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A NEW PROOF OF A CARISTI TYPE FIXED POINT THEOREM

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Abstract. In this note, we give a new proof of a Caristi type fixed point theorem without using Zorn's lemma and transfinite induction.

1. Introduction

Caristi [1] presented the following fixed point theorem in 1976,

Theorem 1. *Let (X, d) be a complete metric space, and assume that $\varphi : X \to R^+$ is lower semi-continuous. Suppose that the mapping $T : X \to X$ satisfies*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx)$$

for every $x \in X$. Then there exists $\overline{x} \in X$ such that $T\overline{x} = \overline{x}$.

It is well know that Caristi's fixed point theorem is a generalisation of the Banach contraction principle. As proved by Kirk in [2], the validity of Caristi's fixed point theorem characterises completeness of (X, d), while this is not the case with Banach's theorem. Caristi's fixed point theorem is equivalent to the Ekeland variational principle, and has extensive applications in the fields of mathematics such as variational inequalities, optimization, control theory and differential equations.

In attempting to generalize Caristi's fixed point theorem, Kirk [3] has raised the problem of whether a map $T : X \to X$ such that for all $x \in X$

$$(d(x, Tx))^p \leq \varphi(x) - \varphi(Tx),$$

for some $p > 1$, has a fixed point.

In 2009, Khamsi [4] gave a negative answer to the question. Moreover, the author assured there exist some positive function $\eta : R^+ \to R^+$ with the property that if $f : X \to X$ satisfies

$$\eta((d(x, Tx)) \leq \varphi(x) - \varphi(Tx), \ x \in X$$

for some lower semicontinuous $\varphi : X \to R^+$, then f has a fixed point.

Then in 2013, following the same line as Khamsi, Zhang and Jiang [5] established a Caristi type fixed point theorem (Theorem 2 in Section 2) which extend the previous results due to Amini-Harandi [6], Khamsi [4], Suzuki [7] and others.

The original proof by Caristi invoked an iterated use of transfinite induction. Most proofs of Caristi's fixed point theorem were based on the use of Zorn's lemma and the axiom of choice (or Zermelo's fixed point theorem) applied to the Brøndsted partial order [8]. The original proof of the Zhang-Jiang's theorem was based on the minimal element method. In 2014, Kirk and Shahzad [9] posed the question weather it is possible to derive Zhang and Jiang's theorem from Brezis-Browder order principle. Dong [10] gave an affirmative answer to this question in 2018.

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In 2014, Khamsi [11] stated that "there are some trials of finding a pure metric proof of Caristi's fixed point theorem (without success so far)". Recently, some new proofs of Catisti's fixed point theorem using only purely metric techniques appear (see [12, 13] and references therein).

Motivated by these works, in this note, we'll give a purely metric proof to a Caristi type fixed point theorem in [5]. The pure metric character is understood in the following sense.

(1) The proof uses only standard methods of metric spaces.

(2) The proof uses standard properties of the set of all real numbers.

(3) The proof uses mathematical induction for the construction of a sequence in a metric space.

(4) No partial order considerations are applied (the natural order in R is the only order used).

(5) No direct use of the axiom of choice, Zorn's lemma or equivalents are used.

Our method is direct and different from other method in literatures. In Section 2, some preliminaries are given, including the full form of the Caristi type fixed point theorem and Cantor's intersection theorem. The purely metric proof to the fixed point theorem is presented in Section 3.

2. Preliminaries

Let $\gamma : [0, \infty) \to [0, \infty)$ be subadditive, i.e.

$$\gamma(t+s) \leqslant \gamma(t) + \gamma(s)$$

for each $s, t \in [0, \infty)$, a increasing continuous map such that $\gamma(\{0\}) = 0$. For example $\gamma(t) = t^p, (0 < p \le 1)$ for $t \in [0, \infty)$. Let Γ consist of all such functions γ.

A denotes the class of all maps

$$\eta : [0, \infty) \to [0, \infty)$$

for which there exist $\bar{\varepsilon} > 0$ and $\gamma \in \Gamma$ such that if $\eta(t) < \bar{\varepsilon}$, then $\eta(t) \ge \gamma(t)$.

Let $F : \mathbb{R} \to \mathbb{R}$, $F(0) = 0$, $F^{-1}[0, \infty) \subset [0, \infty)$, and for $t \ge 0$, F is increasing upper semi-continuous. Moreover

$$F(t) + F(s) \le F(t+s)$$

for $s, t \ge 0$. For example

$$F(t) = \begin{cases} < 0 & t < 0, \\ t^p & 0 \le t < t_0, \\ t^{p+1} & t \ge t_0. \end{cases}$$

where $t_0 > 1$ and $p \ge 1$. The class of all these functions F is denoted by $\mathcal{F}$. If $F(t) = t$, $\forall t \in R$, then trivially $F \in \mathcal{F}$.

Theorem 2 ([5])**.** *Let (X, d) be a complete metric space, $T : X \to X$, φ be a lower semi-continuous on X and bounded below. If there exist $\eta \in A$ and $F \in \mathcal{F}$, such that for any $x \in X$,*

$$\eta(d(x, Tx)) \le F(\varphi(x) - \varphi(Tx)),$$

then T has a fixed point.

To prove Theorem 2 in a pure metric way, we need the following Cantor's intersection theorem.

Lemma 1. *Let (X, d) be a complete metric space. Every decreasing sequence (A_n) of nonempty closed subsets of (X, d) such that there exists a sequence (c_n) with $c_n \to 0$ and $d(x, y) \le c_n$ for $x, y \in A_n$ and $n \in N$, has a nonempty intersection and $\bigcap_{n\in N} A_n$ is a singleton.*

3. A purely metric proof of Theorem 2

Since F is upper-semicontinuous, i.e.

$$\lim_{t\to 0}\sup F(t) \le F(0) = 0,$$

there exists $\delta > 0$ such that $F(t) < \bar{\varepsilon}$ for $0 \le t < \delta$.

Let $\varphi_0 = \inf_{x\in X} \varphi(x)$, $\quad M_\delta = \{x \in X \mid \varphi(x) \le \varphi_0 + \delta\}$.

φ is lower- semicontinuous implies M_δ is a closed set in X. Hence, (M_δ, d) is a complete metric space too.

Now, we'll show that $T(M_\delta) \subset M_\delta$.

In fact, if $x \in M_\delta$, due to,

$$0 \le \eta(d(x, Tx)) \le F(\varphi(x) - \varphi(Tx)).$$

and $F^{-1}[0, \infty) \subset [0, \infty)$, we have

$$\varphi(x) - \varphi(Tx) \ge 0,$$

then

$$\varphi_0 \le \varphi(Tx) \le \varphi(x) \le \varphi_0 + \delta.$$

i.e. $Tx \in M_\delta$.

Let $\mathcal{S}(x) = \{y \in M_\delta \mid \gamma(d(x, y)) \le F(\varphi(x) - \varphi(y))\}$. We assert $\mathcal{S}$ has following properties.

(1) $\mathcal{S}(x)$ is non-empty. Especially $x, Tx \in \mathcal{S}(x)$.

$x \in \mathcal{S}(x)$ is trivial. In what follows, we'll show $Tx \in \mathcal{S}(x)$.

In fact, owing to

$$\varphi_0 \le \varphi(Tx) \le \varphi(x) \le \varphi_0 + \delta,$$

we have

$$0 \le \varphi(x) - \varphi(Tx) \le \delta.$$

Then

$$\eta(d(x, Tx)) \le F(\varphi(x) - \varphi(Tx)) < \bar{\varepsilon}$$

As a consequence, we have

$$\gamma(d(x, Tx)) \le \eta(d(x, Tx)) \le F(\varphi(x) - \varphi(Tx)),$$

which means $Tx \in \mathcal{S}(x)$.

(2)$\mathcal{S}$ is closed.

In fact, if $\{y_n\} \subset \mathcal{S}(x)$ and $y_n \to y$, the lower semi-continuity of φ implies

$$\varphi(y) \le \lim_{n\to\infty}\inf \varphi(y_n)$$

Since F is upper semi-continuous and monotonous increasing,

$$\lim_{n\to\infty}\sup F(\varphi(x) - \varphi(y_n)) \le F(\varphi(x) - \lim_{n\to\infty}\inf \varphi(y_n)) \le F(\varphi(x) - \varphi(y)).$$

Noting that γ is continuous, we have

$$\gamma(d(x, y)) = \lim_{n\to\infty} \gamma(d(x, y_n)) \le \lim_{n\to\infty}\sup F(\varphi(x) - \varphi(y_n)) \le F(\varphi(x) - \varphi(y))$$

which means $y \in \mathcal{S}(x)$. Hence $\mathcal{S}$ is closed.

(3)For $\forall y \in \mathcal{S}(x)$, $\mathcal{S}(y) \subset \mathcal{S}(x)$.

In fact, for $\forall y \in \mathcal{S}(x)$, $\forall z \in \mathcal{S}(y)$, we have

$$\begin{aligned} \gamma(d(x,z)) &\leq \gamma(d(x,y)+d(y,z)) \\ &\leq \gamma(d(x,y))+\gamma(d(y,z)) \\ &\leq F(\varphi(x)-\varphi(y))+F(\varphi(y)-\varphi(z)) \\ &\leq F(\varphi(x)-\varphi(z)) \end{aligned}$$

which implies $z \in \mathcal{S}(x)$. Therefore, $\mathcal{S}(y) \subset \mathcal{S}(x)$.

Let $D(x,y) = \gamma(d(x,y))$, $x, y \in M_\delta$, then (M_δ, D) is a metric space. In fact, for $\forall x, y \in M_\delta$

(1) $D(x,y) = \gamma(d(x,y)) \geq 0$,
$D(x,y) = 0 \Leftrightarrow \gamma(d(x,y)) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$;
(2) $D(x,y) = \gamma(d(x,y)) = \gamma(d(y,x)) = D(y,x)$;
(3)

$$\begin{aligned} D(x,y) &= \gamma(d(x,y)) \\ &\leq \gamma(d(x,z)+d(z,y)) \qquad (\ \forall z \in M_\delta) \\ &\leq \gamma(d(x,z)+\gamma d(z,y)) \\ &= D(x,z)+D(z,x), \end{aligned}$$

.

It is easy to show that (M_δ, D) is a complete metric space. Hence, when $x \in M_\delta$,

$$D(x,Tx) \leq F(\varphi(x)-\varphi(Tx)),$$

$$\mathcal{S}(x) = \{y \in M_\delta \mid D(x,y) \leq F(\varphi(x)-\varphi(y))\}$$

Since $F(0) = 0$,$F^{-1}[0,\infty) \subset [0,\infty)$ and F is increasing, there exists $\varepsilon_1 > 0$, when $0 \leq t < \varepsilon_1$, $F(t) < 1/2^2$.

Given $x \in M_\delta$, since φ is bound below on M_δ and $\mathcal{S}(x) \subset M_\delta$, there exists $x_1 \in \mathcal{S}(x)$ such that

$$\varphi(x_1) < \varepsilon_1 + \inf_{y\in\mathcal{S}(x)} \varphi(y),$$

or

$$\varphi(x_1) - \inf_{y\in\mathcal{S}(x)} \varphi(y) < \varepsilon_1,$$

Now, for any $u, v \in \mathcal{S}(x_1)$, noting that $\mathcal{S}(x_1) \subset \mathcal{S}(x)$, we have

$$\begin{aligned} D(u,v) &\leq D(x_1,u)+D(x_1,v) \\ &\leq F(\varphi(x_1)-\varphi(u))+F(\varphi(x_1)-\varphi(v)) \\ &\leq 2F(\varphi(x_1)-\inf_{y\in\mathcal{S}(x)}\varphi(y)) \\ &< 1/2, \end{aligned}$$

Let $x_2 \in \mathcal{S}(x_1)$ such that

$$\varphi(x_2) < \varepsilon_2 + \inf_{y\in\mathcal{S}(x)} \varphi(y),$$

where ε_2 is a positive constant and $F(t) < 1/2^3$ for $t \in [0, \varepsilon_2)$. Now, for $u, v \in \mathcal{S}(x_2)$, we can get

$$D(u,v) \leq 1/2^2.$$

Continuing this process, we can construct a sequence of nonempty closed bounded subsets $\{\mathcal{S}(x_n)\} \subset X$ satisfying

(1) $\mathcal{S}(x_1) \supset \mathcal{S}(x_2) \supset \mathcal{S}(x_3) \supset ... \supset \mathcal{S}(x_n) \supset ...$;
(2) $D(u,v) \leq 1/2^n$, $\forall u, v \in \mathcal{S}(x_n)$.

Due to Cantor's intersection theorem, there exists a unique $\overline{x} \in M_\delta$ such that

$$\{\overline{x}\} = \cap_{n=1}^{\infty} \mathcal{S}(x_n),$$

for $\overline{x} \in \mathcal{S}(x_n)$, $\forall n \in N$,

$$\mathcal{S}(\overline{x}) \subset \mathcal{S}(x_n), \qquad \forall n \in N,$$

But $T\overline{x} \subset \mathcal{S}(\overline{x})$,
so,

$$T\overline{x} \in \cap_{n=1}^{\infty} \mathcal{S}(x_n),$$

Hence,

$$\overline{x} = T\overline{x}.$$

□

Remark 1. *In this note, we prove a Caristi type fixed point only in metric sense, without using Zorn's lemma and transfinite induction. It should be point out that the partial order principle and pure metric method are equal important in proving this fixed point theorem, they connects the topological structure with the order structure of metric space. Moreover, by a modification of this pure metric method, we can remove the continuity of γ in Theorem 2, and get an extension of Theorem 3.2 in* [14].

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