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ERRATUM TO "INEQUALITIES OF OSTROWSKI AND SIMPSON TYPE FOR MAPPINGS OF TWO VARIABLES WITH BOUNDED VARIATION AND APPLICATIONS"

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ABSTRACT. In recent publication [1], authors prove two inequalities of Ostrowski type and Simpson type for functions of two variables with bounded variation. However, there are some minor errors appeared in main results arising out of using a inexact lemma. These errors are corrected in this Erratum.

In [1, Lemma 1], some Ostrowski type inequalities for functions of two variables with bounded variation is given and applications in cubature formula is provided. However, there are some minor errors appeared in main results of the paper [1] since the Lemma 1 in the published version [1] is inexact. In [2], Moricz has already provided the correct version of the temma in [2, Lemma 2]. Fundamentally, this correction lead to small changings in the left sides of the results given in [1]. The goal of this paper is to correct these errors using the Moricz's results and we now start off to correct them.

Lemma 1. [2, Lemma 2] *If $f(t,s)$ is continuous on renctangle $Q=[a,b]\times[c,d]$ and $\alpha(t,s)\in BV_H(Q)$, then $\alpha(t,s)$ is integrable with respect to $f(t,s)$ over Q in the Riemann-Stieltjes sense, and*

$$\begin{aligned}\int_a^b\int_c^d f(t,s)d_td_s\alpha(t,s) &= \int_a^b\int_c^d \alpha(t,s)d_td_sf(t,s)\\ &- \int_a^b \alpha(t,d)d_tf(t,d)+\int_a^b \alpha(t,c)d_tf(t,c)\\ &- \int_c^d \alpha(b,s)d_sf(b,s)+\int_c^d \alpha(a,s)d_sf(a,s)\\ &+ f(b,d)\alpha(b,d)-f(b,c)\alpha(b,c)-f(a,d)\alpha(a,d)+f(a,c)\alpha(a,c).\end{aligned}$$

Using this lemma, we obtain the correction of the result in [1] as follows:

Correction of Theorem 2: If the function $f:Q=[a,b]\times[c,d]\to R$ is of bounded variation on Q, then for all $(x,y)\in Q$ we have the inequality

$$\left|f(x,y)-\frac{1}{b-a}\int_a^b f(t,y)dt-\frac{1}{d-c}\int_c^d f(x,s)ds+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\left[\frac{1}{2}+\frac{\left|y-\frac{c+d}{2}\right|}{d-c}\right]\bigvee_a^b\bigvee_c^d(f).$$

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Correction to Corollary 1: In Theorem 2, let $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, then we have

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{b-a}\int_a^b f\left(t, \frac{c+d}{2}\right)dt - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, s\right)ds \right. \\ &\left. + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\ \leq\ & \frac{1}{4}\bigvee_Q (f).\end{aligned}$$

Correction to Theorem 3: If the function $f : Q = [a,b] \times [c,d] \to R$ is of bounded variation on Q, then we have the inequality

$$\begin{aligned}&\left| \frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4} - \frac{1}{2(d-c)}\left[\int_c^d f(a,s)ds + \int_c^d f(b,s)ds\right] \right. \\ &\left. - \frac{1}{2(b-a)}\left[\int_a^b f(t,c)dt + \int_a^b f(t,d)dt\right] + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\ \leq\ & \frac{1}{4}\bigvee_a^b\bigvee_c^d (f).\end{aligned}$$

Correction to Theorem 6: If the function $f : Q = [a,b] \times [c,d] \to R$ is of bounded variation on Q, then we have the inequality

$$\begin{aligned}&\left| \frac{1}{9}\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}\right.\right. \\ &+ f\left(a, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(b, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, d\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &- \frac{1}{6(d-c)}\left[\int_c^d f(a,s)ds - 4\int_c^d f\left(\frac{a+b}{2}, s\right)ds - \int_c^d f(b,s)ds\right] \\ &- \frac{1}{6(b-a)}\left[\int_a^b f(t,c)dt - d\int_a^b f\left(t, \frac{c+d}{2}\right)dt - \int_a^b f(t,d)dt\right] \\ &\left. + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\ \leq\ & \frac{1}{9}\bigvee_a^b\bigvee_c^d (f).\end{aligned}$$

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$, and $l_j := y_{j+1} - y_j$,

$$\upsilon(h) := \max\{h_i|\ i = 0, ..., n-1\},$$

$$\upsilon(l) := \max\{l_j|\ j = 0, ..., m-1\}.$$

Correction to Theorem 7: Let $f : Q \to \mathbb{R}$ is of bounded variation on Q and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$. Then we have the cubature formula:

$$\begin{aligned}
&\int_a^b \int_c^d f(t,s)dsdt \\
&= \sum_{i=0}^{n-1} h_i \int_c^d f(\xi_i, s)ds + \sum_{j=0}^{m-1} l_j \int_a^b f(t, \eta_j)dt \\
&\quad - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j) + R(\xi, \eta, I_n, J_m, f).
\end{aligned}$$

The remainder $R(\xi, \eta, I_n, J_m, f)$ satisfies

$$\begin{aligned}
&|R(\xi, \eta, I_n, J_m, f)| \qquad (1) \\
&\leq \left[\frac{1}{2}\upsilon(h) + \max_{i \in \{0,..,n-1\}} \left\{ \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right\} \right] \\
&\quad \times \left[\frac{1}{2}\upsilon(l) + \max_{j \in \{0,..,m-1\}} \left\{ \left| \eta_j - \frac{y_j + y_{j+1}}{2} \right| \right\} \right] \bigvee_a^b \bigvee_c^d (f) \\
&\leq \upsilon(h)\upsilon(l) \bigvee_a^b \bigvee_c^d (f)
\end{aligned}$$

for all $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ and $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$.

Correction to Theorem 8: Let $f : Q \to \mathbb{R}$ is of bounded variation on Q and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$. Then we have the cubature formula:

$$\begin{aligned}
&\int_a^b \int_c^d f(t,s)dsdt \\
&= \frac{1}{6}\left[\sum_{i=0}^{n-1} h_i \int_c^d f(x_i, s)ds + \sum_{i=0}^{n-1} h_i \int_c^d f\left(\frac{x_i + x_{i+1}}{2}, s\right) ds + \sum_{i=0}^{n-1} h_i \int_c^d f(x_{i+1}, s)ds \right] \\
&\quad + \frac{1}{6}\left[\sum_{j=0}^{m-1} l_j \int_a^b f(t, y_j)dt + \sum_{j=0}^{m-1} l_j \int_a^b f\left(t, \frac{y_j + y_{j+1}}{2}\right) dt + \sum_{j=0}^{m-1} l_j \int_a^b f(t, y_{j+1})dt \right] \\
&\quad - \frac{1}{9} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j \left[f\left(x_i, \frac{y_j + y_{j+1}}{2}\right) + f\left(\frac{x_i + x_{i+1}}{2}, y_j\right) + f\left(x_{i+1}, \frac{y_j + y_{j+1}}{2}\right) \right. \\
&\quad \left. + f\left(\frac{x_i + x_{i+1}}{2}, y_{j+1}\right) \right] - \frac{4}{9} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) \\
&\quad - \frac{1}{36} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j \left[f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_j) \right] \\
&\quad + R_S(I_n, J_m, f).
\end{aligned}$$

The remainder $R(\xi,\eta,I_n,J_m,f)$ satisfies

$$\left|R(\xi,\eta,I_n,J_m,f)\right| \leq \frac{1}{9}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

References

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