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ON A CERTAIN SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS DEFINED BY BESSEL FUNCTIONS

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ABSTRACT. The aim of the present paper is to investigate some characterization for generalized Bessel functions of first kind is to be subclass of analytic functions. Furthermore we studied coefficient estimates, radius of starlikeness, convexity, close - to - convexity, convex linear combinations for the class $UB(\gamma, c)$. Finally we proved Integral means inequalities for the class.

1. INTRODUCTION

Let A be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U = \{z \mid z \in \mathcal{C} \text{ and } |z| < 1\}$. We denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

This subclass was introduced and extensively studied by Silverman [4]. Let $T^*(\alpha)$ and $C(\alpha)$ be denote the subclasses of T consisting of starlike and convex functions of order $\alpha, (0 \leq \alpha < 1)$, respectively.

In [2], Kanas and Wisniowska introduced the classes $UCV(\alpha, \beta)$ consists of uniform $\beta-$convex functions of order α and $SP(\alpha, \beta)$ consists parabolic $\beta-$starlike functions of order $\alpha, -1 < \alpha \leq 1, \beta \geq 0$, which generalizes the class UCV and SP respectively.

The function $f \in A$ belongs to $UCV(\alpha, \beta)$ if it satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta \left\{\frac{zf''(z)}{f'(z)}\right\}, \ z \in U. \tag{3}$$

The function $f \in A$ belongs to $SP(\alpha, \beta)$ if it satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta \left\{\frac{zf'(z)}{f(z)} - 1\right\}, \ z \in U. \tag{4}$$

Indeed, it follows from (3) and (4) that $f \in UCV(\alpha, \beta)$ if and only if $zf'(z) \in SP(\alpha, \beta)$. The generalized Bessel functions of the first kind $\omega = \omega_{p,b,c}$ is defined as the particular solution of the second order linear homogeneous differential equation.

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = 0, \ p, b, c \in \mathcal{C}, \tag{5}$$

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which is natural generalization of Bessel's equation. This function has the representation

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p}, \ z \in C, \tag{6}$$

where $p, b, c, z \in \mathcal{C}$ and $c \neq 0$. The differential equation (5) permits the study of Bessel, modified Bessel and spherical Bessel functions all together. Solutions of (5) are referred as generalized Bessel functions of order p. The particular solution given by (6) is called the generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in the open unit disc $U = \{z \in \mathcal{C} \mid |z| < 1\}$. It is worth mentioning that, in particular, when $b = c = 1$.

We reobtain the Bessel function of the first kind $\omega_{p,1,1} = J_p$ and $b = 1, c = -1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function of the first kind I_p.

Now consider the function $u_{p,b,c} : \mathcal{C} \to \mathcal{C}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-\frac{p}{2}} \omega_{p,b,c}(\sqrt{z}) \tag{7}$$

By using the well known Pochhammer symbol (or the shifted factorial) is defined in terms of the Euler Γ function by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0; \\ a(a+1)\cdots(a+n-1), & \text{if } n \in N; \end{cases} \quad (a)_0 = 1.$$

We obtain for the function $u_{p,b,c}(z)$the following representation

$$u_{p,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{\left(\frac{-c}{4}\right)^n}{n!\left(p+\frac{b+1}{2}\right)_n} \tag{8}$$

where $\left(p + \frac{b+1}{2}\right) \neq 0, -1, -2, \cdots$. For convenience, we write $u_{p,b,c}(z) = u_{k,c}(z)$.

We have the given below operator, $S_k^c : A \to A$ defined by the Hadamard product

$$\begin{aligned} S_k^c f(z) = u_{k,c} * f(z) &= z + \sum_{n=1}^{\infty} \frac{(-c)^n a_{n+1}}{4^n (k)_n n!} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1} a_n}{4^{n-1}(k)_{n-1}(n-1)!} z^n = z + \sum_{n=2}^{\infty} D(c,k,n) a_n z^n \end{aligned} \tag{9}$$

$$\text{where } D(c,k,n) = \frac{(-c)^{n-1}}{4^{n-1}(k)_{n-1}(n-1)!}, \ k = \left(p + \frac{b+1}{2}\right) \neq 0, -1, -2, \cdots. \tag{10}$$

We can easily see that from (9) $z\left[S_{k+1}^c f(z)\right]' = kS_k^c f(z) - (k-1)S_{k+1}^c f(z)$.

The function $S_k^c f(z)$ is in (9) is an elementary transformation of the generalized hypergeometric function, so that $kS_k^c f(z) = z_0F_1(k; \frac{-c}{4}z) * f(z)$ and $u_{k,c}\left(\frac{-c}{4}z\right) = z_0F_1(k; z)$. For $f \in A$ is given by (1) and $g \in A$ is given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_{n+1} z^{n+1}$, the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by $(f * g)(z) = (g * f)(z) = z + \sum\limits_{n=2}^{\infty} a_{n+1} b_{n+1} z^{n+1}, z \in E$. In this paper, using the operator, we define the following new subclass motivated by Ramachandran et al. [1].

Definition 1. *Let $c > 1, 0 \leq \gamma < 1$ and $z \in E, f(z) \in UB(\gamma, c)$, where f is in the form (1). Then*

$$Re\left\{\frac{z\left(S_k^c f(z)\right)'}{S_k^c f(z)} - \gamma\right\} > \left|\frac{z\left(S_k^c f(z)\right)'}{S_k^c f(z)} - 1\right|. \tag{11}$$

2. Coefficient estimates

In this section we obtain the coefficient bounds of function $f(z)$.

Theorem 1. *If $f(z) \in UB(\gamma, c)$, where f is in the form* (1) *then*

$$\sum_{n=2}^{\infty}[2n-\gamma-1]D(c,k,n)|a_n| \leq 1-\gamma, \tag{12}$$

where $0 \leq \gamma < 1$ and $D(c,k,n)$ is given by (10)

Proof. It is enough to show that

$$\left|\frac{z\left(S_k^c f(z)\right)'}{S_k^c f(z)} - 1\right| - Re\left\{\frac{z\left(S_k^c f(z)\right)'}{S_k^c f(z)} - \gamma\right\} \leq 1-\gamma. \text{ We have}$$

$$\begin{aligned}
&\left|\frac{z\left(S_k^c f(z)\right)'}{S_k^c f(z)} - 1\right| - Re\left\{\frac{z\left(S_k^c f(z)\right)'}{S_k^c f(z)} - \gamma\right\} \\
\leq & 2\left|\frac{z\left(S_k^c f(z)\right)'}{S_k^c f(z)} - 1\right| \\
\leq & \frac{2\sum\limits_{n=2}^{\infty}(n-1)D(c,k,n)|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}D(c,k,n)|a_n||z|^{n-1}} \\
\leq & \frac{2\sum\limits_{n=2}^{\infty}(n-1)D(c,k,n)|a_n|}{1-\sum\limits_{n=2}^{\infty}D(c,k,n)|a_n|}.
\end{aligned}$$

The last expression is bounded above by $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[2n-\gamma-1]D(c,k,n)|a_n| \leq (1-\gamma)$$

and the proof is complete. □

Theorem 2. *Let $0 \leq \gamma < 1$. Then $f \in UB(\gamma, c)$, where f is in the form* (2)*, if and only if*

$$\sum_{n=2}^{\infty}[2n-\gamma-1]D(c,k,n)|a_n| \leq (1-\gamma), \tag{13}$$

where $D(c,k,n)$ is given by (10).

Proof. In view of the above Theorem, it is enough to prove the necessity. If $f \in UB(\gamma, c)$ and z is real then

$$Re\left\{\frac{1-\sum\limits_{n=2}^{\infty}nD(c,k,n)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}D(c,k,n)a_n z^{n-1}}\gamma\right\} > \left|\frac{\sum\limits_{n=2}^{\infty}(n-1)D(c,k,n)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}D(c,k,n)a_n z^{n-1}}\gamma\right|$$

Along the real axis, $z \to 1$, we get the desired inequality

$$\sum_{n=2}^{\infty}[2n-\gamma-1]D(c,k,n)|a_n| \leq (1-\gamma),$$

where $0 \leq \gamma < 1$ and $D(c,k,n)$ are given by (10) □

Corollary 1. *If $f \in UB(\gamma, c)$ then*

$$|a_n| \leq \frac{(1-\gamma)}{[2n-\gamma-1]D(c,k,n)} z^n, \tag{14}$$

where $0 \leq \gamma < 1$ and $D(c,k,n)$ are given by (10). Equality holds for the function

$$f(z) = z - \frac{(1-\gamma)}{[2n-\gamma-1]D(c,k,n)} z^n. \tag{15}$$

3. Convex Linear Combinations

In this section we prove that the class $UB(\gamma, c)$ is a convex set. And also we prove that if $f \in UB(\gamma, c)$ then $f(z)$ is close-to-convex of order δ $0 \leq \delta < 1$.

Theorem 3. *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{(1-\gamma)}{[2n-\gamma-1]D(c,k,n)} z^n, \; n \geq 2. \tag{16}$$

Then if and only if it can be in the form

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z), \; \omega_n \geq 0, \; \sum_{n=1}^{\infty} \omega_n = 1. \tag{17}$$

Proof. Suppose that $f(z)$ can be written as in (17). Then

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(1-\gamma)}{[2n-\gamma-1]D(c,k,n)} z^n.$$

$$\text{Now } \sum_{n=2}^{\infty} \omega_n \frac{(1-\gamma)[2n-\gamma-1]D(c,k,n)}{(1-\gamma)[2n-\gamma-1]D(c,k,n)} = \sum_{n=2}^{\infty} \omega_n = (1-\omega_1) \leq 1.$$

Thus $f(z) \in UB(\gamma, c)$.

Conversely suppose that $f(z) \in UB(\gamma, c)$. Then by using (14), setting,

$$\omega_n = \frac{[2n-\gamma-1]D(c,k,n)}{1-\gamma} a_n, \; n \geq 2 \text{ and } \omega_1 = 1 - \sum_{n=2}^{\infty} \omega_n.$$

Then we have $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$. Hence the theorem. □

Theorem 4. *The class $UB(\gamma, c)$ is a convex set.*

Proof. Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \; a_{n,j} \geq 0, \; j = 1, 2. \tag{18}$$

be in the class $UB(\gamma, c)$. It is enough to show that the function $h(z)$ defined by $h(z) = \xi f_1(z) + (1-\xi) f_2(z), 0 \leq \xi < 1$ is in the class $UB(\gamma, c)$.

Since $h(z) = z - \sum_{n=2}^{\infty} [\xi a_{n,1} + (1-\xi) a_{n,2}] z^n$, with the help of Theorem 2, and by an easy computation, we get

$$\sum_{n=2}^{\infty} [2n-\gamma-1]\xi D(c,k,n) a_{n,1} + \sum_{n=2}^{\infty} [2n-\gamma-1](1-\xi) D(c,k,n) a_{n,2}$$
$$\leq \xi(1-\gamma) + (1-\xi)(1-\gamma) \leq (1-\gamma),$$

which implies that $h \in UB(\gamma, c)$. Hence the $UB(\gamma, c)$ is convex. □

Theorem 5. *If* $UB(\gamma, c)$, *where* $f(z)$ *is in the form* (2) *then it is close-to-convex of order* $\delta, (0 \leq \delta < 1)$ *in the disc* $|z| < r_1$, *where*

$$r_1 = \inf_{n \geq 2} \left[\frac{(1-\delta) \sum_{n=2}^{\infty} [2n - \gamma - 1] D(c, k, n)}{n(1-\gamma)} \right]^{\frac{1}{n-1}}, \ n \geq 2. \tag{19}$$

The result is sharp with the extremal function $f(z)$ *by* (16).

Proof. Given $f \in T$ and f is close-to-convex of order δ, we have

$$|f'(z) - 1| < (1 - \delta). \tag{20}$$

For the left hand side of (20), we have $|f'(z) - 1| < \sum_{n=2}^{\infty} n a_n |z|^{n-1}$.
The right hand side of the above inequality is less than $(1 - \delta)$. Then

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n |z|^{n-1} \leq 1.$$

We have $f(z) \in UB(\gamma, c)$ if and only if $\sum_{n=2}^{\infty} \frac{[2n-\gamma-1]D(c,k,n)}{(1-\gamma)}$.
We can (20) is true if $\frac{n}{1-\delta} z|^{n-1} \leq \frac{[2n-\gamma-1]D(c,k,n)}{(1-\gamma)}$
or equivalently $|z| \leq \left[\frac{(1-\delta)[2n-\gamma-1]D(c,k,n)}{n(1-\gamma)} \right]^{\frac{1}{n-1}}$
and hence the proof of the theorem. □

Theorem 6. *If* $UB(\gamma, k, c)$, *then* $f(z)$ *is starlike of order* $\delta, (0 \leq \delta < 1)$ *in the disc* $|z| < r_2$, *where*

$$r_2 = \inf_{n \geq 2} \left[\frac{(1-\delta) \sum_{n=2}^{\infty} [2n - \gamma - 1] D(c, k, n)}{(n-\delta)(1-\gamma)} \right]^{\frac{1}{n-1}}, \ n \geq 2. \tag{21}$$

The result is sharp with the extremal function given by (15)

Proof. Given $f \in T$ and f is starlike of order δ, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < (1 - \delta). \tag{22}$$

For the left hand side of (22), we have $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}$.
The right hand side of the above inequality is less than $(1 - \delta)$ if

$$\sum_{n=2}^{\infty} \frac{(n-\delta)}{(1-\delta)} a_n |z|^{n-1} < 1.$$

We have $f(z) \in UB(\gamma, c)$ if and only if $\sum_{n=2}^{\infty} \frac{[2n-\gamma-1]D(c,k,n)}{(1-\gamma)} a_n \leq 1$, (22) is true if $\frac{(n-\delta)}{(1-\delta)} |z|^{n-1} \leq \frac{[2n-\gamma-1]D(c,k,n)}{(1-\gamma)}$
or equivalently $|z|^{n-1} \leq \frac{(1-\delta)[2n-\gamma-1]D(c,k,n)}{(n-\delta)(1-\gamma)}$.
It yield starlikeness of the family. □

4. Integral means inequalities

In [5], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjectured [5] and settled in [6], that

$$\int_0^{2\Pi} \left|f(re^{i\phi})\right|^\eta d\phi \leq \int_0^{2\Pi} \left|f_2(re^{i\phi})^\eta\right| d\phi,$$

for all $f \in T, \eta > 0$ and $0 < r < 1$. In [6], he also proved his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now we prove Silverman's conjecture for the class of functions $UB(\gamma, c)$. We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [3]. Two functions f and g, which are analytic in E, the function ω is said to be subordinate to g in E if there exists a function ω analytic in E with $\omega(0) = 0, |\omega(z)| < 1, (z \in E)$ such that $f(z) = g(\omega(z)),\ (z \in E)$. We denote this subordination by $f(z) \prec g(z)$.

Lemma 1. [3] *If the functions f and g are analytic in E with $f(z) \prec g(z)$, then for $\eta > 0$ and $z = re^{i\phi}, 0 < r < 1$, $\int_0^{2\Pi} \left|g(re^{i\phi})\right|^\eta d\phi \leq \int_0^{2\Pi} \left|f(re^{i\phi})\right|^\eta d\phi$.*

Now we discuss the integral means inequalities for functions $f \in UB(\gamma, c)$ and

$$\int_0^{2\Pi} \left|g(re^{i\phi})\right|^\eta d\phi \leq \int_0^{2\Pi} \left|f(re^{i\phi})\right|^\eta d\phi.$$

Theorem 7. *Let $f(z) \in UB(\gamma, c), 0 \leq \gamma < 1$ and $f_2(z)$ be defined by*

$$f_2(z) = z - \frac{1-\gamma}{\phi_2(\gamma)} z^2. \tag{23}$$

Then $\int_0^{2\Pi} |f(z)|^\eta d\phi \leq \int_0^{2\Pi} |f_2(z)|^\eta d\phi$, where $z = re^{i\phi}, 0 < r < 1$.

Proof. For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, (23) is equivalent to

$$\int_0^{2\Pi} |f(z)|^\eta d\phi \leq \int_0^{2\Pi} |f_2(z)|^\eta d\phi.$$

By Lemma 1, it is enough to prove that $1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\phi_2(\gamma)} z$.
Assuming $1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\phi_2(\gamma)} \omega(z)$ and using (13), we obtain

$$\omega(z) = \left|\sum_{n=2}^{\infty} \frac{\phi_2(\gamma)}{1-\gamma} a_n z^{n-1}\right| \leq |z| \sum_{n=2}^{\infty} \frac{\phi_2(\gamma, k)}{1-\gamma} a_n \leq |z|,$$

where $\phi_n(\gamma) = [2n - \gamma - 1] D(c, k, n)$.
Hence the proof is completed. □

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