A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS
DEFINED BY A LINEAR OPERATOR

HASAN BAYRAM AND SIBEL YALÇIN

Abstract. In the present paper, we investigate some basic properties of a subclass of harmonic functions defined by multiplier transformations. Such as, coefficient inequalities, distortion bounds, convolutions, convex combinations and extreme points.

1. Introduction

Let \( D = \{ z : |z| < 1 \} \) denotes the open unit disk and let \( H \) denotes the family of continuous complex valued harmonic functions which are harmonic in \( D \) and let \( A \) be the subclass of \( H \) consisting of functions which are analytic in \( D \). A function harmonic in \( D \) may be written as \( f = h + \overline{g} \), where \( h \) and \( g \) are members of \( A \). We call \( h \) the analytic part and \( g \) co-analytic part of \( f \). Note that \( H \) reduces to the class of normalized analytic functions if the co-analytic part of its members are zero. A necessary and sufficient condition for \( f \) to be locally univalent and sense-preserving in \( D \) is that \( |h'(z)| > |g'(z)|, z \in D \) (see Clunie and Sheil-Small [1]). Denote by \( SH \) the class of functions \( f = h + \overline{g} \) that are harmonic univalent and sense-preserving in the unit disk \( D \) for which \( f(0) = f_{\overline{z}}(0) - 1 = 0 \). Also note that \( SH \) reduces to the class \( S \) of normalized analytic univalent functions in \( D \), if the co-analytic part of \( f \) is identically zero. Then for \( f = h + \overline{g} \) we may express the analytic functions \( h \) and \( g \) as

\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k.
\]

One shows easily that the sense-preserving property implies that \( |b_1| < 1 \). The subclass \( SH^0 \) of \( SH \) consists of all functions in \( SH \) which have the additional property \( f_{\overline{z}}(0) = 0 \).

In 1984 Clunie and Sheil-Small [1] investigated the class \( SH \) as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on \( SH \) and its subclasses.

For \( f \in S \), the differential operator \( D^n (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \) of \( f \) was introduced by Salagean [2]. This operator was developed and modified by many researchers over time. As a simple example for \( f = h + \overline{g} \) given by [1], Jahangiri et al. [3] defined the modified Salagean operator of \( f \)

\[
D^n f(z) = D^n h(z) + (-1)^n D^n \overline{g}(z),
\]

where

\[
D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.
\]

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Next, Darus and Ibrahim [4] introduce a differential operator defined as follows: $D^{n,\alpha}_{\lambda,\delta}: A \rightarrow A$ by

$$D^{n,\alpha}_{\lambda,\delta} f(z) = z + \sum_{k=2}^{\infty} [k^\alpha + (k-1)k^\alpha \lambda]^n C(\delta, k) a_k z^k$$  \hspace{1cm} (2)

where

$$C(\delta, k) = \binom{\delta + k - 1}{\delta} = \frac{\Gamma(k + \delta)}{\Gamma(k)\Gamma(\delta + 1)}$$

and $n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\lambda \geq 0$, $\delta \geq 0$.

Now, we introduce a differential operator defined as follows $D^{n,\alpha}_{\lambda,\delta}: SH \rightarrow SH$ by

$$D^{n,\alpha}_{\lambda,\delta} f(z) = z + \sum_{k=2}^{\infty} [k^\alpha + (k-1)k^\alpha \lambda]^n C(\delta, k) a_k z^k$$

$$+ (-1)^n \sum_{k=2}^{\infty} [-k^\alpha + (k+1)k^\alpha \lambda]^n C(\delta, k) b_k z^k,$$  \hspace{1cm} (3)

$n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\lambda \geq 1$, $\delta \geq 0$.

With choosing special numbers for parameters $n, \alpha$, $\lambda$ and $\delta$ we obtain the following operators studied by various authors:

- (i) $D^{n,1,0}_{1,0} f(z) = D^{n,0}_{1,0} f(z) = D^n f(z)$ (2),
- (ii) $D^{n,0}_{\lambda,0} f(z) = D^n f(z)$ (4),
- (iii) $D^{n,0}_{\lambda,\delta} f(z) = D^n f(z)$ (6),
- (iv) $D^{n,0}_{\lambda,\delta} f(z) = D^n f(z)$ (7),
- (v) $D^{n,1}_{0,\delta} f(z) = D^{n,0}_{1,\delta} f(z) = D^n f(z)$ (15),

for $f \in A$,

- (vi) $D^{n,1,0}_{0,0} f(z) = P^{1,0}_{0,0} f(z) = D^n f(z)$ (8, 13),
- (vii) $D^{n,0,0}_{\lambda,0} f(z) = D^n f(z)$ (9),
- (viii) $D^{n,1,0}_{\lambda,0} f(z) = D^n f(z)$ (10).

Denote by $SH(\alpha, \lambda, \delta, n, \mu)$ the subclass of $SH$ consisting of functions $f$ of the form\footnote{[4]} that satisfy the condition

$$\text{Re} \left( \frac{D^{n+1,\alpha}_{\lambda,\delta} f(z)}{D^n f(z)} \right) \geq \mu, \hspace{0.5cm} 0 \leq \mu < 1$$  \hspace{1cm} (4)

where $D^{n,\alpha}_{\lambda,\delta} f(z)$ is defined by [3].

We let the subclass $SH(\alpha, \lambda, \delta, n, \mu)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ in $SH$ so that $h$ and $g_n$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \ b_k \geq 0.$$  \hspace{1cm} (5)

By suitably specializing the parameters, the classes $SH(\alpha, \lambda, \delta, n, \mu)$ reduces to the various subclasses of harmonic univalent functions. Such as,
(i) $SH(0,1,0,0,0) = SH^0(0)$ \[ (11, 12, 13) \],
(ii) $SH(0,1,0,0,\mu) = SH^\mu(\mu)$ \[ (13) \],
(iii) $SH(0,1,0,1,0) = KH(0)$ \[ (11, 12, 13) \],
(iv) $SH(0,1,0,1,\mu) = KH(\mu)$ \[ (13) \],
(v) $SH(0,1,0,\mu,\nu) = H(\nu,\mu)$ \[ (3) \],
(vi) $SH(0,\lambda,\nu,\mu) = SH(\lambda,\mu,\nu)$, $\nu \in \mathbb{N}$ \[ (15) \].

Define

$$SH^0(\alpha, \lambda, \delta, n, \mu) := H(\alpha, \lambda, \delta, n, \mu) \cap SH^0$$

and

$$\overline{SH}^0(\alpha, \lambda, \delta, n, \mu) := \overline{SH}(\alpha, \lambda, \delta, n, \mu) \cap SH^0.$$  

The object of the present paper is to investigate the various properties of harmonic univalent functions belonging to the subclasses $SH(\alpha, \lambda, \delta, n, \mu)$ and $\overline{SH}(\alpha, \lambda, \delta, n, \mu)$. Necessary and sufficient coefficient conditions, distortion bounds, extreme points and convex combination of above mentioned class are derived.

2. Main results

In the first theorem, we introduce a sufficient coefficient condition for harmonic functions in $SH(\alpha, \lambda, \delta, n, \mu)$.

**Theorem 1.** Let $f = h + \overline{g}$ be so that $h$ and $g$ are given by (1) with $b_1 = 0$. Let

$$\sum_{k=2}^{\infty} |k^\alpha + (k-1)k^\alpha \lambda|^n |k^\alpha + (k-1)k^\alpha \lambda - \mu| C(\delta,k) |a_k| +$$

$$\sum_{k=2}^{\infty} [-k^\alpha + (k+1)k^\alpha \lambda]^{n-1} [-k^\alpha + (k+1)k^\alpha \lambda + \mu] C(\delta,k) |b_k| \leq 1 - \mu,$$

where $n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\lambda \geq 1, \delta \geq 0$, $0 \leq \mu < 1$ and $C(\delta,k) = \binom{k+\delta-1}{\delta}$. Then $f$ is sense-preserving, harmonic univalent in $D$ and $f \in SH^0(\alpha, \lambda, \delta, n, \mu)$.

**Proof.** If $z_1 \neq z_2$,

$$\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \sum_{k=2}^{\infty} b_k \left( z_1^k - z_2^k \right) \right| \frac{1}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k \left( z_1^k - z_2^k \right)}$$

$$> 1 - \frac{\sum_{k=2}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|}$$

$$\geq 1 - \frac{\sum_{k=2}^{\infty} [-k^\alpha + (k+1)k^\alpha \lambda]^{n-1} [-k^\alpha + (k+1)k^\alpha \lambda + \mu] C(\delta,k) |b_k|}{1 - \sum_{k=2}^{\infty} [k^\alpha + (k-1)k^\alpha \lambda]^{n-1} [k^\alpha + (k-1)k^\alpha \lambda - \mu] C(\delta,k) |a_k|} \geq 0,$$

which proves univalence.
Note that $f$ is sense preserving in $\mathbb{D}$. This is because
\[
|h'(z)| \geq 1 - \sum_{k=2}^{\infty} |a_k| |z|^{k-1} \\
> 1 - \sum_{k=2}^{\infty} \frac{|k^\alpha + (k - 1)k^\alpha \lambda|^n |k^\alpha + (k - 1)k^\alpha \lambda - \mu| C(\delta, k)}{1 - \mu} |a_k| \\
\geq \sum_{k=2}^{\infty} \frac{|-k^\alpha + (k + 1)k^\alpha \lambda|^n |k^\alpha + (k + 1)k^\alpha \lambda + \mu| C(\delta, k)}{1 - \mu} |b_k| \\
> \sum_{k=2}^{\infty} |b_k| |z|^{k-1} \\
\geq |g'(z)|.
\]

Using the fact that $\text{Re} f \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that
\[
\left| (1 - \mu)D^{\mu}_{\lambda, \delta}f(z) + D^{\mu+1}_{\lambda, \delta}f(z) \right| - \left| (1 + \mu)D^{\mu}_{\lambda, \delta}f(z) - D^{\mu+1}_{\lambda, \delta}f(z) \right| \geq 0. \tag{7}
\]

Substituting for $D^{\mu}_{\lambda, \delta}f(z)$ and $D^{\mu+1}_{\lambda, \delta}f(z)$ in (7), we obtain
\[
\left| (1 - \mu)D^{\mu}_{\lambda, \delta}f(z) + D^{\mu+1}_{\lambda, \delta}f(z) \right| - \left| (1 + \mu)D^{\mu}_{\lambda, \delta}f(z) - D^{\mu+1}_{\lambda, \delta}f(z) \right| \\
\geq 2(1 - \mu) |z| - \sum_{k=2}^{\infty} \left[ k^\alpha + (k - 1)k^\alpha \lambda \right]^n \left[ k^\alpha + (k - 1)k^\alpha \lambda + 1 - \mu \right] C(\delta, k) |a_k| |z|^k \\
- \sum_{k=2}^{\infty} \left[ -k^\alpha + (k + 1)k^\alpha \lambda \right]^n \left[ -k^\alpha + (k + 1)k^\alpha \lambda - 1 + \mu \right] C(\delta, k) |b_k| |z|^k \\
- \sum_{k=2}^{\infty} \left[ k^\alpha + (k - 1)k^\alpha \lambda \right]^n \left[ k^\alpha + (k - 1)k^\alpha \lambda - 1 - \mu \right] C(\delta, k) |a_k| |z|^k \\
- \sum_{k=2}^{\infty} \left[ -k^\alpha + (k + 1)k^\alpha \lambda \right]^n \left[ -k^\alpha + (k + 1)k^\alpha \lambda + 1 + \mu \right] C(\delta, k) |b_k| |z|^k \\
> 2(1 - \mu) |z| \left\{ 1 - \sum_{k=2}^{\infty} \left[ k^\alpha + (k - 1)k^\alpha \lambda \right]^n \left[ k^\alpha + (k - 1)k^\alpha \lambda - \mu \right] C(\delta, k) |a_k| \\
- \sum_{k=2}^{\infty} \left[ -k^\alpha + (k + 1)k^\alpha \lambda \right]^n \left[ -k^\alpha + (k + 1)k^\alpha \lambda + \mu \right] C(\delta, k) |b_k| \right\}.
\]

This last expression is non-negative by (7), and so the proof is complete. □

**Theorem 2.** Let $f_n = h + \gamma_n$ be given by (5) with $b_1 = 0$.

Then $f_n \in \mathcal{ST}^{\mu_{\alpha}}(\alpha, \lambda, \delta, n, \mu)$ if and only if
\[
\sum_{k=2}^{\infty} \left[ k^\alpha + (k - 1)k^\alpha \lambda \right]^n \left[ k^\alpha + (k - 1)k^\alpha \lambda - \mu \right] C(\delta, k) |a_k| + \\
\sum_{k=2}^{\infty} \left[ -k^\alpha + (k + 1)k^\alpha \lambda \right]^n \left[ -k^\alpha + (k + 1)k^\alpha \lambda + \mu \right] C(\delta, k) |b_k| \leq 1 - \mu, \tag{8}
\]

where $n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\lambda \geq 1, \delta \geq 0, 0 \leq \mu < 1$ and $C(\delta, k) = \binom{n+k-1}{\delta}$. 
Proof. The “if” part follows from Theorem 1 upon noting that \( \overrightarrow{SH}^0(\alpha, \lambda, \delta, n, \mu) \subset SH^0(\alpha, \lambda, \delta, n, \mu) \). For the “only if” part, we show that \( f_n \notin \overrightarrow{SH}^0(\alpha, \lambda, \delta, n, \mu) \) if the condition (8) does not hold. Note that a necessary and sufficient condition for \( f_n = h + g_n \) given by (5), to be in \( \overrightarrow{SH}^0(\alpha, \lambda, \delta, n, \mu) \) is that the condition (4) to be satisfied. This is equivalent to

The above condition must hold for all values of \( z \), \( |z| = r < 1 \). Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \) we must have

\[
\Re \left\{ \frac{(1 - \mu)z - \sum_{k=2}^{\infty} [k^{\alpha} + (k - 1)k^{\alpha}]n[k^{\alpha} + (k - 1)k^{\alpha} \lambda - \mu]C(\delta, k)a_k z^k}{z - \sum_{k=2}^{\infty} [k^{\alpha} + (k - 1)k^{\alpha}]nC(\delta, k)a_k z^k + \sum_{k=2}^{\infty} [-k^{\alpha} + (k + 1)k^{\alpha} \lambda + \mu]C(\delta, k)b_k z^k} \right\} \geq 0.
\]

The above condition must hold for all values of \( z \), \( |z| = r < 1 \). Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \) we must have

\[
\Re \left\{ \frac{(1 - \mu)z - \sum_{k=2}^{\infty} [k^{\alpha} + (k - 1)k^{\alpha}]n[k^{\alpha} + (k - 1)k^{\alpha} \lambda - \mu]C(\delta, k)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} [k^{\alpha} + (k - 1)k^{\alpha}]nC(\delta, k)a_k z^{k-1} + \sum_{k=2}^{\infty} [-k^{\alpha} + (k + 1)k^{\alpha} \lambda + \mu]C(\delta, k)b_k z^{k-1}} \right\} \geq 0. \tag{9}
\]

If the condition (5) does not hold, then the expression in (9) is negative for \( r \) values approaching 1. Hence there exist \( z_0 = r_0 \in (0, 1) \) for which the quotient in (9) is negative. This shows the required condition for \( f_n \in \overrightarrow{SH}^0(\alpha, \lambda, \delta, n, \mu) \) and so the proof is complete. \( \square \)

**Theorem 3.** Let \( f_{nk} \) be given by (5). Then \( f_{nk} \in \overrightarrow{SH}^0(\alpha, \lambda, \delta, n, \mu) \) if and only if

\[
f_{nk}(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{nk}(z)),
\]

where

\[
h_1(z) = z, \quad h_k(z) = z - \frac{(1 - \mu)^{k-1}}{[k^{\alpha} + (k - 1)k^{\alpha} \lambda - \mu]C(\delta, k)} z^k \quad (k = 2, 3, ...),
\]

and for \( k = 2, 3, ... \)

\[
g_{n1}(z) = z, \quad g_{nk}(z) = z + (1)^{n-1} \frac{(1 - \mu)^{k-1}}{[-k^{\alpha} + (k + 1)k^{\alpha} \lambda + \mu]C(\delta, k)} z^k
\]

\[
X_k \geq 0, Y_k \geq 0, \sum_{k=1}^{\infty} (X_k + Y_k) = 1; \quad n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \lambda \geq 1, \delta \geq 0, 0 \leq \mu < 1 \text{ and } C(\delta, k) = \left(\frac{\alpha + k - 1}{\delta}\right).
\]

In particular, the extreme points of \( \overrightarrow{SH}^0(\alpha, \lambda, \delta, n, \mu) \) are \{\( h_k \)\} and \{\( g_{nk} \)\}.
Proof. For functions $f_{n_k}$ of the form (5) we have

$$f_{n_k}(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z))$$

$$= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \left[ k^\alpha + (k - 1) k^\alpha \lambda \right]^n \left[ k^\alpha + (k - 1) k^\alpha \lambda - \mu \right] \frac{1 - \mu}{X_k z^k}$$

$$+ (-1)^n \sum_{k=2}^{\infty} \left[ -k^\alpha + (k + 1) k^\alpha \lambda \right]^n \left[ -k^\alpha + (k + 1) k^\alpha \lambda + \mu \right] \frac{1 - \mu}{Y_k z^k}.$$

Then

$$\sum_{k=2}^{\infty} \left[ k^\alpha + (k - 1) k^\alpha \lambda \right]^n \left[ k^\alpha + (k - 1) k^\alpha \lambda - \mu \right] \frac{1 - \mu}{X_k}$$

$$+ \sum_{k=2}^{\infty} \left[ -k^\alpha + (k + 1) k^\alpha \lambda \right]^n \left[ -k^\alpha + (k + 1) k^\alpha \lambda + \mu \right] \frac{1 - \mu}{Y_k}$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=2}^{\infty} Y_k = 1 - X_1 - Y_1 \leq 1$$

and so $f_{n_k} \in SH^0(\alpha, \lambda, \delta, n, \mu)$. Conversely, if $f_{n_k} \in SH^0(\alpha, \lambda, \delta, n, \mu)$, then

$$a_k \leq \frac{1 - \mu}{k^\alpha + (k - 1) k^\alpha \lambda - \mu} \frac{1 - \mu}{X_k}$$

and

$$b_k \leq \frac{1 - \mu}{-k^\alpha + (k + 1) k^\alpha \lambda + \mu} \frac{1 - \mu}{Y_k}.$$

Set

$$X_k = \left[ k^\alpha + (k - 1) k^\alpha \lambda \right]^n \left[ k^\alpha + (k - 1) k^\alpha \lambda - \mu \right] \frac{1 - \mu}{C(\delta, k)} a_k, \ (k = 2, 3, \ldots)$$

$$Y_k = \left[ -k^\alpha + (k + 1) k^\alpha \lambda \right]^n \left[ -k^\alpha + (k + 1) k^\alpha \lambda + \mu \right] \frac{1 - \mu}{C(\delta, k)} b_k, \ (k = 2, 3, \ldots)$$

and

$$X_1 + Y_1 = 1 - \left( \sum_{k=2}^{\infty} X_k + Y_k \right)$$

where $X_k, Y_k \geq 0$. Then, as required, we obtain

$$f_{n_k}(z) = (X_1 + Y_1) z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=2}^{\infty} Y_k g_{n_k}(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z)).$$

\[\Box\]

**Theorem 4.** Let $f_n \in SH^0(\alpha, \lambda, \delta, n, \mu)$. Then for $|z| = r < 1$ and $n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \lambda \geq 1, \delta \geq 0, 0 \leq \mu < 1$ and $C(\delta, 2) = \delta + 1$, we have

$$|f_n(z)| \leq r + \frac{1 - \mu}{(2^n(1 + \lambda))^n \left[ 2^n(1 + \lambda) - \mu \right] (\delta + 1)} r^2,$$
and

$$|f_n(z)| \geq r - \frac{1 - \mu}{[2^\alpha(1 + \lambda)]^n [2^\alpha(1 + \lambda) - \mu](\delta + 1)^2}.$$ 

**Proof.** We will only prove the right side of the inequality. The left side of the inequality can be shown in a similar way. Let $f'_n \in \mathcal{SH}^\mu(\alpha, \lambda, \delta, n, \mu)$. Taking the absolute value of $f_n$ we have

$$|f_n(z)| \leq r + \sum_{k=2}^{\infty} (a_k + b_k) r^n$$

$$\leq r + \frac{(1 - \mu) r^2}{[2^\alpha(1 + \lambda)]^n [2^\alpha(1 + \lambda) - \mu](\delta + 1)} \times \sum_{k=2}^{\infty} \frac{[k^\alpha + (k - 1)k^\alpha \lambda]^n [k^\alpha + (k - 1)k^\alpha \lambda - \mu] C(\delta, k)}{1 - \mu} a_k$$

$$+ \frac{(1 - \mu) r^2}{[2^\alpha(1 + \lambda)]^n [2^\alpha(1 + \lambda) - \mu](\delta + 1)} \times \sum_{k=2}^{\infty} \frac{[-k^\alpha + (k + 1)k^\alpha \lambda]^n [-k^\alpha + (k + 1)k^\alpha \lambda + \mu] C(\delta, k)}{1 - \mu} b_k$$

$$\leq r + \frac{(1 - \mu) r^2}{[2^\alpha(1 + \lambda)]^n [2^\alpha(1 + \lambda) - \mu](\delta + 1)^2}.$$ 

The following covering result follows from the left hand inequality in Theorem 4. □

**Corollary 1.** Let $f_n$ of the form $(5)$ be so that $f_n \in \mathcal{SH}^\mu(\alpha, \lambda, \delta, n, \mu)$, where $n, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \lambda \geq 1, \delta \geq 0, 0 \leq \mu < 1$ and $C(\delta, 2) = \delta + 1$.

$$\left\{ w : |w| < 1 \right\} - \frac{1 - \mu}{[2^\alpha(1 + \lambda)]^n [2^\alpha(1 + \lambda) - \mu](\delta + 1)} \subset f_n(\mathbb{D}).$$

**Theorem 5.** The class $\mathcal{SH}^\mu(\alpha, \lambda, \delta, n, \mu)$ is closed under convex combinations.

**Proof.** Let $f_{n_i} \in \mathcal{SH}^\mu(\alpha, \lambda, \delta, n, \mu)$ for $i = 1, 2, \ldots$, where $f_{n_i}$ is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=2}^{\infty} b_k z^k.$$

Then by (8),

$$\sum_{k=2}^{\infty} \frac{[k^\alpha + (k - 1)k^\alpha \lambda]^n [k^\alpha + (k - 1)k^\alpha \lambda - \mu] C(\delta, k)}{1 - \mu} a_{ki}$$

$$+ \sum_{k=2}^{\infty} \frac{[-k^\alpha + (k + 1)k^\alpha \lambda]^n [-k^\alpha + (k + 1)k^\alpha \lambda + \mu] C(\delta, k)}{1 - \mu} b_{ki} \leq 1.$$  

(10)

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_{n_i}$ may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i a_{ki} \right) z^k + (-1)^n \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i b_{ki} \right) z^k.$$
Then by (10),

\[
\sum_{k=2}^{\infty} \frac{[k^\alpha + (k-1)k^\alpha \lambda]_n \left[k^\alpha + (k-1)k^\alpha \lambda - \mu\right] C(\delta, k)}{1 - \mu} \left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) + \sum_{k=2}^{\infty} \frac{[-k^\alpha + (k+1)k^\alpha \lambda]_n \left[-k^\alpha + (k+1)k^\alpha \lambda + \mu\right] C(\delta, k)}{1 - \mu} \left(\sum_{i=1}^{\infty} t_i b_{k_i}\right)
\]

\[
= \sum_{i=1}^{\infty} t_i \sum_{k=2}^{\infty} \frac{[k^\alpha + (k-1)k^\alpha \lambda]_n \left[k^\alpha + (k-1)k^\alpha \lambda - \mu\right] C(\delta, k)}{1 - \mu} a_{k_i}
\]

\[
+ \sum_{i=1}^{\infty} t_i \sum_{k=2}^{\infty} \frac{[-k^\alpha + (k+1)k^\alpha \lambda]_n \left[-k^\alpha + (k+1)k^\alpha \lambda + \mu\right] C(\delta, k)}{1 - \mu} b_{k_i} \leq \sum_{i=1}^{\infty} t_i = 1.
\]

This is the condition required by (8) and so \(\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \mathcal{SH}_0^{\alpha}(\alpha, \lambda, \delta, n, \mu). \)

References


