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SOME SERIES OF DISTRIBUTIONALLY INTEGRABLE FUNCTIONS

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Abstract. We study several properties of series of distributionally integrable functions. We consider the case of a nowhere dense closed subset K of $[a,b]$, with $a,b\in K$, and complementary open intervals (a_n,b_n) . We prove that if f is distributionally integrable in $[a,b]$ and if the primitives of order m of f, F_m, are such that their restrictions $F_m|_K$ are absolutely continuous in K for all $m\geq 1$, then $f\chi_K$ is Lebesgue integrable and

$$f\chi_{[a,b]}=f\chi_K+\sum_{n=1}^{\infty}f\chi_{[a_n,b_n]}\,,$$

in the space $\mathcal{D}'(\mathbb{R})$. We also prove that under these conditions several related results hold, in particular the generalized integration by parts formula

$$\int_K f(x)\,\phi(x)\,\mathrm{d}x=-\int_K F(x)\,\phi'(x)\,\mathrm{d}x+F\phi|_a^b-\sum_{n=1}^{\infty}F\phi|_{a_n}^{b_n}\,,$$

where ϕ is a test function. Furthermore, we prove that

$$\int_a^b f(x)\,\mathrm{d}x=\int_K f(x)\,\mathrm{d}x+\sum_{n=1}^{\infty}\int_{a_n}^{b_n}f(x)\,\mathrm{d}x\,,$$

if f is distributionally integrable and the restriction of its first order primitive to K is absolutely continuous. We give examples that show that the formulas may fail when the hypotheses are not satisfied.

1. Introduction

If f is a Lebesgue integrable function in $\mathbb{R}$ and if $\{E_n\}_{n=1}^{\infty}$ is a sequence of mutually disjoint measurable sets with $[a,b]=\bigcup_{n=1}^{\infty}E_n$ then in the space of distributions $\mathcal{D}'(\mathbb{R})$

$$f\chi_{[a,b]}=\sum_{n=1}^{\infty}f(x)\,\chi_{E_n}\,, \tag{1}$$

where χ_A is the characteristic function of a set A. Our main aim in this article is to consider the corresponding question when f is integrable in the sense of a more general integral, such as the Denjoy-Perron-Henstock-Kurzweil integral or the recently introduced distributional integral. It is very simple to see that for non-absolute integrals it is not even true that $f\chi_E$ has to be integrable for f integrable and E measurable, so we consider a particular but very important decomposition of $E=[a,b]$ as a disjoint union, namely as

$$[a,b]=K\cup\bigcup_{n=1}^{\infty}(a_n,b_n)\,, \tag{2}$$

where K is a closed nowhere dense set with $a,b\in K$ and where the (a_n,b_n) are the complementary intervals. We shall actually be able to show in the Theorem 1 that under

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certain restrictions on a distributionally integrable function f then

$$f\chi_{[a,b]} = f\chi_K + \sum_{n=1}^{\infty} f\chi_{[a_n,b_n]} \,, \tag{3}$$

in the space $\mathcal{D}'(\mathbb{R})$. Nevertheless when those conditions are not satisfied then (3) does not need to hold, as we show by several examples in the Section 6.

A simpler but closely related question is whether the formula

$$\int_a^b f(x)\ \mathrm{d}x = \int_K f(x)\ \mathrm{d}x + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)\ \mathrm{d}x \,, \tag{4}$$

which is clearly valid if f is Lebesgue integrable, remains true for other integrals. In fact this question already appears in the work of Lebesgue on the reconstruction of a function from its derivative [10, Chp. 8] since he established that if F has a derivative everywhere in $[a,b]$ then, even if F' is not Lebesgue integrable, if $F'\chi_K$ is Lebesgue integrable and if the series $\sum_{n=1}^{\infty} |F(b_n) - F(a_n)|$ converges then

$$F(b) - F(a) = \int_K F'(x)\ \mathrm{d}x + \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right) \,, \tag{5}$$

which is basically (4) for $f = F'$. This formula was pivotal for the construction of the Denjoy totalization [10, Chp. 8]. Actually it is well known that (4) holds if f is Denjoy integrable in $[a,b]$, $f\chi_K$ is Lebesgue integrable, and the series $\sum_{n=1}^{\infty} \left|\int_{a_n}^{b_n} f(x)\ \mathrm{d}x\right|$ converges [6]. Notice that if f is Denjoy integrable in $[a,b]$ then so are the functions $f\chi_{[a_n,b_n]}$ but in general $f\chi_K$ is not, and thus one needs to assume this explicitly; a similar situation is encountered with the distributional integral.

The plan of the article is the following. In section 2 we explain the basic properties of the distributional integral. Sections 3 and 4 contain the necessary technical developments needed to prove in Section 5 that under suitable hypotheses then (3), (4), and (5) are valid for distributionally integrable functions. Then in Section 6 we give examples that show that the formulas may fail if the hypotheses are not satisfied.

2. The distributional integral

We refer to the texts for the basic ideas about distributions [7, 13]. Here we would like to discuss some not so well known topics [2, 4, 11, 12, 17], topics needed to understand the distributional integral [5], which we explain after them.

In [8] Łojasiewicz defined the value of a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ at the point x_0 as the limit

$$\mathsf{f}(x_0) = \lim_{\varepsilon \to 0} \mathsf{f}(x_0 + \varepsilon x) \,, \tag{6}$$

if the limit exists in $\mathcal{D}'(\mathbb{R})$, that is, if

$$\lim_{\varepsilon \to 0} \langle \mathsf{f}(x_0 + \varepsilon x), \phi(x) \rangle = \mathsf{f}(x_0) \int_{-\infty}^{\infty} \phi(x)\, \mathrm{d}x \,, \tag{7}$$

for each $\phi \in \mathcal{D}(\mathbb{R})$. It was proved in [8] that the distributional point value $\mathsf{f}(x_0)$ exists and equals L if and only if there exists m and a primitive of order m, F_m, $(\mathsf{F}_m)^{(m)} = \mathsf{f}$, that comes from a function F_m that is continuous in a neighborhood of x_0 and such that

$$\lim_{x \to x_0} \frac{F_m(x)}{(x - x_0)^m} = \frac{L}{m!} \,. \tag{8}$$

In this case we say that the point value is order m at the most.

A distribution f is a *Łojasiewicz distribution* if the distributional point value $\mathsf{f}(x_0)$ exists for every $x_0 \in \mathbb{R}$. A function f defined in $\mathbb{R}$ is called a *Łojasiewicz function* if there exists a Łojasiewicz distribution f such that

$$\mathsf{f}(x) = f(x) \qquad \forall x \in \mathbb{R}\,. \tag{9}$$

The Łojasiewicz functions can be considered as a distributional generalization of continuous functions. They are defined at all points, and furthermore the value at each given point is not arbitrary but the (distributional) limit of the function as one approaches that point.

The question of whether a distribution can be considered a function is the theme of [3]. In general distributions are not functions, but, interestingly, sometimes they are. In fact, locally integrable Lebesgue functions f give rise in a unique fashion to associated regular distributions f, $f \leftrightarrow \mathsf{f}$. Actually Denjoy-Perron-Henstock-Kurzweil integrable functions also have canonically associated distributions [9]. The correspondence $f \leftrightarrow \mathsf{f}$ is clearly defined in the case of Łojasiewicz functions and distributions. The class of the locally distributionally integrable functions is a large class that includes both the locally Denjoy-Perron-Henstock-Kurzweil integrable functions the Łojasiewicz functions, and many more, for which the correspondence $f \leftrightarrow \mathsf{f}$ is well defined [5]. Notice that in this section it is better to say that f and f are *associated* and employ different notations for the function and the distribution, instead of the standard practice of saying that f "is" f, *but we will employ the standard practice in the rest of the article.*

In [5] we constructed and studied the properties of a general integration operator that can be applied to functions of one variable, $f : [a,b] \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, denoted in this section as

$$(\mathfrak{dist}) \int_a^b f(x)\, \mathrm{d}x\,, \tag{10}$$

but as $\int_a^b f(x)\, \mathrm{d}x$ *in the rest of the article.* We call it the *distributional integral* of f. The space of distributionally integrable functions is a vector space and the operator (10) is a linear functional in this space.

Any Denjoy-Perron-Henstock-Kurzweil integrable function is also distributionally integrable, and the integrals coincide; if the Denjoy-Perron-Henstock-Kurzweil integral can be assigned the value ∞ then the distributional integral can also be assigned the value ∞. If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is a function that is distributionally integrable over any compact interval, and if $\psi \in \mathcal{D}(\mathbb{R})$ is a test function, then the formula

$$\langle \mathsf{f}(x), \psi(x) \rangle = (\mathfrak{dist}) \int_{-\infty}^{\infty} f(x)\, \psi(x)\, \mathrm{d}x\,, \tag{11}$$

defines a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$. This distribution f has distributional point values almost everywhere and

$$\mathsf{f}(x) = f(x) \qquad (\text{a.e.})\,. \tag{12}$$

If we start with a distribution $\mathsf{f}_0 \in \mathcal{D}'(\mathbb{R})$ that has values everywhere, then construct the function f given by those values, and then define a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ by formula (11) then we recover the initial distribution: $\mathsf{f} = \mathsf{f}_0$.

Let us now look at the indefinite integral

$$F(x) = (\mathfrak{dist}) \int_a^x f(t)\, \mathrm{d}t\,, \tag{13}$$

of a distributionally integrable function f. For the Denjoy-Perron-Henstock-Kurzweil integral F is continuous while for the Lebesgue integral F is absolutely continuous, but for a general distributionally integrable function f the function F is a Łojasiewicz function.

Let F be the distribution associated to F, $F \leftrightarrow \mathsf{F}$, and let $\mathsf{f} = \mathsf{F}'$. Then $\mathsf{f} = \mathsf{F}'$ is the same distribution given by (11). Furthermore, F' has distributional values almost everywhere and actually $\mathsf{F}'(x) = f(x)$ (a.e), a precise statement of the idea that f is the derivative of F almost everywhere.

One can think of a simple procedure for the construction of primitives of *functions* by using the fact that *distributions* are known to have primitives. Indeed, start with a function f, associate to it a distribution f, construct the distributional primitive F, that is, $\mathsf{F}' = \mathsf{f}$, and then construct the function F associated to F. Then F would be a primitive of f. Unfortunately, this procedure fails, in general, because there is no unique way to assign a distribution f to a given function f, as follows from a well known theorem of Lusin [5, Thm. 7.1]. The method works sometimes, however. When? It works for the distributionally integrable functions!. It is important to emphasize that the distributional integral is a method to find the integral of *functions* not integrals of *distributions;* for the integrals of distributions see [1, 14, 15, 16].

If f is distributionally integrable over $[a,b]$ then so are $(x-a)^{\alpha}(b-x)^{\beta} f(x)$ for any real numbers $\alpha > 0$ and $\beta > 0$. For a general distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ there is no canonical way to define $\chi_{[a,b]}\mathsf{f}$, but if f corresponds to a locally distributionally integrable function then $\chi_{[a,b]}\mathsf{f}$ is well defined. In particular, if F is the distribution corresponding to F, the primitive of f, then $\chi_{[a,b]}\mathsf{F}$ is well defined and we have the ensuing version of the well known formula,

$$\left(\chi_{[a,b]}(x)\,\mathsf{F}(x)\right)' = \chi_{[a,b]}\mathsf{f} + F(a)\,\delta(x-a) - F(b)\,\delta(x-b)\ . \tag{14}$$

3. Some series of deltas

We shall employ the following conventions. The set $K \subset [a,b]$ is a closed nowhere dense set, such that $a, b \in K$. We will denote by (a_n, b_n), $n = 1, 2, 3, \ldots$ the components of $[a,b] \setminus K$, so that $[a,b]$ is decomposed as the disjoint union

$$[a,b] = K \cup \bigcup_{n=1}^{\infty} (a_n, b_n)\ . \tag{15}$$

We start with the derivative formulas for some Lebesgue integrable functions.

Lemma 1. *Let $f \in L^1(\mathbb{R})$ and let F be any primitive of f. Then the series of delta functions $\sum_{n=1}^{\infty}(F(a_n)\,\delta(x-a_n) - F(b_n)\,\delta(x-b_n))$ converges in $\mathcal{D}'(\mathbb{R})$ and*

$$\begin{aligned}\sum_{n=1}^{\infty}&\left(F(a_n)\,\delta(x-a_n) - F(b_n)\,\delta(x-b_n)\right)\\ &= f(x)\,\chi_K(x) - (F(x)\,\chi_K(x))' + F(a)\,\delta(x-a) - F(b)\,\delta(x-b)\ .\end{aligned} \tag{16}$$

Proof. Notice that since both f and F are locally Lebesgue integrable then

$$f(x)\,\chi_{[a,b]}(x) = f(x)\,\chi_K(x) + \sum_{n=1}^{\infty} f(x)\,\chi_{[a_n,b_n]}(x)\ , \tag{17}$$

and

$$F(x)\,\chi_{[a,b]}(x) = F(x)\,\chi_K(x) + \sum_{n=1}^{\infty} F(x)\,\chi_{[a_n,b_n]}(x)\ . \tag{18}$$

Distributional differentiation of the last relation, taking the first into consideration, yields

$$f(x)\chi_{[a,b]}(x) + F(a)\delta(x-a) - F(b)\delta(x-b) =$$
$$(F(x)\chi_K(x))' + \sum_{n=1}^{\infty}\left\{f(x)\chi_{[a_n,b_n]}(x) + F(a_n)\delta(x-a_n) - F(b_n)\delta(x-b_n)\right\} =$$
$$(F(x)\chi_K(x))' + \big(f(x)\chi_{[a,b]}(x) - f(x)\chi_K(x)\big) + \sum_{n=1}^{\infty}\left(F(a_n)\delta(x-a_n) - F(b_n)\delta(x-b_n)\right),$$

and (16) is obtained after a rearrangement of terms.

Observe also that the series $\sum_{n=1}^{\infty}(F(a_n)\delta(x-a_n) - F(b_n)\delta(x-b_n))$ must be convergent but in general both series $\sum_{n=1}^{\infty}F(a_n)\delta(x-a_n)$ and $\sum_{n=1}^{\infty}F(b_n)\delta(x-b_n)$ are divergent. □

Two special cases of this formula will be particularly important in our analysis.

Lemma 2. *Let* $g \in L^1(\mathbb{R})$ *with* $\operatorname{supp} g \subset K$. *Let* $G(x) = G(a) + \int_a^x g(t)\,\mathrm{d}t$ *be a primitive of* g. *Then*

$$\sum_{n=1}^{\infty}G(a_n)(\delta(x-a_n) - \delta(x-b_n)) = g(x) - (G(x)\chi_K(x))' + G(a)\delta(x-a) - G(b)\delta(x-b)\,, \quad (19)$$

in the space $\mathcal{D}'(\mathbb{R})$.

Proof. Notice that in this case $G(a_n) = G(b_n)$, so that the result follows from (16). □

Lemma 3. *Let* $f \in L^1(\mathbb{R})$, *let* $h = f - f\chi_K$, *and let* H *be a primitive of* h. *Then*

$$\sum_{n=1}^{\infty}(H(a_n)\delta(x-a_n) - H(b_n)\delta(x-b_n)) = -\widetilde{h}(x) + H(a)\delta(x-a) - H(b)\delta(x-b)\,, \quad (20)$$

where

$$\widetilde{h}(x) = (W(x)\chi_K(x))'\,, \quad (21)$$

W *being any continuous function in* $\mathbb{R}$ *such that* $W|_K = H|_K$.

Proof. This is another particular case of the Lemma 1, but we worded it in such a way that it is clear that $\widetilde{h}$ depends only on $H|_K$ but not on the values of H in the intervals (a_n, b_n). □

The particular cases when K has measure 0 deserve to be mentioned. In the Lemma 2 G must be a constant, which we may take to be 1. We thus obtain that in $\mathcal{D}'(\mathbb{R})$

$$\sum_{n=1}^{\infty}(\delta(x-a_n) - \delta(x-b_n)) = \delta(x-a) - \delta(x-b)\,. \quad (22)$$

Notice that both series $\sum_{n=1}^{\infty}\delta(x-a_n)$ and $\sum_{n=1}^{\infty}\delta(x-b_n)$ are divergent. In the Lemma 3 we have $\widetilde{h} = 0$ and thus

$$\sum_{n=1}^{\infty}(H(a_n)\delta(x-a_n) - H(b_n)\delta(x-b_n)) = H(a)\delta(x-a) - H(b)\delta(x-b)\,, \quad (23)$$

a formula that, as shall see in the next section, actually holds for *any* function H whose restriction to K is absolutely continuous.

It is interesting to note that in general such series of delta functions are not Radon measures. Indeed, if K has positive measure the formula

$$\sum_{n=1}^{\infty} \left(\delta\left(x-a_n\right)-\delta\left(x-b_n\right)\right)=-\left(\chi_K\left(x\right)\right)'+\delta\left(x-a\right)-\delta\left(x-b\right)\,, \tag{24}$$

obtained from (16) if $F=1$, shows that the series on the left is not a measure but rather a distributionally convergent series of measures which does not converge in the space of measures.

4. Decomposition of continuous functions on K

Let W be a function defined and continuous in K. The jump sequence of W is defined as

$$S_n=W\left(b_n\right)-W\left(a_n\right)\,,\qquad n=1,2,3,\ldots\,. \tag{25}$$

We say that W is jump-less if this sequence vanishes.

Let us observe that the usual definitions of bounded variation and absolute continuity can be also applied to functions defined only on K [6, Chp. 6].

Lemma 4. *Let $W\in C\left(K\right)$ be such that*

$$\sum_{n=1}^{\infty}\left|S_n\right|<\infty\,. \tag{26}$$

Then W can be written in a unique way as

$$W=L+S\,, \tag{27}$$

where L is jump-less and where S is a saltus function,

$$S\left(x\right)=\sum_{b_n\leq x}S_n\,. \tag{28}$$

If W is absolutely continuous then (26) holds and there exists $l\in L^1\left(K\right)$ such that

$$L\left(x\right)=L\left(a\right)+\int_a^x l\left(t\right)\,\mathrm{d}t\,,\qquad x\in K\,. \tag{29}$$

Proof. The results are quite clear. Notice that W being absolutely continuous means that L is, and this is equivalent to the fact that $\widetilde{L}$, its extension to $[a,b]$ constructed by asking it to be constant in each interval $\left[a_n,b_n\right]$, is absolutely continuous and thus there exists $l\in L^1\left[a,b\right]$ with $\operatorname{supp} l\subset K$ such that $\widetilde{L}\left(x\right)=L\left(a\right)+\int_a^x l\left(t\right)\,\mathrm{d}t$, $x\in\left[a,b\right]$. □

It is well known that a saltus function has the property that $S'\left(x\right)=0$ almost everywhere [6, 10]. Hence if $\widetilde{W}$ is an extension to a nearly everywhere differentiable function in $[a,b]$ then

$$\left(\widetilde{W}\left(x\right)\right)'=l\left(x\right)\,,\qquad x\in K\,. \tag{30}$$

It also follows that

$$l\left(x\right)=\lim_{y\to x,y\in K}\frac{W\left(y\right)-W\left(x\right)}{y-x}\,,\ \text{almost everywhere in } K\,. \tag{31}$$

The following result will also be useful.

Lemma 5. *Let W be a continuous function defined on K that satisfies (26). Then the series*

$$\sum_{n=1}^{\infty}\left(W\left(a_{n}\right)\delta\left(x-a_{n}\right)-W\left(b_{n}\right)\delta\left(x-b_{n}\right)\right)\,, \tag{32}$$

converges in $\mathcal{D}'(\mathbb{R})$.

Proof. Let $\phi\in\mathcal{D}(\mathbb{R})$. For each n there exists $\theta_n\in(a_n,b_n)$ such that

$$\phi\left(b_{n}\right)=\phi\left(a_{n}\right)+\left(b_{n}-a_{n}\right)\phi'\left(\theta_{n}\right)\,. \tag{33}$$

Therefore the series

$$\sum_{n=1}^{\infty}\left\langle W\left(a_{n}\right)\delta\left(x-a_{n}\right)-W\left(b_{n}\right)\delta\left(x-b_{n}\right),\phi\left(x\right)\right\rangle=\sum_{n=1}^{\infty}\left(W\left(a_{n}\right)\phi\left(a_{n}\right)-W\left(b_{n}\right)\phi\left(b_{n}\right)\right)$$
$$=\sum_{n=1}^{\infty}W\left(b_{n}\right)\left(a_{n}-b_{n}\right)\phi'\left(\theta_{n}\right)-\sum_{n=1}^{\infty}S_{n}\phi\left(a_{n}\right)\,,$$

converges since both series on the right converge. □

We now proceed to compute the sum of the series (32) when W is absolutely continuous.

Lemma 6. *Let L be a jump-free absolutely continuous function defined on K. Then*

$$\sum_{n=1}^{\infty}L\left(a_{n}\right)\left(\delta\left(x-a_{n}\right)-\delta\left(x-b_{n}\right)\right)=$$
$$l\left(x\right)-\left(L\left(x\right)\chi_{K}\left(x\right)\right)'+L\left(a\right)\delta\left(x-a\right)-L\left(b\right)\delta\left(x-b\right)\,. \tag{34}$$

Proof. It follows from the Lemma 2 since $\left.\widetilde{L}\right|_{K}=L,$ $\widetilde{L}$ is absolutely continuous and l is its derivative. □

Lemma 7. *Let S be the saltus function defined in (28) where $\sum_{n=1}^{\infty}\left|S_{n}\right|<\infty$. Then*

$$\sum_{n=1}^{\infty}\left(S\left(a_{n}\right)\delta\left(x-a_{n}\right)-S\left(b_{n}\right)\delta\left(x-b_{n}\right)\right)=$$
$$-\left(S\left(x\right)\chi_{K}\left(x\right)\right)'+S\left(a\right)\delta\left(x-a\right)-S\left(b\right)\delta\left(x-b\right)\,. \tag{35}$$

Proof. Let H be the extension of $S|_K$ to $[a,b]$ that is linear in each interval $[a_n,b_n]$. Then H is absolutely continuous and thus (35) is obtained from the Lemma 3. □

If we now combine the last two lemmas, we obtain the desired formula.

Proposition 1. *Let W be an absolutely continuous function defined on K. Then*

$$\sum_{n=1}^{\infty}\left(W\left(a_{n}\right)\delta\left(x-a_{n}\right)-W\left(b_{n}\right)\delta\left(x-b_{n}\right)\right)=$$
$$l\left(x\right)-\left(W\left(x\right)\chi_{K}\left(x\right)\right)'+W\left(a\right)\delta\left(x-a\right)-W\left(b\right)\delta\left(x-b\right)\,, \tag{36}$$

where $l\in L^1$ is given by (31) in K and vanishes in $[a,b]\setminus K$.

Notice that even though W is defined only on K, the function $W\chi_K$ is a well defined function in all of $\mathbb{R}$, an element of $\mathcal{D}'(\mathbb{R})$ in fact.

We would also like to point out that (36) does not hold if W is just continuous but not absolutely continuous. An example is provided if K has measure 0 and W is the restriction to K of ω, a continuous singular function whose derivative vanishes in $[a,b]\setminus K$. Indeed in this case W is jump-free but

$$\sum_{n=1}^{\infty} W(a_n)\left(\delta(x-a_n)-\delta(x-b_n)\right)=\omega'(x)+W(a)\,\delta(x-a)-W(b)\,\delta(x-b)\ , \quad (37)$$

where ω' is a singular measure, concentrated on K; thus neither (34) nor (36) hold.

5. The main results

Let f be distributionally integrable in $[a,b]$. Let us denote by F_m some fixed primitives of order m of f, $(F_m)^{(m)}=f$. There exists m_0 such that F_m is absolutely continuous in $[a,b]$ for $m\geq m_0$. In fact if F_m is absolutely continuous in $[a,b]$ then so are the $F_{m'}$ for $m'>m$. However, as we show in the Example 3, it is possible for $F_m|_K$ to be absolutely continuous while $F_{m+1}|_K$ is not. Hence in our main results we must assume that $F_m|_K$ is absolutely continuous for all $m\geq 1$; notice however that this holds for any K whenever $m\geq m_0$.

We start with an auxiliary result.

Lemma 8. *Let f be distributionally integrable in $[a,b]$. Suppose*

$$f\chi_{[a,b]}=\sum_{n=1}^{\infty} f\chi_{[a_n,b_n]}\,, \quad (38)$$

in the space $\mathcal{D}'(\mathbb{R})$. Then

$$f(x)=0\,, \qquad \textit{almost everywhere in } K\,, \quad (39)$$

so that

$$f\chi_K=0\,, \quad (40)$$

in the space $\mathcal{D}'(\mathbb{R})$.

Proof. If $x\in K$ then the distributional point value $f(x)$ may or may not exist, but even if it does it does not have to vanish. Consider for example the case when K has measure zero and f is a continuous function in $[a,b]$ that does not vanish in K : here the point values exist for all $x\in K$ but they are never zero. Nevertheless, our aim is to show that the distributional point values values vanish *almost everywhere* in K.

Let us now observe that (39) and (40) certainly hold for Lebesgue integrable functions. Let m be a positive integer. Let us find a $g\in L^1$ such that

$$\int_{a_n}^{b_n} f(x)\,x^j\,\mathrm{d}x=\int_{a_n}^{b_n} g(x)\,x^j\,\mathrm{d}x\,, \qquad 0\leq k\leq m\,, \quad (41)$$

for all n. Then if F_m and G_m are primitives of order m of f and g, respectively, that coincide of order m at one point of K, then they will coincide in all of K. Let $z\in K$ be a point where the point value exists and is of order m at the most, so that for the primitive $F_m(x)=\int_z^x (x-t)^{m-1} f(t)\,\mathrm{d}t/(m-1)!$ we have

$$f(z)=\lim_{y\to z}\frac{m!F_m(y)}{(y-z)^m}\,. \quad (42)$$

If $G_m(x) = \int_z^x (x-t)^{m-1} g(t)\,\mathrm{d}t/(m-1)!$ then $F_m(x) = G_m(x)$ for all $x \in K$. If the distributional point value $g(z)$ exists and is of order m at the most then

$$f(z) = \lim_{y\to z} \frac{m!F_m(y)}{(y-z)^m} = \lim_{y\to z,\, y\in K} \frac{m!F_m(y)}{(y-z)^m}$$
$$= \lim_{y\to z,\, y\in K} \frac{m!G_m(y)}{(y-z)^m} = g(z)\ . \tag{43}$$

But g is Lebesgue integrable and thus the distributional point value $g(z)$ is of order 1 almost everywhere and vanishes almost everywhere in K. Hence $f(z) = 0$ almost everywhere in $K_m = \{z \in K : f(z) \text{ exists of order } m \text{ at the most}\}$. The result follows because $\cup_{m=1}^{\infty} K_m$ is of full measure in K. □

We now have all the preliminaries to prove our main theorem.

Theorem 1. *Let f be distributionally integrable in $[a,b]$. Let F_m be some fixed primitives of order m of f. Suppose that $F_m|_K$ is absolutely continuous in K for all $m \geq 1$. Then $f\chi_K$ is Lebesgue integrable and*

$$f\chi_{[a,b]} = f\chi_K + \sum_{n=1}^{\infty} f\chi_{[a_n,b_n]}\,, \tag{44}$$

in the space $\mathcal{D}'(\mathbb{R})$.

Proof. We can easily prove a special case: If the formula

$$F\chi_{[a,b]} = F\chi_K + \sum_{n=1}^{\infty} F\chi_{[a_n,b_n]}\,, \tag{45}$$

is true for $F = F_1$ and the hypotheses of the theorem hold, then (44) is satisfied. Namely, if we take the Proposition 1 into account, then we obtain

$$\begin{aligned} f\chi_{[a,b]} &= \left(F\chi_{[a,b]}\right)' - F(a)\,\delta_a + F(b)\,\delta_b \\ &= (F\chi_K)' + \sum_{n=1}^{\infty}\left(F\chi_{[a_n,b_n]}\right)' - F(a)\,\delta_a + F(b)\,\delta_b \\ &= (F\chi_K)' + \sum_{n=1}^{\infty}\left(f\chi_{[a_n,b_n]} + F(a_n)\,\delta_{a_n} - F(b_n)\,\delta_{b_n}\right) - F(a)\,\delta_a + F(b)\,\delta_b \\ &= l\chi_K + \sum_{n=1}^{\infty} f\chi_{[a_n,b_n]}\,, \end{aligned}$$

where $l\chi_K \in L^1$. But applying the Lemma 8 to $f\chi_{[a,b]} - l\chi_K$ it follows that $f\chi_K = l\chi_K$ and thus we obtain (44).

We therefore obtain the general result by iteration of this special case. Indeed, if m is large enough then F_m is continuous in $[a,b]$. Consequently F_m satisfies the decomposition formula (45). Then the special case yields that the decomposition formula also holds for $(F_m)'$, and thus for $(F_m)''$, and so on, until we obtain that it holds for $(F_m)^{(m)} = f$. □

Let us observe that if f satisfies the hypotheses of the Theorem 1 then evaluation at a test function ϕ gives

$$\int_a^b f(x)\,\phi(x)\,\mathrm{d}x = \int_K f(x)\,\phi(x)\,\mathrm{d}x + \sum_{n=1}^{\infty}\int_{a_n}^{b_n} f(x)\,\phi(x)\,\mathrm{d}x\,, \tag{46}$$

for all $\phi \in \mathcal{D}(\mathbb{R})$. If those hypotheses are not satisfied then (46) might not be true for all ϕ, but maybe it holds for *some* test functions. We are particularly interested in when it holds for $\phi = 1$. We first need a preliminary result similar to the Lemma 8.

Lemma 9. *Let f be distributionally integrable and let F be a primitive. Suppose $F|_K$ is absolutely continuous in K. Then*

$$f(x) = \lim_{y \to x, y \in K} \frac{F(y) - F(x)}{y - x}, \text{ almost everywhere in } K, \tag{47}$$

and f is Lebesgue integrable in K.

Proof. It is enough to prove the result in the special case when $F|_K$ vanishes since in the general case, decomposing $F|_K$ as $L + S$ where L is jump-less and S is a jump function, we obtain $F = \widetilde{L} + \widetilde{S} + V$, where $\widetilde{L}$ is the extension of L to $[a, b]$ that is constant in each $[a_n, b_n]$, $\widetilde{S}$ is the extension of S to $[a, b]$ that is linear in each $[a_n, b_n]$, and where V is the primitive of a distributionally integrable function with $V(x) = 0$ for all $x \in K$, and both $\widetilde{L}$ and $\widetilde{S}$ satisfy the result corresponding to (47). Therefore from now on we assume that $F(x) = 0$ for $x \in K$ and if $x \leq a$ or $x \geq b$.

Notice that if F is Lebesgue integrable then $F = \sum_{n=1}^{\infty} F\chi_{[a_n, b_n]}$ and differentiation yields $f = \sum_{n=1}^{\infty} f\chi_{[a_n, b_n]}$ and the Lemma 8 gives that $f(x) = 0$ almost everywhere in K. As the examples in the next section show, for a distributionally integrable f the series $\sum_{n=1}^{\infty} f\chi_{[a_n, b_n]}$ does not have to converge to f, so that for the general case we need to employ a different argument.

Let U be a harmonic representation of F, that is, $U(x, y)$ is harmonic in the upper half plane $y > 0$, vanishes angularly at infinity and $\lim_{y \to 0} U(x, y) = F(x)$ distributionally. Since F has point values at each point, we may extend U to the closed half plane $y \geq 0$ by putting $U(x, 0) = F(x)\chi_{[a,b]}$; while in general U is not continuous at a point $(x_0, 0)$, $U(x, y) \to U(x_0, 0)$ if $(x, y) \to (x_0, 0)$ angularly. Notice also that $u = \partial U/\partial x$ is a harmonic representation of f and consequently if the point value $f(x_0)$ exists then $u(x, y) \to f(x_0)$ as $(x, y) \to (x_0, 0)$ angularly.

Let $M > 0$. Let K_M be the set of points $x_0 \in K$ such that $|U(x, y)| \leq M$ for $(x, y) \in \Omega_{M, x_0}$ where Ω_{M, x_0} is the region $y > \varphi_{x_0}(x) = \min\{|x - x_0|, 1\}$. Let Ω_M be the region $\cup_{x_0 \in K_M} \Omega_{M, x_0}$. Then $|U(x, y)| \leq M$ for $(x, y) \in \overline{\Omega}_M$. On the other hand, Ω_M is the set of points (x, y) with $y > \varphi_{x_0}(x)$ for some $x_0 \in K_M$, that is, $y > \lambda_M(x)$ where

$$\lambda_M(x) = \inf_{x_0 \in K_M} \varphi_{x_0}(x) = \min\{d(x, K_M), 1\}, \tag{48}$$

$d(x, K_M)$ being the distance from x to K_M. Observe that λ_M is a Lipschitz function and thus a set $X \subset \mathbb{R}$ has measure zero if and only if the set $\{(x, \lambda_M(x)) : x \in X\}$ has measure zero as a subset of the boundary $\partial\Omega_M$, with the measure induced by a conformal equivalence map of Ω_M and the upper half palne.

Consider now the function $F_M(x) = U(x, \lambda_M(x))$. It is a bounded function and $F_M(x) = 0$ in K_M. Hence $F_M'(x) = 0$ almost everywhere in K_M, as a subset of $\mathbb{R}$ or as subset of $\partial\Omega_M$. The distributional boundary value of u on the curve $\partial\Omega_M$ has point values almost everywhere and vanishes if $x_0 \in K_M$ and $F_M'(x_0) = 0$ and thus $u(x, y) \to 0$ as $(x, y) \to (x_0, 0)$ angularly inside the region Ω_M. This does not imply that $f(x_0) = 0$, but suppose now that also $f(x_0)$ exists. Then $u(x, y) \to f(x_0)$ if $(x, y) \to (x_0, 0)$ angularly in the upper half plane, too. Hence $f(x_0) = 0$. It follows that $f(x) = 0$ almost everywhere in K_M. But $K = \cup_{M > 0} K_M$ so that $f(x) = 0$ almost everywhere in K. □

We immediately obtain the ensuing result.

Theorem 2. *Let f be distributionally integrable in $[a,b]$ with primitive F. Suppose that $F|_K$ is absolutely continuous in K. Then $f\chi_K$ is Lebesgue integrable and*

$$\int_a^b f(x)\,\mathrm{d}x = \int_K f(x)\,\mathrm{d}x + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)\,\mathrm{d}x\,. \tag{49}$$

Proof. The Lemma 9 says that in the Proposition 1 if $W = F|_K$ then $l = f\chi_K$, so that $f\chi_K \in L^1$. Evaluation of (37) at $\phi = 1$ then yields (49). □

At this point it is worth to point out that while of course if W is absolutely continuous in K then W is of bounded variation and continuous and $\sum_{n=1}^{\infty} |W(b_n) - W(a_n)|$ converges, the converse also holds for some primitives of distributionally integrable functions.

Lemma 10. *Suppose f and $f\chi_K$ are distributionally integrable. If $F|_K$ is of bounded variation and continuous and $\sum_{n=1}^{\infty} |F(b_n) - F(a_n)|$ converges then $F|_K$ is absolutely continuous in K. Also $f\chi_K$ is Lebesgue integrable.*

Proof. Indeed, let $g = f\chi_K$ and $h = f - g$. Let G and H be the corresponding primitives. Since $\sum_{n=1}^{\infty} |H(b_n) - H(a_n)| = \sum_{n=1}^{\infty} |F(b_n) - F(a_n)|$, the restriction $H|_K$ is a pure jump function and thus absolutely continuous. On the other hand, G is constant in each $[a_n, b_n]$ and thus it is also continuous and of bounded variation, hence of the form $G_{\mathrm{ac}} + G_{\mathrm{sg}}$, an absolutely continuous part and a singular part. But the singular measure $G'_{\mathrm{sg}} = G' - G'_{\mathrm{ac}}$ is a distributionally integrable function and hence $G'_{\mathrm{sg}} = 0$. Thus $G|_K = G_{\mathrm{ac}}|_K$ is absolutely continuous and $g = G'_{\mathrm{ac}}$ is Lebesgue integrable. □

One may also express the formula (49) as a way to reconstruct a function from the point values of its distributional derivative.

Theorem 3. *Let F be the primitive of a distributionally integrable function. Suppose that $F|_K$ is absolutely continuous in K. Then $F'\chi_K$ is Lebesgue integrable and*

$$F(b) - F(a) = \int_K F'(x)\,\mathrm{d}x + \sum_{n=1}^{\infty} (F(b_n) - F(a_n))\,. \tag{50}$$

Let us observe that the particular case when K has measure zero is actually a corollary of the Proposition 1 that tell us that if $F|_K$ is absolutely continuous in K then

$$F(b) - F(a) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))\,.$$

Another interesting result is the following extended integration by parts formula.

Theorem 4. *Let f be distributionally integrable and let F be a primitive. Suppose $F|_K$ is absolutely continuous in K. Then*

$$\int_K f(x)\,\phi(x)\,\mathrm{d}x = -\int_K F(x)\,\phi'(x)\,\mathrm{d}x + F\phi|_a^b - \sum_{n=1}^{\infty} F\phi|_{a_n}^{b_n}\,. \tag{51}$$

Proof. Indeed, the Proposition 1 gives, after a slight rearrangement, remembering that $l = f\chi_K$ if $W = F|_K$,

$$f\chi_K = (F\chi_K)' - F(a)\,\delta_a + F(b)\,\delta_b + \sum_{n=1}^{\infty} (F(a_n)\,\delta_{a_n} - F(b_n)\,\delta_{b_n})\,, \tag{52}$$

and (51) follows by evaluation at a test function ϕ. □

6. Examples

Let us start by considering examples of Łojasiewicz functions. The oscillatory functions

$$e_{\alpha,\beta}(x) = \begin{cases} |x|^{\alpha} e^{i|x|^{-\beta}}, & x \neq 0\,, \\ 0\,, & x = 0\,, \end{cases} \tag{53}$$

for $\alpha \in \mathbb{C}$ and $\beta > 0$, are Łojasiewicz functions. If H is the Heaviside function, then so are the functions $H(\pm x)\, s_{\alpha,\beta}(x)$ and their linear combinations.

In fact, instead of the exponential e^{ix} one may employ any continuous periodic function with zero mean ψ and obtain Łojasiewicz functions as

$$\Psi_{\alpha,\beta}(x) = \begin{cases} |x|^{\alpha} \psi\left(|x|^{-\beta}\right), & x \neq 0\,, \\ 0\,, & x = 0\,, \end{cases} \tag{54}$$

for any $\alpha \in \mathbb{C}$ and $\beta > 0$. This is true because for any periodic distribution f, in particular for ψ,

$$\left\langle x^{\alpha} f\left(\lambda x^{\beta}\right), \phi(x)\right\rangle = o\left(\lambda^{-N}\right), \qquad \text{as } \lambda \to \infty\,, \tag{55}$$

for any $\phi \in \mathcal{K}(0,\infty)$ and any $N > 0$ [4].

We can construct Łojasiewicz functions that oscillate at every point of K as follows,

$$\Psi_{\alpha,\beta,K}(x) = \begin{cases} \Psi_{\alpha,\beta}\left((x-a_n)(b_n-x)\right), & x \in (a_n,b_n)\,, \\ 0\,, & x \in K\,. \end{cases} \tag{56}$$

It is interesting that this type of construction has been employed since a long time ago. In fact, Lebesgue considers $\Psi_{2,2,K}$ for $\psi(x) = \sin x$ in writings from 1902 [10, pg. 134] and his proof of the fact that $\Psi'_{2,2,K}(x)$ exists everywhere in $[a,b]$ and vanishes in K gives with little modification the argument needed to show that for any continuous ψ the function $\Psi_{\alpha,\beta,K}$ is a Łojasiewicz function.

Suppose now that ψ is a C^{∞} function. Then all derivatives of any order $\Psi^{(m)}_{\alpha,\beta,K}$ are also Łojasiewicz functions that vanish in K. Consequently they are distributionally integrable over $[a,b]$ and actually they are also primitives of distributionally integrable functions.

Example 1. *The function $\Psi_{1/2,1,K}$, say for $\psi(x) = \sin x$, is a continuous function that is the primitive of a distributionally integrable function, namely $\Psi'_{1/2,1,K}$. However, if K has positive measure then $\Psi'_{1/2,1,K}$ is not Denjoy-Perron-Henstock-Kurzweil integrable because the* ordinary *derivative of $\Psi_{1/2,1,K}$ does not exist in K so that it does not exist almost everywhere.*

Example 2. *Let ψ be a C^{∞} function periodic of period 1. Suppose $\psi(1) = L \neq 0$. Let $F = \Psi_{0,1}$ and $f = F'$. Let $K = \{0\} \cup \{1/n : n = 1,2,3,\ldots\}$. Let $(a_n,b_n) = \left(\frac{1}{n+1}, \frac{1}{n}\right)$. Then f is distributionally integrable in $[a,b] = [0,1]$ and*

$$\int_0^1 f(x)\,\mathrm{d}x = F(1) - F(0) = L\,, \tag{57}$$

but $\int_K f(x)\,\mathrm{d}x = 0$ and

$$\sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n)\right) = 0\,. \tag{58}$$

Actually the series $\sum_{n=1}^{\infty} f\chi_{[a_n,b_n]}$ converges in $\mathcal{D}'(\mathbb{R})$ but it does not converge to $f\chi_{[0,1]} - f\chi_K = f\chi_{[0,1]}$ but rather

$$f(x)\,\chi_K(x) + \sum_{n=1}^{\infty} f(x)\,\chi_{[a_n,b_n]}(x) = f(x)\,\chi_{[0,1]}(x) + L\delta(x)\,, \tag{59}$$

since

$$\int_K f(x)\,\phi(x)\ \mathrm{d}x + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)\,\phi(x)\ \mathrm{d}x = \int_0^1 f(x)\,\phi(x)\ \mathrm{d}x + L\phi(0)\ , \tag{60}$$

for all test functions ϕ.

Example 3. *Let* ψ *be a* C^∞ *function periodic of period* 1 *that satisfies* $\psi(x) = 1$ *for* $|x| \leq \varepsilon$ *for some* $\varepsilon > 0$. *Consider the functions* $G_2 = \Psi_{0,1}$, $G_1 = G_2'$, *and* $g = G_2''$. *Let* $K = \{0\} \cup \{1/n : n = 1, 2, 3, \ldots\}$. *Then the restriction of the first order primitive of* g *to* K *vanishes there and thus* $G_1|_K$ *absolutely continuous but the restriction of the second order primitive* $G_2|_K$ *is not absolutely continuous since* $F_2(1/n) = 1$ *for all* n *while* $F_2(0) = 0$. *We also easily obtain that*

$$g(x)\,\chi_K(x) + \sum_{n=1}^{\infty} g(x)\,\chi_{[a_n,b_n]}(x) = g\chi_{[0,1]}(x) + L\delta'(x)\ . \tag{61}$$

This shows that the absolute continuity of the first primitive on K *is not enough to obtain the result of the Theorem 1.*

More examples are easy to construct by using these ideas. For example, if K_0 has measure zero and ν is any Radon measure concentrated on K_0 then we can find $K = [a,b] \setminus \bigcup_{n=1}^{\infty}(a_n, b_n)$, with $K_0 \subset K$, and f distributionally integrable such that

$$f\chi_K + \sum_{n=1}^{\infty} f\chi_{[a_n,b_n]} = f\chi_{[a,b]} + \nu\,. \tag{62}$$

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