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A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY A LINEAR OPERATOR

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Abstract. In the present paper, we investigate some basic properties of a subclass of harmonic functions defined by multiplier transformations. Such as, coefficient inequalities, distortion bounds, convolutions, convex combinations and extreme points.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denotes the open unit disk and let H denotes the family of continuous complex valued harmonic functions which are harmonic in $\mathbb{D}$ and let A be the subclass of H consisting of functions which are analytic in $\mathbb{D}$. A function harmonic in $\mathbb{D}$ may be written as $f = h + \overline{g}$, where h and g are members of A. We call h the analytic part and g co-analytic part of f. Note that H reduces to the class of normalized analytic functions if the co-analytic part of its members are zero. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$, $z \in \mathbb{D}$ (see Clunie and Sheil-Small [1]). Denote by SH the class of functions $f = h + \overline{g}$ thar are harmonic univalent and sense-preserving in the unit disk $\mathbb{D}$ for which $f(0) = f_z(0) - 1 = 0$. Also note that SH reduces to the class S of normalized analytic univalent functions in $\mathbb{D}$, if the co-analytic part of f is identically zero. Then for $f = h + \overline{g}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1}$$

One shows easily that the sense-preserving property implies that $|b_1| < 1$. The subclass SH^0 of SH consists of all functions in SH which have the additional property $f_{\bar{z}}(0) = 0$.

In 1984 Clunie and Sheil-Small [1] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses.

For $f \in S$, the differential operator D^n $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ of f was introduced by Salagean [2]. This operator was developed and modified by many researchers over time. As a simple example for $f = h + \overline{g}$ given by (1), Jahangiri et al. [3] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)},$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

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Next, Darus and Ibrahim [4] introduce a differential operator defines as follows: $D_{\lambda,\delta}^{n,\alpha}$: $A \longrightarrow A$ by

$$D_{\lambda,\delta}^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} [k^{\alpha} + (k-1)k^{\alpha}\lambda]^n C(\delta,k) a_k z^k \tag{2}$$

where

$$C(\delta,k) = \binom{\delta+k-1}{\delta} = \frac{\Gamma(k+\delta)}{\Gamma(k)\Gamma(\delta+1)}$$

and $n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \lambda \geq 0, \delta \geq 0$.
Now, we introduce a differential operator defines as follows $D_{\lambda,\delta}^{n,\alpha} : SH \longrightarrow SH$ by

$$\begin{aligned} D_{\lambda,\delta}^{n,\alpha} f(z) &= z + \sum_{k=2}^{\infty} [k^{\alpha} + (k-1)k^{\alpha}\lambda]^n C(\delta,k) a_k z^k \\ &+ (-1)^n \sum_{k=2}^{\infty} [-k^{\alpha} + (k+1)k^{\alpha}\lambda]^n C(\delta,k) \overline{b_k z^k}, \end{aligned} \tag{3}$$

$n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \lambda \geq 1, \delta \geq 0$.

With choosing special numbers for parameters n, α, λ and δ we obtain the following operators studied by various authors:

for $f \in A$,
(i) $D_{0,0}^{n,1} f(z) = D_{1,0}^{n,0} f(z) = D^n f(z)$ ([2]),
(ii) $D_{\lambda,0}^{n,0} f(z) = D_{\lambda}^{n} f(z)$ ([5]),
(iii) $D_{\lambda,\delta}^{0,\alpha} f(z) = D^{\delta} f(z)$ ([6]),
(iv) $D_{\lambda,\delta}^{n,0} f(z) = D_{\lambda,\delta}^{n} f(z)$ ([7]),
(v) $D_{0,\delta}^{n,1} f(z) = D_{1,\delta}^{n,0} f(z) = D_{\delta}^{n} f(z)$ ([15]),

for $f \in H$,
(vi) $D_{0,0}^{n,1} f(z) = I_{1,0}^{n} f(z) = D^n f(z)$ ([3], [8]),
(vii) $D_{\lambda,0}^{n,0} f(z) = D_{\lambda}^{n} f(z)$ ([9]),
(viii) $D_{0,\delta}^{n,1} f(z) = D_{\delta}^{n} f(z)$ ([10]),

Denote by $SH(\alpha, \lambda, \delta, n, \mu)$ the subclass of SH consisting of functions f of the form (1) that satisfy the condition

$$\operatorname{Re}\left(\frac{D_{\lambda,\delta}^{n+1,\alpha} f(z)}{D_{\lambda,\delta}^{n,\alpha} f(z)} \right) \geq \mu, \quad 0 \leq \mu < 1 \tag{4}$$

where $D_{\lambda,\delta}^{n,\alpha} f(z)$ is defined by (3).

We let the subclass $\overline{SH}(\alpha, \lambda, \delta, n, \mu)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ in SH so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \; b_k \geq 0. \tag{5}$$

By suitably specializing the parameters, the classes $SH(\alpha, \lambda, \delta, n, \mu)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH(0,1,0,0,0) = SH^*(0)$ ([11], [12], [13]),
(ii) $SH(0,1,0,0,\mu) = SH^*(\mu)$ ([14]),
(iii) $SH(0,1,0,1,0) = KH(0)$ ([11], [12], [13]),
(iv) $SH(0,1,0,1,\mu) = KH(\mu)$ ([14]),
(v) $SH(0,1,0,n,\mu) = H(n,\mu)$ ([3]),
(vi) $SH(0,\lambda,0,n,\mu) = SH(\lambda,n,\mu)$, $m \in \mathbb{N}$ ([16]) .

Define

$$SH^0(\alpha,\lambda,\delta,n,\mu) := H(\alpha,\lambda,\delta,n,\mu) \cap SH^0$$

and

$$\overline{SH}^0(\alpha,\lambda,\delta,n,\mu) := \overline{SH}(\alpha,\lambda,\delta,n,\mu) \cap SH^0.$$

The object of the present paper is to investigate the various properties of harmonic univalent functions belonging to the subclasses $SH(\alpha,\lambda,\delta,n,\mu)$ and $\overline{SH}(\alpha,\lambda,\delta,n,\mu)$. Necessary and sufficient coefficient conditions, distortion bounds, extreme points and convex combination of above mentioned class are derived.

2. Main results

In the first theorem, we introduce a sufficient coefficient condition for harmonic functions in $SH(\alpha,\lambda,\delta,n,\mu)$.

Theorem 1. *Let $f = h + \overline{g}$ be so that h and g are given by (1) with $b_1 = 0$. Let*

$$\sum_{k=2}^{\infty} [k^\alpha + (k-1)k^\alpha\lambda]^n [k^\alpha + (k-1)k^\alpha\lambda - \mu] C(\delta,k) |a_k| + \sum_{k=2}^{\infty} [-k^\alpha + (k+1)k^\alpha\lambda]^n [-k^\alpha + (k+1)k^\alpha\lambda + \mu] C(\delta,k) |b_k| \leq 1-\mu, \tag{6}$$

where $n,\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \lambda \geq 1, \delta \geq 0,\ 0 \leq \mu < 1$ and $C(\delta,k) = \binom{\delta+k-1}{\delta}$. Then f is sense-preserving, harmonic univalent in $\mathbb{D}$ and $f \in SH^0(\alpha,\lambda,\delta,n,\mu)$.

Proof. If $z_1 \neq z_2$,

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum_{k=2}^{\infty} b_k\left(z_1^k - z_2^k\right)}{(z_1-z_2) + \sum_{k=2}^{\infty} a_k\left(z_1^k - z_2^k\right)}\right| \\ &> 1 - \frac{\sum_{k=2}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|} \\ &\geq 1 - \frac{\sum_{k=2}^{\infty} \frac{[-k^\alpha+(k+1)k^\alpha\lambda]^n[-k^\alpha+(k+1)k^\alpha\lambda+\mu]C(\delta,k)}{1-\mu}|b_k|}{1-\sum_{k=2}^{\infty} \frac{[k^\alpha+(k-1)k^\alpha\lambda]^n[k^\alpha+(k-1)k^\alpha\lambda-\mu]C(\delta,k)}{1-\mu}|a_k|} \geq 0,\end{aligned}$$

which proves univalence.

Note that f is sense preserving in $\mathbb{D}$. This is because

$$\begin{aligned} |h'(z)| &\geq 1-\sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} \\ &> 1-\sum_{k=2}^{\infty} \frac{[k^\alpha+(k-1)k^\alpha\lambda]^n\,[k^\alpha+(k-1)k^\alpha\lambda-\mu]\,C(\delta,k)}{1-\mu}\,|a_k| \\ &\geq \sum_{k=2}^{\infty} \frac{[-k^\alpha+(k+1)k^\alpha\lambda]^n\,[-k^\alpha+(k+1)k^\alpha\lambda+\mu]\,C(\delta,k)}{1-\mu}\,|b_k| \\ &> \sum_{k=2}^{\infty} k\,|b_k|\,|z|^{k-1} \\ &\geq |g'(z)|\,. \end{aligned}$$

Using the fact that $\mathrm{Re}w \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|(1-\mu)D_{\lambda,\delta}^{n,\alpha}f(z)+D_{\lambda,\delta}^{n+1,\alpha}f(z)\right|-\left|(1+\mu)D_{\lambda,\delta}^{n,\alpha}f(z)-D_{\lambda,\delta}^{n+1,\alpha}f(z)\right| \geq 0. \tag{7}$$

Substituting for $D_{\lambda,\delta}^{n,\alpha}f(z)$ and $D_{\lambda,\delta}^{n+1,\alpha}f(z)$ in (7), we obtain

$$\begin{aligned} &\left|(1-\mu)D_{\lambda,\delta}^{n,\alpha}f(z)+D_{\lambda,\delta}^{n+1,\alpha}f(z)\right|-\left|(1+\mu)D_{\lambda,\delta}^{n,\alpha}f(z)-D_{\lambda,\delta}^{n+1,\alpha}f(z)\right| \\ \geq\ & 2(1-\mu)\,|z|-\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^n\,[k^\alpha+(k-1)k^\alpha\lambda+1-\mu]\,C(\delta,k)\,|a_k|\,|z|^k \\ &-\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^n\,[-k^\alpha+(k+1)k^\alpha\lambda-1+\mu]\,C(\delta,k)\,|b_k|\,|z|^k \\ &-\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^n\,[k^\alpha+(k-1)k^\alpha\lambda-1-\mu]\,C(\delta,k)\,|a_k|\,|z|^k \\ &-\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^n\,[-k^\alpha+(k+1)k^\alpha\lambda+1+\mu]\,C(\delta,k)\,|b_k|\,|z|^k \\ >\ & 2(1-\mu)\,|z|\left\{1-\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^n\,[k^\alpha+(k-1)k^\alpha\lambda-\mu]\,C(\delta,k)\,|a_k|\right. \\ &\left.-\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^n\,[-k^\alpha+(k+1)k^\alpha\lambda+\mu]\,C(\delta,k)\,|b_k|\right\}. \end{aligned}$$

This last expression is non-negative by (6), and so the proof is complete. □

Theorem 2. *Let* $f_n = h+\overline{g}_n$ *be given by (5) with* $b_1=0$.
Then $f_n \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$ *if and only if*

$$\begin{aligned} &\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^n\,[k^\alpha+(k-1)k^\alpha\lambda-\mu]\,C(\delta,k)\,|a_k|+ \\ &\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^n\,[-k^\alpha+(k+1)k^\alpha\lambda+\mu]\,C(\delta,k)\,|b_k| \leq 1-\mu, \end{aligned} \tag{8}$$

where $n,\alpha \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}, \lambda \geq 1, \delta \geq 0,\ 0 \leq \mu < 1$ *and* $C(\delta,k)=\binom{\delta+k-1}{\delta}$.

Proof. The "if "part follows from Theorem 1 upon noting that $\overline{SH}^0(\alpha,\lambda,\delta,n,\mu) \subset SH^0(\alpha,\lambda,\delta,n,\mu)$. For the "only if "part, we show that $f_n \notin \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$ if the condition (8) does not hold. Note that a necessary and sufficient condition for $f_n = h + \overline{g}_n$ given by (5), to be in $\overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$ is that the condition (4) to be satisfied. This is equivalent to

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$ we must have

$$\mathrm{Re}\left\{\frac{(1-\mu)z - \sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^n\left[k^\alpha+(k-1)k^\alpha\lambda-\mu\right]C(\delta,k)a_kz^k}{z-\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^nC(\delta,k)a_kz^k+\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^nC(\delta,k)b_k\overline{z}^k}\right.$$
$$\left.-\frac{\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^n\left[-k^\alpha+(k+1)k^\alpha\lambda+\mu\right]C(\delta,k)b_k\overline{z}^k}{z-\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^nC(\delta,k)a_kz^k+\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^nC(\delta,k)b_k\overline{z}^k}\right\} \ge 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$ we must have

$$\left\{\frac{(1-\mu)-\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^n[k^\alpha+(k-1)k^\alpha\lambda-\mu]C(\delta,k)a_kr^{k-1}}{1-\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^nC(\delta,k)a_kr^{k-1}+\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^nC(\delta,k)b_kr^{k-1}}\right.$$
$$\left.-\frac{\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^n[-k^\alpha+(k+1)k^\alpha\lambda+\mu]C(\delta,k)b_kr^{k-1}}{1-\sum_{k=2}^{\infty}[k^\alpha+(k-1)k^\alpha\lambda]^nC(\delta,k)a_kr^{k-1}+\sum_{k=2}^{\infty}[-k^\alpha+(k+1)k^\alpha\lambda]^nC(\delta,k)b_kr^{k-1}}\right\} \ge 0. \qquad (9)$$

If the condition (8) does not hold, then the expression in (9) is negative for r values approaching 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in (9) is negative. This shows the required condition for $f_n \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$ and so the proof is complete. □

Theorem 3. *Let f_{n_k} be given by (3). Then $f_{n_k} \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$ if and only if*

$$f_{n_k}(z) = \sum_{k=1}^{\infty}\left(X_kh_k(z) + Y_kg_{n_k}(z)\right),$$

where

$$h_1(z) = z,\ h_k(z) = z - \frac{1-\mu}{[k^\alpha+(k-1)k^\alpha\lambda]^n[k^\alpha+(k-1)k^\alpha\lambda-\mu]C(\delta,k)}z^k\ (k=2,3,...),$$

and for $k = 2,3,...$

$$g_{n_1}(z) = z,\ g_{n_k}(z) = z + (-1)^n\frac{1-\mu}{[-k^\alpha+(k+1)k^\alpha\lambda]^n[-k^\alpha+(k+1)k^\alpha\lambda+\mu]C(\delta,k)}\overline{z}^k$$

$X_k \ge 0, Y_k \ge 0, \sum_{k=1}^{\infty}(X_k+Y_k) = 1;\ n,\alpha \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}; \lambda \ge 1, \delta \ge 0,\ 0 \le \mu < 1$ *and* $C(\delta,k) = \binom{\delta+k-1}{\delta}$.

In particular, the extreme points of $\overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$ *are* $\{h_k\}$ *and* $\{g_{n_k}\}$.

Proof. For functions f_{n_k} of the form (5) we have

$$\begin{aligned} f_{n_k}(z) &= \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right) \\ &= \sum_{k=1}^{\infty}\left(X_k + Y_k\right) z - \sum_{k=2}^{\infty}\frac{1-\mu}{\left[k^{\alpha}+(k-1)k^{\alpha}\lambda\right]^n\left[k^{\alpha}+(k-1)k^{\alpha}\lambda-\mu\right]C(\delta,k)}X_k z^k \\ &\quad +(-1)^n\sum_{k=2}^{\infty}\frac{1-\mu}{\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda\right]^n\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda+\mu\right]C(\delta,k)}Y_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{\left[k^{\alpha}+(k-1)k^{\alpha}\lambda\right]^n\left[k^{\alpha}+(k-1)k^{\alpha}\lambda-\mu\right]C(\delta,k)}{1-\mu} \\ &\cdot\left(\frac{1-\mu}{\left[k^{\alpha}+(k-1)k^{\alpha}\lambda\right]^n\left[k^{\alpha}+(k-1)k^{\alpha}\lambda-\mu\right]C(\delta,k)}X_k\right) \\ &+\sum_{k=2}^{\infty}\frac{\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda\right]^n\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda+\mu\right]C(\delta,k)}{1-\mu} \\ &\cdot\left(\frac{1-\mu}{\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda\right]^n\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda+\mu\right]C(\delta,k)}Y_k\right) \\ =\ &\sum_{k=2}^{\infty}X_k + \sum_{k=2}^{\infty}Y_k = 1 - X_1 - Y_1 \leq 1 \end{aligned}$$

and so $f_{n_k} \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$. Conversely, if $f_{n_k} \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$, then

$$a_k \leq \frac{1-\mu}{\left[k^{\alpha}+(k-1)k^{\alpha}\lambda\right]^n\left[k^{\alpha}+(k-1)k^{\alpha}\lambda-\mu\right]C(\delta,k)}$$

and

$$b_k \leq \frac{1-\mu}{\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda\right]^n\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda+\mu\right]C(\delta,k)}.$$

Set

$$\begin{aligned} X_k &= \frac{\left[k^{\alpha}+(k-1)k^{\alpha}\lambda\right]^n\left[k^{\alpha}+(k-1)k^{\alpha}\lambda-\mu\right]C(\delta,k)}{1-\mu}a_k,\ (k=2,3,...) \\ Y_k &= \frac{\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda\right]^n\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda+\mu\right]C(\delta,k)}{1-\mu}b_k,\ (k=2,3,...) \end{aligned}$$

and

$$X_1 + Y_1 = 1 - \left(\sum_{k=2}^{\infty}X_k + Y_k\right)$$

where $X_k,\ Y_k \geq 0$. Then, as required, we obtain

$$f_{n_k}(z) = (X_1+Y_1)z + \sum_{k=2}^{\infty}X_k h_k(z) + \sum_{k=2}^{\infty}Y_k g_{n_k}(z) = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right).$$

□

Theorem 4. *Let* $f_n \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$. *Then for* $|z| = r < 1$ *and* $n,\alpha \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}, \lambda \geq 1, \delta \geq 0, 0 \leq \mu < 1$ *and* $C(\delta,2) = \delta+1$. *we have*

$$|f_n(z)| \leq r + \frac{1-\mu}{\left[2^{\alpha}(1+\lambda)\right]^n\left[2^{\alpha}(1+\lambda)-\mu\right](\delta+1)}r^2,$$

and

$$|f_n(z)| \geq r - \frac{1-\mu}{[2^\alpha(1+\lambda)]^n\,[2^\alpha(1+\lambda)-\mu]\,(\delta+1)}r^2.$$

Proof. We will only prove the right side of the inequality. The left side of the inequality can be shown in a similar way. Let $f_n \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$. Taking the absolute value of f_n we have

$$\begin{aligned}
&|f_n(z)| \leq r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^n \\
\leq\ & r + \frac{(1-\mu)\,r^2}{[2^\alpha(1+\lambda)]^n\,[2^\alpha(1+\lambda)-\mu]\,(\delta+1)} \\
\times\ & \sum_{k=2}^{\infty}\frac{[k^\alpha+(k-1)k^\alpha\lambda]^n\,[k^\alpha+(k-1)k^\alpha\lambda-\mu]\,C(\delta,k)}{1-\mu}a_k \\
+\ & \frac{(1-\mu)\,r^2}{[2^\alpha(1+\lambda)]^n\,[2^\alpha(1+\lambda)-\mu]\,(\delta+1)} \\
\times\ & \sum_{k=2}^{\infty}\frac{[-k^\alpha+(k+1)k^\alpha\lambda]^n\,[-k^\alpha+(k+1)k^\alpha\lambda+\mu]\,C(\delta,k)}{1-\mu}b_k \\
\leq\ & r + \frac{(1-\mu)}{[2^\alpha(1+\lambda)]^n\,[2^\alpha(1+\lambda)-\mu]\,(\delta+1)}r^2.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 4. □

Corollary 1. *Let f_n of the form (5) be so that $f_n \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$, where $n,\alpha \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}, \lambda \geq 1, \delta \geq 0, 0 \leq \mu < 1$ and $C(\delta,2) = \delta+1$.*

$$\left\{w : |w| < 1 - \frac{1-\mu}{[2^\alpha(1+\lambda)]^n\,[2^\alpha(1+\lambda)-\mu]\,(\delta+1)}\right\} \subset f_n(\mathbb{D}).$$

Theorem 5. *The class $\overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$ is closed under convex combinations.*

Proof. Let $f_{n_i} \in \overline{SH}^0(\alpha,\lambda,\delta,n,\mu)$ for $i = 1,2,...$, where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty}a_{k_i}z^k + (-1)^n\sum_{k=2}^{\infty}b_{k_i}\overline{z}^k.$$

Then by (8),

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{[k^\alpha+(k-1)k^\alpha\lambda]^n\,[k^\alpha+(k-1)k^\alpha\lambda-\mu]\,C(\delta,k)}{1-\mu}a_{k_i} \\
+&\sum_{k=2}^{\infty}\frac{[-k^\alpha+(k+1)k^\alpha\lambda]^n\,[-k^\alpha+(k+1)k^\alpha\lambda+\mu]\,C(\delta,k)}{1-\mu}b_{k_i} \leq 1.
\end{aligned} \tag{10}$$

For $\sum\limits_{i=1}^{\infty}t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty}t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i a_{k_i}\right)z^k + (-1)^n\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i b_{k_i}\right)\overline{z}^k.$$

Then by (10),

$$\sum_{k=2}^{\infty}\frac{[k^{\alpha}+(k-1)k^{\alpha}\lambda]^{n}\left[k^{\alpha}+(k-1)k^{\alpha}\lambda-\mu\right]C(\delta,k)}{1-\mu}\left(\sum_{i=1}^{\infty}t_i a_{k_i}\right)$$
$$+\sum_{k=2}^{\infty}\frac{[-k^{\alpha}+(k+1)k^{\alpha}\lambda]^{n}\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda+\mu\right]C(\delta,k)}{1-\mu}\left(\sum_{i=1}^{\infty}t_i b_{k_i}\right)$$
$$=\sum_{i=1}^{\infty}t_i\sum_{k=2}^{\infty}\frac{[k^{\alpha}+(k-1)k^{\alpha}\lambda]^{n}\left[k^{\alpha}+(k-1)k^{\alpha}\lambda-\mu\right]C(\delta,k)}{1-\mu}a_{k_i}$$
$$+\sum_{i=1}^{\infty}t_i\sum_{k=2}^{\infty}\frac{[-k^{\alpha}+(k+1)k^{\alpha}\lambda]^{n}\left[-k^{\alpha}+(k+1)k^{\alpha}\lambda+\mu\right]C(\delta,k)}{1-\mu}b_{k_i}\leq\sum_{i=1}^{\infty}t_i=1.$$

This is the condition required by (8) and so $\sum_{i=1}^{\infty}t_i f_{n_i}(z)\in\overline{SH}^{0}(\alpha,\lambda,\delta,n,\mu)$. □

References

[1] Clunie, J. and Sheil-Small, T., *Harmonic univalent function*, Complex Variables, Annales Academica Scientiarum Fennicae Mathematica, **9**, 3–24, 1984.

[2] Salagean, G.S., *Subclasses of univalent functions in complex analysis* -Fifth Romanian- Finish seminar, of lecture notes in Mathematics, Springer, New York, USA. **1013**, 362–372, 1983.

[3] Jahangiri, J.M., Murugusundaramoorthy, G., Vijaya, K., *Salagean-type harmonic univalent functions*, South J. Pure Appl. Math. **2**, 77–82, 2002.

[4] Darus, M. and Ibrahim, R.W., *New classes containing generalization of differential operator*, Appl. Math. Sci. **3**, 2507–2515, 2009.

[5] Al-Aboudi, F.M., *On univalent functions defined by a generalized Salagean operator*, J. Math. Sci. **27**, 1429–1436, 2004.

[6] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49**, 109–115, 1975.

[7] Darus, M. and Ibrahim, R.W., *Generalization of differential operator*, J. Math. Stat. **4**, 138–144, 2008.

[8] Bayram, H. and Yalçın, S., *A Subclass of harmonic univalent functions defined by a linear operator*, Pal. Jour. Math. **6** (Special Issue II), 320–326, 2017.

[9] Yasar, E. and Yalçın, S. *Certain Subclasses Of Harmonic Univalent Functions Associated With Generalized Salagean Operator*, Acta Universitatis Apulensis **29**, 153–163, 2012.

[10] Al-Shaqsi, K. and Darus, M., *On Harmonic Functions Defined by Derivative Operator*, Journal of Inequalities and Applications, 2008: 263413, 2007.

[11] Avcı, Y. and Zlotkiewicz, E., *On harmonic univalent mappings*, Ann. Univ. Mariae Curie-Sklodowska Sect. A **44**, 1–7, 1990.

[12] Silverman, H., *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl. **220**, 283–289, 1998.

[13] Silverman, H. and Silvia, E.M., *Subclasses of harmonic univalent functions*, N. Z. J. Math. **28**, 275–284, 1999.

[14] Jahangiri, J.M., *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl. **235**, 470–477, 1999.

[15] Al-Shaqsi, K. and Darus, M., *An operator defined by convolution involving polylogarithms functions*, Journal of Mathematics and Statistics **4** (1), 46–50, 2008.

[16] Yasar, E. and Yalçın, S., *Generalized Salagean type harmonic univalent functions*, Stud. Univ. Babeș-Bolyai Math. **57** (3), 395–403, 2012.

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