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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF DELAY DYNAMIC EQUATIONS

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Abstract. This article concerns the existence of nonoscillatory solutions for a dynamic equation on time scales. The main tool employed here is the Schauder's fixed point theorem. The results obtained here extend the work of Kubjatkova, Olach and Stoberova [12].

1. Introduction

In 1988, Stephan Hilger [9] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger's initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems; see [6, 7, 8, 13] and references therein.

Let $\mathbb{T}$ be a time scale such that $t_0 \in \mathbb{T}$. In this article, we are interested in the analysis of qualitative theory of nonoscillatory solutions of delay dynamic equations. Motivated by the papers [1]-[5], [8], [10]-[12], [14], [16]-[19] and the references therein, we consider the following delay dynamic equation

$$x^{\triangle}(t) + p(t)\, x^{\sigma}(t) + q(t)\, x(\tau(t)) = 0,\ t \geq t_0, \tag{1}$$

Throughout this paper we assume that $p : [t_0, \infty) \cap \mathbb{T} \to \mathbb{R}$ and $q : [t_0, \infty) \cap \mathbb{T} \to (0, \infty)$ are rd-continuous, $\tau : \mathbb{T} \to \mathbb{T}$ is increasing so that the function $x(\tau(t))$ is well defined over $\mathbb{T}$. We also assume that $\tau : [t_0, \infty) \cap \mathbb{T} \to [0, \infty) \cap \mathbb{T}$ is rd-continuous, $\tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

A solution of (1) is called oscillatory if it has arbitrarily large zeros and otherwise it is nonoscillatory. To reach our desired end we have to transform (1) into an integral equation and then use Schauder's fixed point theorem to show the existence of solutions which are bounded by positive functions.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary material needed in later sections. We will state some facts about the exponential function on a time scale as well as the Schauder's fixed point theorem. For details on Schauder theorem we refer the reader to [15]. In Section 3, we establish our main results for positive solutions by applying the Schauder's fixed point theorem. The results presented in this paper extend the main results in [12].

2. Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1]-[5], [10], [11], [18] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that

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the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [6, 7, 13] most of the material needed to read this paper.

Definition 1 ([6]). *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right-dense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 2 ([6]). *For $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)\right| < \varepsilon |\sigma(t) - s| \text{ for all } s \in U.$$

The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of f on $\mathbb{T}^k$.

Definition 3 ([6]). *A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \ \forall t \in \mathbb{T}\}.$$

Definition 4 ([6]). *Let $p \in \mathcal{R}$, then the generalized exponential function e_p is defined as the unique solution of the initial value problem*

$$x^{\Delta}(t) = p(t)x(t), \ x(s) = 1, \ \text{where } s \in \mathbb{T}.$$

An explicit formula for $e_p(t, s)$ is given by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(v)}(p(v))\Delta v\right), \text{ for all } s, t \in \mathbb{T},$$

with

$$\xi_h(v) = \begin{cases} \frac{\log(1+hv)}{h} & \text{if } h \neq 0, \\ v & \text{if } h = 0, \end{cases}$$

where log *is the principal logarithm function.*

Lemma 1 ([6]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$,
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,
(iv) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$,
(v) $e_p(t, s)e_p(s, r) = e_p(t, r)$,
(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Lemma 2 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(v)\Delta v\right), \ \forall t \in \mathbb{T}.$$

The proof of the main results in the next section is based upon an application of the following Schauder's fixed point theorem.

Theorem 1 (Schauder's fixed point theorem [15]). *Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is there exists an $x \in \Omega$ such that $Sx = x$.*

3. Existence of positive solutions

In this section we shall investigate the existence of positive solutions for equation (1). The main result is in the following theorem.

Theorem 2. *Suppose that there exist functions* $k_1, k_2 \in C_{rd}\left([t_0,\infty)\cap\mathbb{T},(0,\infty)\right)$ *such that for* $t \geq t_0$

$$k_1(t) \leq k_2(t), \ p(t) + k_1(t)\, q(t) \geq 0,$$

and

$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k_1(s)\, q(s)\right)\right]\Delta s \geq \log k_1(t),$$
$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k_2(s)\, q(s)\right)\right]\Delta s \leq \log k_2(t). \tag{2}$$

Then equation (1) has a solution which is bounded by positive functions.

Proof. We choose $T \geq t_0 + \tau(T)$ and set

$$u(t) = \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + k_2(s)\, q(s)\right)\right]\Delta s\right),$$
$$v(t) = \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + k_1(t)\, q(s)\right)\right]\Delta s\right), \quad t \geq T.$$

Let $C_{rd}\left([t_0,\infty)\cap\mathbb{T},\mathbb{R}\right)$ be the set of all bounded rd-continuous functions with the norm

$$\|x\| = \sup_{t\geq t_0} |x(t)| < \infty.$$

Then $C_{rd}\left([t_0,\infty)\cap\mathbb{T},\mathbb{R}\right)$ is a Banach space. We define a close, bounded and convex subset Ω of $C_{rd}\left([t_0,\infty)\cap\mathbb{T},\mathbb{R}\right)$ as follows

$$\begin{aligned}\Omega = \{x \in C_{rd}\left([t_0,\infty)\cap\mathbb{T},\mathbb{R}\right) :& \\ u(t) \leq x(t) \leq v(t), &\quad t \geq T, \\ x(\tau(t)) \leq k_2(t)\, x^{\sigma}(t), &\quad t \geq T, \\ x(\tau(t)) \geq k_1(t)\, x^{\sigma}(t), &\quad t \geq T, \\ x(t) = 1, &\quad \tau(T) \leq t \leq T\}.\end{aligned}$$

Define the map $S : \Omega \to C_{rd}\left([t_0,\infty),\mathbb{R}\right)$ as follows

$$(Sx)(t) = \begin{cases} \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^{\sigma}(s)}\right)\right]\Delta s\right), & t \geq T, \\ 1, & \tau(T) \leq t \leq T. \end{cases}$$

We shall show that for any $x \in \Omega$ we have $Sx \in \Omega$. For every $x \in \Omega$ and $t \geq T$ we get

$$(Sx)(t) \leq \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + k_1(s)\, q(s)\right)\right]\Delta s\right) = v(t).$$

Furthermore for $t \geq T$ we have

$$(Sx)(t) \geq \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + k_2(s)\, q(s)\right)\right]\Delta s\right) = u(t).$$

For $t \in [\tau(T), T] \cap \mathbb{T}$ we obtain $(Sx)(t) = 1$. Further for every $x \in \Omega$ and $\tau(t) \geq T$ we get

$$\begin{aligned}(Sx)(\tau(t)) &= \exp\left(\int_T^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)\\ &= (Sx)(\sigma(t))\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right). \qquad (3)\end{aligned}$$

With regard to (2) and (3) we have

$$\begin{aligned}(Sx)(\tau(t)) &\leq (Sx)(\sigma(t))\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k_2(s)\,q(s)\right)\right]\Delta s\right)\\ &\leq k_2(t)\,(Sx)(\sigma(t)), \quad \tau(t) \geq T,\end{aligned}$$

and

$$\begin{aligned}(Sx)(\tau(t)) &\geq (Sx)(\sigma(t))\exp\left(\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k_1(s)\,q(s)\right)\right]\Delta s\right)\\ &\geq k_1(t)\,(Sx)(\sigma(t)), \quad \tau(t) \geq T.\end{aligned}$$

For $\tau(T) \leq \tau(t) \leq T$ we obtain $(Sx)(\tau(t)) = 1$. Thus we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

We now show that S is continuous. Let $x_i \in \Omega$ be such that $x_i \to x$ as $i \to \infty$. Because Ω is closed, $x \in \Omega$. For $t \geq T$ we have

$$\begin{aligned}&|(Sx_i)(t) - (Sx)(t)|\\ &= \Bigg|\exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x_i(\tau(s))}{x_i^\sigma(s)}\right)\right]\Delta s\right)\\ &- \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + q(s)\frac{x(\tau(s))}{x^\sigma(s)}\right)\right]\Delta s\right)\Bigg|.\end{aligned}$$

So we conclude that

$$\lim_{i\to\infty} \|Sx_i - Sx\| = 0.$$

For $t \in [\tau(T), T] \cap \mathbb{T}$ the relation above is also valid. This means that S is continuous.

The family of functions $\{Sx : x \in \Omega\}$ is uniformly bounded on $[\tau(T), \infty) \cap \mathbb{T}$. It follows from the definition of Ω. This family is also equicontinuous on $[\tau(T), \infty) \cap \mathbb{T}$. Then by Arsela-Ascoli theorem the $S\Omega$ is relatively compact subset of $C_{rd}([t_0, \infty) \cap \mathbb{T}, \mathbb{R})$. By Theorem 1 there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that x_0 is a positive solution of the equation (1). The proof is complete. □

Corollary 1. *Suppose that for* $t \geq t_0$

$$0 < k_1 \leq k_2,\ p(t) + k_1 q(t) \geq 0,$$

and

$$\begin{aligned}&\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k_1 q(s)\right)\right]\Delta s \geq \log k_1,\\ &\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k_2 q(s)\right)\right]\Delta s \leq \log k_2.\end{aligned}$$

Then equation (1) has a solution which is bounded by positive functions.

Proof. We put $k_1(t) = k_1$, $k_2(t) = k_2$ and apply Theorem 2. □

Corollary 2. *Suppose that there exists a function $k \in C_{rd}([t_0,\infty)\cap\mathbb{T},(0,\infty))$ such that for $t \geq t_0$*

$$p(t) + k(t)\,q(t) \geq 0,$$

and

$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + k(s)\,q(s)\right)\right]\Delta s = \log k(t)\,.$$

Then equation (1) has a solution

$$x(t) = \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + k(s)\,q(s)\right)\right]\Delta s\right), \quad t \geq T.$$

Proof. We put $k_1(t) = k_2(t) = k(t)$ and apply Theorem 2. □

Corollary 3. *Suppose that for $t \geq t_0$*

$$k > 0,\ p(t) + kq(t) \geq 0,$$

and

$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(p(s) + kq(s)\right)\right]\Delta s = \log k.$$

Then equation (1) has a solution

$$x(t) = \exp\left(\int_T^t \xi_{\mu(s)}\left[\ominus\left(p(s) + kq(s)\right)\right]\Delta s\right), \quad t \geq T.$$

Proof. We put $k(t) = k$ and apply Corollary 2. □

Corollary 4. *Suppose that there exist functions $k_1, k_2 \in C_{rd}([t_0,\infty)\cap\mathbb{T},(0,\infty))$ and $\alpha \in C_{rd}([t_0,\infty)\cap\mathbb{T},[0,\infty))$ such that for $t \geq t_0$*

$$\alpha(t) \leq k_1(t) \leq k_2(t)\,,\ p(t) + \alpha(t)\,q(t) = 0,$$

and

$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(\left[k_1(s) - \alpha(s)\right]q(s)\right)\right]\Delta s \geq \log k_1(t)\,,$$
$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(\left[k_2(s) - \alpha(s)\right]q(s)\right)\right]\Delta s \leq \log k_2(t)\,.$$

Then equation (1) has a solution which is bounded by positive functions.

Proof. We put $p(t) = -\alpha(t)\,q(t)$ into (2) and apply Theorem 2. □

Corollary 5. *Suppose that for $t \geq t_0$*

$$0 \leq \alpha < k_1 \leq k_2,\ p(t) + \alpha q(t) = 0,$$

and

$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(\left[k_1 - \alpha\right]q(s)\right)\right]\Delta s \geq \log k_1,$$
$$\int_{\sigma(t)}^{\tau(t)} \xi_{\mu(s)}\left[\ominus\left(\left[k_2 - \alpha\right]q(s)\right)\right]\Delta s \leq \log k_2.$$

Then equation (1) has a solution which is bounded by positive functions.

Proof. We put $\alpha(t) = \alpha$, $k_1(t) = k_1$, $k_2(t) = k_2$ and apply Corollary 4. □

References

[1] Adivar, M., Raffoul, Y.N., *Existence of periodic solutions in totally nonlinear delay dynamic equations*, Electronic Journal of Qualitative Theory of Differential Equations **2009(1)**, 1–20 (2009).

[2] Ardjouni, A., Djoudi, A., *Existence of positive periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale*, Malaya Journal of Matematik **2(1)**, 60–67 (2013).

[3] Ardjouni, A., Djoudi, A., *Existence of periodic solutions for nonlinear neutral dynamic equations with functional delay on a time scale*, Acta Univ. Palacki. Olomnc., Fac. rer. nat., Mathematica **52(1)**, 5–19 (2013).

[4] Ardjouni, A., Djoudi, A., *Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale*, Commun Nonlinear Sci Numer Simulat **17**, 3061–3069 (2012).

[5] Ardjouni, A., Djoudi, A., *Periodic solutions in totally nonlinear dynamic equations with functional delay on a time scale*, Rend. Sem. Mat. Univ. Politec. Torino Vol. **68(4)**, 349–359 (2010).

[6] Bohner, M., Peterson, A., *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, (2001).

[7] Bohner, M., Peterson, A., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, (2003).

[8] Dorociakova, B., Olach, R., *Existence of positive solutions of delay differential equations*, Tatra Mt. Math. Publ. **43**, 63–70 (2009).

[9] Hilger, S., *Ein Masskettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, PhD thesis, Universität Würzburg, (1988).

[10] Kaufmann, E.R., Raffoul, Y.N., *Periodic solutions for a neutral nonlinear dynamical equation on a time scale*, J. Math. Anal. Appl. **319**, 315–325 (2006).

[11] Kaufmann, E.R., Raffoul, Y.N., *Periodicity and stability in neutral nonlinear dynamic equation with functional delay on a time scale*, Electronic Journal of Differential Equations **2007(27)**, 1–12 (2007).

[12] Kubjatkova, M., Olach, R., Stoberova, J., *Existence of nonoscillory solutions of delay differential equations*, Journal of Applied Mathematics **5(2)**, 103–109 (2012).

[13] Lakshmikantham, V., Sivasundaram, S., Kaymarkcalan, B., *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Dordrecht, (1996).

[14] Olach, R., *Positive periodic solutions of delay differential equations*, Applied Mathematics Letters **26**, 1141–1145 (2013).

[15] Smart, D.R., *Fixed Points Theorems*, Cambridge Univ. Press, Cambridge, UK, (1980).

[16] Yu, Y., Wang, H., *Nonoscillatory solutions of second-order nonlinear neutral delay equations*, J. Math. Anal. Appl. **311**, 445–456 (2005).

[17] Zhang, W., Feng, W., Yan, J., Song, J., *Existence of nonoscillatory solutions of first-order linear neutral delay differential equations*, Comput. Math. Appl. **49**, 1021–1027 (2005).

[18] Zhang, C., Agarwal, R., Bohner, M., Li, T., *Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators*, Bull. Malays. Math. Sci. Soc. **38**, 761–778 (2015).

[19] Zhou, Y., Zhang, B., *Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients*, Appl. Math. Lett. **15**, 867–874 (2002).

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