{6})^7}{3 \times 6! \times \sqrt[3]{a}} \Gamma\left(\frac{7}{3}\right) \right)^{-\frac{3}{2}} = \frac{0.47288}{\sqrt{a}}, \tag{60}$$

both of which are consistent with Lemma 4.

Similarly, for large values of η, the first components of the solution $f(\eta;\ a)$ are

$$\begin{cases} f_0(\eta) &=& \eta, \\ f_1(\eta) &=& -\frac{C_1}{a} \int_0^\eta \int_s^\infty A_0(\zeta) d\zeta ds,\ A_0(\zeta) = e^{-\frac{\zeta^2}{2a}} \\ f_1(\eta) &=& -\frac{C_1\sqrt{2a}}{a} \frac{\sqrt{\pi}}{2}\eta + \frac{C_1\sqrt{2a}}{a} \frac{\sqrt{\pi}}{2} \left[\eta \operatorname{erf}(\eta) + \frac{e^{-\eta^2}}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}} \right], \\ & \dots & . \end{cases} \tag{61}$$

and

$$f_L(\eta;\ a) = f_0(\eta) + f_1(\eta) + ..., \tag{62}$$

where $f_L(\eta;\ a)$ is $f(\eta;\ a)$ for large values of η.

Hence, we have

$$f(\eta;\ a) = \begin{cases} f_s(\eta;\ a), & \text{for small values of } \eta, \\ f_L(\eta;\ a), & \text{for large values of } \eta. \end{cases}$$

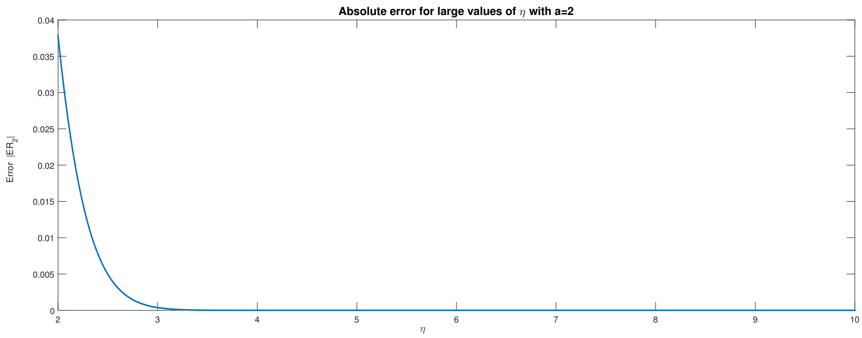

FIGURE 5. Errors for large values of η with a=2

Figure 3 shows the absolute error , with $a = 2$, when we consider n terms $(n = 2, \ldots, 5)$ in the expression of f_s (Eq. 52). We observe, first, that the error tends to zero when more terms are considered in the expansion of f_s , and then, the error increases in a exponential manner beginning from a certain value. Table 2 gives the error term for each $n = 1, \ldots, 5$.

Figure 4 compares the absolute error obtained in [1] with the absolute error obtained by our Adomian decomposition approach in the neighborhood of the origin for two values of the parameter a ($a = 1$ and $a = 2$). We can see that for small values of η, the ADM method proposed in this paper gives better results by considering the same order (order 11) in the polynomial approximation of $f_s(\eta; a)$ in both methods.

Figure 5 shows the absolute error for large values of η with $a = 2$. We observe that the error is very small and converges to zero for larger values of η.

Figure 6 shows the derivatives for small and large values of η with $a = 2$. We notice that at the vicinity of the origin the curve is a straight line and, far from the origin, the curve is a constant equal to 1.

6.2. **Case 2: $a(\eta)$ is a function.** We know that the fixed point theorem requires a special choice of the function $a(\eta)$. Hence, let us consider the case of $a(\eta) = 1 + \frac{\eta^2}{2}$. Thus, the Taylor expansion of the first several components of the solution $f_s(\eta)$ are given by

$$\left\{ \begin{array}{lcl} f_0(\eta) & = & 0, \\ f_1(\eta) & = & C_1 \left[\ln 2 - \ln(\eta^2+2) + (\sqrt{2}\,\eta)\arctan\left(\sqrt{2}\,\eta/2\right) \right] \\ & = & C_1 \frac{\eta^{10}}{1440} - C_1 \frac{\eta^8}{448} + C_1 \frac{\eta^6}{120} - C_1 \frac{\eta^4}{24} + C_1 \frac{\eta^2}{2}, \\ f_2(\eta) & = & -13C_1^2 \frac{\eta^9}{10080} + 17C_1^2 \frac{\eta^7}{5040} - C_1^2 \frac{\eta^5}{120}, \\ f_3(\eta) & = & -41C_1^3 \frac{\eta^{10}}{201600} + 11C_1^3 \frac{\eta^8}{40320}, \\ & \ldots & . \end{array} \right. \tag{63}$$

Therefore

$$f_s(\eta) = -\frac{C_1\eta^2}{201600}(41C_1^2\eta^8 - 55C_1^2\eta^6 + 260C_1\eta^7 - 680C_1\eta^5 + 1680C_1\eta^3 - 140\eta^8 + 450\eta^6 - 1680\eta^4 + 8400\eta^2 - 100800)\ldots . \tag{64}$$

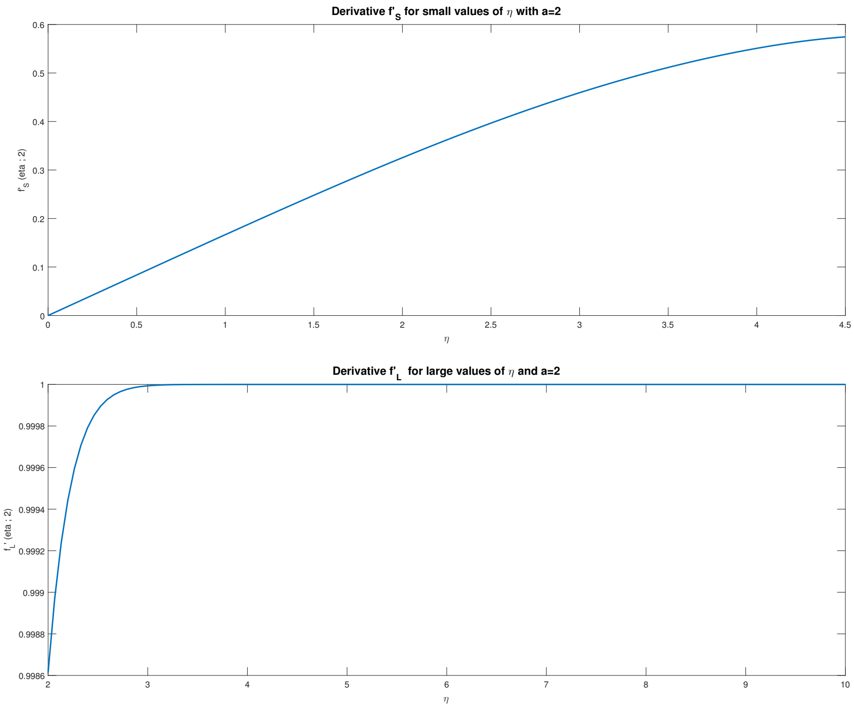

FIGURE 6. Derivatives f_s' (above) and f_L' (below) with $a = 2$

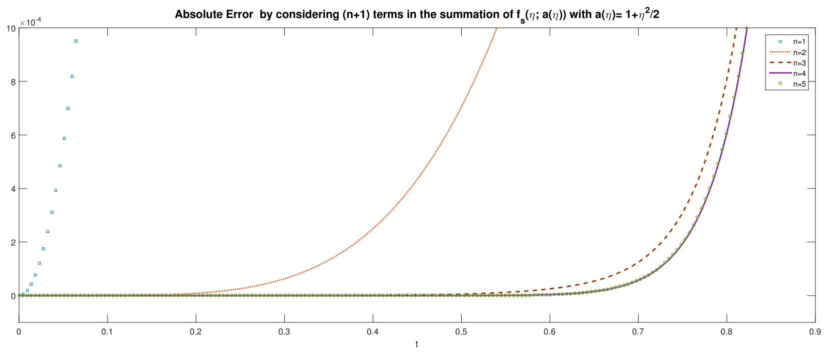

FIGURE 7. Absolute Error by considering (n+1) terms ($n = 3, 4, 5$) in the summation of $f_s(\eta; a(\eta))$ with $a(\eta) = 1 + \eta^2/2$

Figure 7 shows the absolute error by considering $2, 3, 4, 5$ and 6 terms in the summation of $f_s(\eta; a(\eta))$. We observe that the more the number of terms increases, the more the curve tends to zero on a broader interval, that means, the error, which converges to zero, is improved. In this example, the value of the constant $C_1 = 0.675$ is chosen according to Eq. (11) from Lemma 1 which states that $\frac{\sqrt{2}}{\pi} \leq C_1 \leq \frac{2\sqrt{2}}{\pi}$.

7. Conclusions

In this paper we have attempted to use the fixed point method to establish the existence and uniqueness of the generalized Blasius equation in which the coefficient $a(eta)$ plays an important role. Furthermore, we have presented the analytic approximate solution for both the classical and the generalized Blasius equations as solved by the Adomian decomposition method; our solution of the classical Blasius equation is shown to be a special case of the generalized Blasius equation. Our solutions are specialized further to both small-magnitude and large-magnitude cases demonstrating rapid rate of convergence. For the solutions in the small and in the large, we have derived the corresponding nonlinear Volterra integral equation of the first and second kind, respectively. Furthermore, the accuracy of our approximate solution has been validated through error analysis with our results presented graphically.

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