TJMM
10 (2018), No. 1, 31-41

SECOND AND THIRD HANKEL DETERMINANT FOR A CLASS DEFINED BY GENERALIZED POLYLOGARITHM FUNCTIONS

MOHD NAZRAN MOHAMMED PAUZI, MASLINA DARUS, AND SAIBAH SIREGAR

Abstract. By making use of $\mathfrak{D}_\lambda^m f(z)$, the generalized Polylogarithms derivative operator introduced by Al-Saqsi and Darus [4] defined by

$$\mathfrak{D}_\lambda^m f(z) = z + \sum_{n=2}^{\infty} \frac{n^m (n+\lambda-1)!}{\lambda!(n-1)!} a_n z^n,$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$. The sharp upper bound for the second Hankel determinant $H_{2,2}(f)$ and third Hankel determinant $H_{3,1}(f)$ is obtained. Relevant connections of the results presented here with those given in earlier works are also indicated.

1. Introduction

Let $\mathcal{A}$ denotes the family of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A function f is said to be univalent in the domain $\mathbb{U}$, if it is one-to-one in $\mathbb{U}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{U}$.

The Hankel determinant $H_{q,n}(f)$ for $q \geq 1$ and $n \geq 1$ of Taylor's coefficients of function $f \in \mathcal{A}$ of the form (1) defined by Noonan and Thomas [25] defined as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q+2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $a_1 = 1$ and $n, q \in \mathbb{N} = 1, 2, ...$.

The application of Hankel determinant have been investigate by various researchers. For example, Wilson [33] study the application of Hankel determinant in meromorphic functions and Cantor in [9] shows the application of Hankel determinant in showing that a function of bounded characteristic in $\mathbb{U}$, i.e. a function which a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational. Since then, the study of $|H_{q,n}(f)|$ have been investigated by several authors. Pommerenke [29] investigated the Hankel determinant of a really mean p-valent functions as well as of starlike functions and prove that the determinants of univalent functions satisfy

$$|H_{q,n}(f)| < K n^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad \text{for} \quad (n = 1, 2, ...) \quad \text{and} \quad (q = 2, 3, ...),$$

where $\beta > 1/4000$ and K depends only on q. Later, Hayman [14] showed that $|H_{2,n}(f)| < An^{1/2}$ (n=1,2,...; A an absolute constant) for a really mean univalent functions. Noor in

2010 *Mathematics Subject Classification.* 30C45; 30C50.

Key words and phrases. Hankel determinant, positive real functions, polylogarithms functions.

[26] have determined the rate of growth of as with bounded boundary and also studied the Hankel determinant for Bazilevic functions in [27]. The Hankel determinant of exponential polynomials were studied by Ehrenborg [11], and Layman in [20] discussed some of its properties.

It is easily observe that for $q = 2$ and $n = 1$, we will have a classical theorem of Fekete and Szegö given by,

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2. \tag{2}$$

They in [12] made an early study for the estimates of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ and μ real. The well-known result due to them states that if $f \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & \text{if} \quad \mu \geq 1, \\ 1 + 2e^{\left(\frac{-2\mu}{1-\mu}\right)} & \text{if} \quad 0 \leq \mu \leq 1, \\ 3 - 4\mu & \text{if} \quad \mu \leq 0. \end{cases}$$

Several author have investigated problem involving $H_{2,1}(f)$. For example, Keogh and Merkes [17] discussed the sharp estimates for $|a_3 - \mu a_2^2|$ when f is close-to-convex and starlike in $\mathbb{U}$. The functional (2) is studied, among others, by Koepf [18], London [23], Srivastava et. al [32] and others.

Hankel determinant of $f \in \mathcal{A}$ for $q = 2$ and $n = 2$, known as the second Hankel determinant, given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{3}$$

This second Hankel determinant has been considered by many researchers. Such as, Janteng et. al [15] have studied the sharp bound for the function f in (1), consisting the functions which derivative has a positive real part and have the result $|a_2 a_4 - a_3^2| \leq 4/9$. The same author [16] obtained the result for the sharp upper bounds for starlike and convex functions as $|a_2 a_4 - a_3^2| \leq 1$ and $|a_2 a_4 - a_3^2| \leq 1/8$ respectively. Further, various authors studied and investigated the second Hankel determinant for a certain class of analytic functions such as Al-Refai and Darus [1], Abubaker and Darus [2], Al-Abbadi and Darus [3] and Bansal [7].

The third Hankel determinant $H_{3,1}(f)$ is defined by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2). \tag{4}$$

for $f \in A$ and $a_1 = 1$. By applying triangle inequality, we obtain

$$|H_{3,1}(f)| \leq |a_3||(a_2 a_4 - a_3^2)| - |a_4||(a_4 - a_2 a_3)| + |a_5||(a_3 - a_2^2)|. \tag{5}$$

Recently, the study on $|H_{3,1}(f)|$ have been investigated by Babalola [5], Shanmugam et. al [31], Prajabat et. al [28], Bansal et. al [8], Krishna et. al [19] and Zaparwa [34].

In the present paper we will use the generalized polylogarithms derivative operator $\mathfrak{D}_\lambda^m f(z)$ introduced by Al-Saqsi and Darus [4] defined as follow:

Definition 1. [4] *For $f \in \mathcal{A}$, the generalized polylogarithms defined by $\mathfrak{D}_\lambda^m f(z) : \mathcal{A} \to \mathcal{A}$*

$$\mathfrak{D}_\lambda^m f(z) = z + \sum_{n=2}^{\infty} \frac{n^m (n + \lambda - 1)!}{\lambda!(n-1)!} a_n z^n, \tag{6}$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}, z \in \mathbb{U}$. It is clear that the operator $\mathfrak{D}_\lambda^m f(z)$ included two unknown derivative operators. Note that $\mathfrak{D}_0^m = \mathfrak{D}^m$ which are Sălăgean and $\mathfrak{D}_\lambda^0 = \mathfrak{D}_\lambda$ is the Ruscheweyh derivative operators respectively.

Motivated by the results obtained by various authors in this direction mentioned above, we investigate the upper bound for functional $|a_2a_4 - a_3^2|$, $|a_4 - a_2a_3|$ and $|a_3 - a_2^2|$ to find $|H_{2,2}(f)|$ and $|H_{3,1}(f)|$.

The subclass $\mathfrak{D}_\lambda^m f(z)$ is defined as the following.

Definition 2. *Let f be given by (1). Then f is said to be the class $\mathfrak{D}_\lambda^m f(z)$ if it is satisfies the inequality*

$$\operatorname{Re}\{[\mathfrak{D}_\lambda^m f(z)]'\} > 0, \quad (z \in \mathbb{U}) \tag{7}$$

We first state some preliminary lemmas required for proving our results.

2. Preliminary Results

Lemma 1. [30] *Let $\mathcal{P}$ be the family of all functions p analytic in $\mathbb{U}$ for which* $\operatorname{Re}\{p(z)\} > 0$*. If $p \in \mathcal{P}$ is of the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \tag{8}$$

for $z \in \mathbb{U}$, then

$$|c_n| \leq 2 \quad for \quad n \in \mathbb{N} := \{1, 2, ...\}. \tag{9}$$

The inequality in (9) is sharp and the equality holds for the function $\varphi(z) = (1+z)/(1-z)$ (see Duren [10]).

Lemma 2. [13]. *The power series for $p(z)$ given in (2.1) converges in $\mathbb{U}$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$T_n(p) = \begin{vmatrix} 2 & c_1 & c_2 & \dots & c_n \\ c_{-1} & 2 & c_1 & \dots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \dots & 2 \end{vmatrix}, \quad n = 1, 2, 3, ... \tag{10}$$

and $c_k = \bar{c}_k$, are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^{l} \varrho_k p_0(e^{it_k} z)$, $\varrho_k > 0$, t_k real and $t_k \neq t_j$ for $k \neq j$; in this case $T_n(p) > 0$ for $n < (l-1)$ and $T_n(p) = 0$ for $n \geq l$.

This necessary and sufficient condition is due to Carathéodory and Toeplitz and can be found in Grenander and Szegö [13].

The Toeplitz determinant can be use to find the estimate of the upper bound on the coefficients functional for analytic function introduced by Janteng et al. [15]. By referring to method introduced by Libera and Zlotkiewicz [21, 22]. We may assume without restriction that $c_1 > 0$. For the case $n = 2$, then from (10) we obtain

$$T_2(p) = \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \bar{c_2} & c_1 & 2 \end{vmatrix} = 8 + 2\operatorname{Re}\{c_1^2 c_2\} - 2|c_2| - 4c_1^2 \geq 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{11}$$

for some x,$|x| \leq 1$. Then for $n = 3$, $T_3(p) \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|2c_2 - c_1^2|^2;$$

and this, with (11), provides the relation

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{12}$$

for some value of z, $|z| \leq 1$.

3. Main Result

Our main result is the following:

Theorem 1. *Let the function f given by (1) be in the class $\mathfrak{D}_\lambda^m f(z)$. Then*

$$|a_2 a_4 - a_3^2| \leq \frac{16}{9^{m+1}(\lambda+1)^2(\lambda+2)^2}. \tag{13}$$

The inequality in result (13) obtained is sharp.

Proof. Since $f \in \mathfrak{D}_\lambda^m f(z)$, by virtue of (7) there exists an analytic function $p \in \mathcal{P}$ in the unit disk $\mathbb{U}$ with $p(0) = 1$ and $[\mathrm{Re} p(z)] > 0$ such that

$$[\mathfrak{D}_\lambda^m f(z)]' = p(z). \tag{14}$$

Replacing $[\mathfrak{D}_\lambda^m f(z)]'$ and $p(z)$ with their equivalent series expressions in 14, we have for some $z \in \mathbb{U}$.

$$1 + \sum_{n=2}^{\infty} \frac{n^{m+1}(n+\lambda-1)!}{\lambda!(n-1)!} a_n z^{n-1} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots.$$

Upon simplification, we have

$$1 + 2 \cdot 2^m(\lambda+1)a_2 z + \frac{3 \cdot 3^m(\lambda+2)(\lambda+1)}{2} a_3 z^2$$

$$+ \frac{2 \cdot 4^m(\lambda+3)(\lambda+2)(\lambda+1)}{3} a_4 z^3 + \ldots = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots. \tag{15}$$

Equating coefficients in (15) of the like powers z^0, z and z^2, respectively, yields

$$\left\{ a_2 = \frac{1}{2}\frac{c_1}{2^m(\lambda+1)}, a_3 = \frac{2}{3}\frac{c_2}{3^m(\lambda+2)(\lambda+1)}, a_4 = \frac{3}{2}\frac{c_3}{4^m(\lambda+3)(\lambda+2)(\lambda+1)} \right\}. \tag{16}$$

Substituting the values of a_2, a_3 and a_4 from (16) in the second Hankel functional $|a_2 a_4 - a_3^2|$, it can be easily established that

$$|a_2 a_4 - a_3^2| = \frac{1}{(\lambda+1)^2(\lambda+2)} \left| \frac{3}{4}\frac{c_1 c_3}{2^m 4^m(\lambda+3)} - \frac{4}{9}\frac{c_2^2}{(3^m)^2(\lambda+2)} \right|.$$

We make use of Lemma 2 to obtain the proper bound on

$$\frac{1}{(\lambda+1)^2(\lambda+2)} \left| \frac{3}{4}\frac{c_1 c_3}{2^m 4^m(\lambda+3)} - \frac{4}{9}\frac{c_2^2}{(3^m)^2(\lambda+2)} \right| \tag{17}$$

Now, to simplify our calculation, we let

$$\{u = 2^m, v = 3^m \quad \text{and} \quad w = 4^m\}. \tag{18}$$

Thus, equation (17) can be written as

$$\frac{1}{(\lambda+1)^2(\lambda+2)} \left| \frac{3}{4}\frac{c_1 c_3}{uw(\lambda+3)} - \frac{4}{9}\frac{c_2^2}{v^2(\lambda+2)} \right|.$$

By substituting the values of c_2 and c_3 from (11) along with (12) from Lemma 2 in (17), we get

$$\frac{1}{(\lambda+1)^2(\lambda+2)}\left|\frac{3}{4}\frac{c_1c_3}{uw(\lambda+3)}-\frac{4}{9}\frac{c_2^2}{v^2(\lambda+2)}\right|$$
$$=\frac{1}{(\lambda+1)^2(\lambda+2)}\left|c^4\left[\frac{27v^2(\lambda+2)-16uw(\lambda+3)}{144uv^2w(\lambda+3)(\lambda+2)}\right]\right.$$
$$+c^2(4-c^2)x\left[\frac{27v^2(\lambda+2)-16uw(\lambda+3)}{72uv^2w(\lambda+3)(\lambda+2)}\right]$$
$$\left.-(4-c^2)x^2\left\{\frac{[27v^2(\lambda+2)-16uw(\lambda+3)]c^2-64uw(\lambda+3)}{144uv^2w(\lambda+3)(\lambda+2)}\right\}+\frac{3c(4-c^2)(1-|x|^2)z}{8uw(\lambda+3)}\right|.$$

By using the facts $|z|<1$ and triangle inequality with taking $c_1=c$ and $c\in[0,2]$ shows that

$$\frac{1}{(\lambda+1)^2(\lambda+2)}\left|\frac{3}{4}\frac{c_1c_3}{uw(\lambda+3)}-\frac{4}{9}\frac{c_2^2}{v^2(\lambda+2)}\right|$$
$$\leq\frac{1}{(\lambda+1)^2(\lambda+2)}\left\{\frac{|27v^2(\lambda+2)-16uw(\lambda+3)|c^4}{144uv^2w(\lambda+3)(\lambda+2)}+\frac{3c(4-c^2)}{8uw(\lambda+3)}\right.$$
$$+c^2(4-c^2)\rho\frac{|27v^2(\lambda+2)-16uw(\lambda+3)|}{72uv^2w(\lambda+3)(\lambda+2)}$$
$$\left.+(4-c^2)(c-2)\rho^2\left[\frac{27v^2(\lambda+2)c-16uw(\lambda+3)(c+2)}{144uv^2w(\lambda+3)(\lambda+2)}\right]\right\}$$
$$=F(c,\rho),\quad\text{for}\quad 0\leq\rho=|x|\leq 1. \tag{19}$$

We assume that the upper bound for (19) attains at the interior point of $\rho\in[0,1]$ and $c\in[0,2]$. Next, we maximize the function $F(c,\rho)$ on the closed square $[0,2]\times[0,1]$. Differentiating (19) with respect to ρ, we obtain

$$\frac{\partial F}{\partial\rho}=c^2(4-c^2)\frac{|27v^2(\lambda+2)-16uw(\lambda+3)|}{72uv^2w(\lambda+3)(\lambda+2)}+(4-c^2)(c-2)\rho\left[\frac{27v^2(\lambda+2)-16uw(\lambda+3)(c+2)}{72uv^2w(\lambda+3)(\lambda+2)}\right]. \tag{20}$$

From (20) we observe that, $\frac{\partial F}{\partial\rho}>0$ for $\rho>0$. Thus, (20) is an increasing function of ρ and hence it cannot have a maximum in the interior of the closed region $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$ we have

$$\max_{0\leq\rho\leq 1}F(c,\rho)=F(c,1). \tag{21}$$

Therefore, by substituting $\rho=1$ in (19), upon simplification we obtain

$$F(c,1)=\frac{1}{144vw(uv^2w\lambda+1)^2(\lambda+2)^2(\lambda+3)}\Big\{54v^2(\lambda+2)(4-c^2)c+|27v^2(\lambda+2)-16uw(\lambda+3)|[c^4+2c^2(4-c^2)]+(4-c^2)(c-2)[27v^2(\lambda+2)c-16uw(\lambda+3)(c+2)]\Big\}, \tag{22}$$

then

$$\begin{aligned} F'(c,1) &= \frac{1}{144uv^2w(\lambda+1)^2(\lambda+2)^2(\lambda+3)}\Big\{(4-3c^2)[54v^2(\lambda+2)] \\ &\quad +4c(4-c^2)|27v^2(\lambda+2)-16t^2uw(\lambda+3)| \\ &\quad +(8-3c^2)[27v^2(\lambda+2)c-16t^2uw(\lambda+3)(c+2)] \\ &\quad +(4-c^2)(c-2)[27v^2(\lambda+2)c-162uw(\lambda+3)]\Big\}. \end{aligned} \tag{23}$$

From (23), we note that $F'(c,1) \leq 0$ for every $c \in [0,2]$. Therefore, $F(c,1)$ is a decreasing function of c in the interval $c \in [0,2]$, whose maximum values occurs at points of F must be on the boundary of $c \in [0,2]$. However, $F(0,1) \geq F(2,1)$ and thus F has maximum value at $c = 0$.
The upper bound for (19) corresponds to $\rho = 1$ and $c = 0$, in which case

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{1}{(\lambda+1)^2(\lambda+2)}\left|\frac{3}{4}\frac{c_1c_3}{uw(\lambda+3)} - \frac{4}{9}\frac{c_2^2}{v^2(\lambda+2)}\right| \\ &\leq \frac{16}{9v^2(\lambda+1)^2(\lambda+2)^2}. \end{aligned} \tag{24}$$

By substituting $v = 3^m$. We have the upper bound

$$|a_2a_4 - a_3^2| \leq \frac{16}{9^{m+1}(\lambda+1)^2(\lambda+2)^2}. \tag{25}$$

By setting $c_1 = 0$ and choosing $x = 1$ in (11) and (12), we find that $c_2 = 2$ and $c_3 = 0$. Substitute these values in (24), the equality is attained, which shows that our result is sharp. This concludes the proof of our theorem. □

Remark 1. *For the choice of $m = 0$ and $\lambda = 0$ into Theorem 1, we will obtained the result coincides with Janteng et. al* [15] *which stated that $|a_2a_4 - a_3^2| \leq 4/9$.*

Theorem 2. *Let the function f given by (1) be in the class $\mathfrak{D}_\lambda^m f(z)$. Then*

$$|a_2a_3 - a_4| \leq \frac{3}{4^m(\lambda+2)(\lambda+3)}. \tag{26}$$

The result obtained in is sharp.

Proof. Substituting the values of a_2, a_3 and a_4 from (16) in the nonlinear functional $|a_2a_3 - a_4|$, we obtain

$$|a_2a_3 - a_4| = \frac{1}{(\lambda+1)(\lambda+2)}\left|\frac{1}{3}\frac{c_1c_2}{2^m3^m(\lambda+1)} - \frac{3}{2}\frac{c_3}{4^m(\lambda+3)}\right|. \tag{27}$$

We substitute (18) into (27) to simplfy our calculation. Thus, equation (27) can be written as

$$\frac{1}{(\lambda+1)(\lambda+2)}\left|\frac{1}{3}\frac{c_1c_2}{uv(\lambda+1)} - \frac{3}{2}\frac{c_3}{w(\lambda+3)}\right|.$$

By Lemma 2 and substituting the values of c_2 and c_3 from (11) along with (12) in (27), we get

$$\begin{aligned}
&\frac{1}{(\lambda+1)(\lambda+2)}\left|\frac{1}{3}\frac{c_1c_2}{uv(\lambda+1)}-\frac{3}{2}\frac{c_3}{w(\lambda+3)}\right| \\
&=\frac{1}{(\lambda+1)(\lambda+2)}\Bigg|\left(\frac{[8w(\lambda+3)-18uv(\lambda+1)]}{48uvw(\lambda+1)(\lambda+3)}\right)c^3 \\
&+\left(\frac{4w(\lambda+3)-18uv(\lambda+1)}{24uvw(\lambda+1(\lambda+3)}\right)c(4-c^2)x+\frac{3}{8}\frac{(4-c^2)x^2}{w(\lambda+3)}-\frac{3}{4}\frac{(4-c^2)(1-|x|^2)z}{w(\lambda+3)}\Bigg|,
\end{aligned}$$

for some x and z such that $|x|\leq 1$ and $|z|\leq 1$. Using the triangle inequality with $c_1=c$ and $c\in[0,2]$, we have

$$\begin{aligned}
&\frac{1}{(\lambda+1)(\lambda+2)}\left|\frac{1}{3}\frac{c_1c_2}{uv(\lambda+1)}-\frac{3}{2}\frac{c_3}{w(\lambda+3)}\right| \\
&\leq\frac{1}{(\lambda+1)(\lambda+2)}\Bigg\{\left(\frac{[8w(\lambda+3)-18uv(\lambda+1)]}{48uvw(\lambda+1)(\lambda+3)}\right)c^3+\frac{3}{4}\frac{(4-c^2)}{w(\lambda+3)} \\
&+c(4-c^2)\rho\left(\frac{[4w(\lambda+3)-18uv(\lambda+1)]}{24uvw(\lambda+1)(\lambda+3)}\right)+(4-c^2)\rho^2\left[\frac{3c-6}{8w(\lambda+3)}\right]\Bigg\} \\
&\leq\frac{1}{48uvw(\lambda+1)^2(\lambda+2)(\lambda+3)}\Big\{[8w(\lambda+3)-18uv(\lambda+1)]c^3 \\
&+36uv(4-c^2)(\lambda+1)+2\rho(4-c^2)[4w(\lambda+3)-18uv(\lambda+1)]c \\
&+6\rho^2(4-c^2)(3c-6)uv(\lambda+1)]\Big\}=G(c,\rho),\quad\text{for}\quad 0\leq\rho=|x|\leq 1. \qquad (28)
\end{aligned}$$

We assume that the upper bound for (27) attains at the interior point of $\rho\in[0,1]$ and $c\in[0,2]$. Next, we maximize the function $G(c,\rho)$ on the closed square $[0,2]\times[0,1]$. Since

$$\begin{aligned}
\frac{\partial G}{\partial\rho} &= \frac{1}{48uvw(\lambda+1)^2(\lambda+2)(\lambda+3)}\Big\{2(4-c^2)[4w(\lambda+3)-18uv(\lambda+1)]c \\
&+12\rho(4-c^2)(3c-6)uv(\lambda+1)]\Big\}. \qquad (29)
\end{aligned}$$

with elementary calculus, we can show that $\frac{\partial G}{\partial\rho}>0$ for $\rho>0$. Thus, $G(c,\rho)$ is an increasing function of ρ and hence it cannot have a maximum in the interior of the closed region $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$ we have

$$\max_{0\leq\rho\leq 1}G(c,\rho)=G(c,0). \qquad (30)$$

Therefore, by substituting $\rho=0$ in (28), upon simplification we obtain

$$\begin{aligned}
G(c,0) &= \frac{1}{48uvw(\lambda+1)^2(\lambda+2)(\lambda+3)}\Big\{[8w(\lambda+3)-18uv(\lambda+1)]c^3 \\
&+36uv(4-c^2)(\lambda+1)\Big\} \qquad (31)
\end{aligned}$$

then

$$\begin{aligned}
G'(c,0) &= \frac{1}{48uvw(\lambda+1)^2(\lambda+2)(\lambda+3)}\Big\{3[8w(\lambda+3)-18uv(\lambda+1)]c^2 \\
&-72uv(\lambda+1)c\Big\}. \qquad (32)
\end{aligned}$$

From (32), we note that $G'(c,0) \leq 0$ for every $c \in [0,2]$. Therefore, $G(c,0)$ is a decreasing function of c in the interval $c \in [0,2]$, whose maximum values occurs at points of G must be on the boundary of $c \in [0,2]$. However, $G(c) \geq G(2)$ and thus the local maximum at $G(0,0)$.
The upper bound for (27) corresponds to $\rho = 0$ and $c = 0$, in which case

$$|a_2a_3 - a_4| \leq \frac{3}{w(\lambda+2)(\lambda+3)}. \tag{33}$$

By substituting $w = 4^m$. We have the upper bound

$$|a_2a_4 - a_3^2| \leq \frac{3}{4^m(\lambda+2)(\lambda+3)}. \tag{34}$$

By setting $c_1 = 0$ and choosing $x = |1|$ in (11) and (12), we find that $c_2 = 2$ and $c_3 = 0$. Substitute these values in (33), the equality is attained, which shows that our result is sharp. This concludes the proof of our theorem. □

Remark 2. *For the choice of $m = 0$ and $\lambda = 0$ into Theorem 2, we will obtained the result coincides with Bansal et. al* [8] *which $|a_2a_4 - a_3^2| \leq 1/2$.*

Theorem 3. *Let the function f given by (1) be in the class $\mathfrak{D}_\lambda^m f(z)$. Then*

$$|a_3 - a_2^2| \leq \frac{4}{3^{m+1}(\lambda+1)(\lambda+2)}. \tag{35}$$

The result obtained in (35) is sharp.

Proof. Substituting the values of a_2, a_3 and a_4 from (16) in functional $|a_3 - a_2^2|$, we obtain

$$|a_3 - a_2^2| = \frac{1}{(\lambda+1)}\left|\frac{2}{3}\frac{c_2}{3^m(\lambda+2)} - \frac{1}{4}\frac{c_1^2}{(2^m)^2(\lambda+1)}\right|. \tag{36}$$

To simplfy our calculation, we substitute (18) into (36). Thus, equation (36) become

$$|a_3 - a_2^2| = \frac{1}{(\lambda+1)}\left|\frac{2}{3}\frac{c_2}{v(\lambda+2)} - \frac{1}{4}\frac{c_1^2}{u^2(\lambda+1)}\right|.$$

We assume $c_1 = c$ and $c \in [0,2]$ and substituting the values of c_2 and c_3 from (11) along with (12) and make use Lemma 2 in (36), we have

$$\begin{aligned} &\frac{1}{(\lambda+1)}\left|\frac{2}{3}\frac{c_2}{v(\lambda+2)} - \frac{1}{4}\frac{c_1^2}{u^2(\lambda+1)}\right| \\ &= \frac{1}{(\lambda+1)}\left|\frac{1}{3}\frac{c^3}{v(\lambda+2)} + \frac{1}{3}\frac{c(4-c^2)x}{v(\lambda+2)} - \frac{1}{4}\frac{c^2}{u^2(\lambda+1)}\right|. \end{aligned} \tag{37}$$

for some x and z such that $|x| \leq 1$ and $|z| \leq 1$. Using the triangle inequality , we have

$$\begin{aligned} &\frac{1}{(\lambda+1)}\left|\frac{2}{3}\frac{c_2}{v(\lambda+2)} - \frac{1}{4}\frac{c_1^2}{u^2(\lambda+1)}\right| \\ &\leq \frac{1}{(\lambda+1)}\left\{\frac{1}{3}\frac{c^3}{v(\lambda+2)} + \frac{1}{3}\frac{c(4-c^2)\rho}{v(\lambda+2)} - \frac{1}{4}\frac{c^2}{u^2(\lambda+1)}\right\} \\ &\leq \frac{1}{12u^2v(\lambda+1)(\lambda+2)}\left\{4c^3u^2(\lambda+1) + 4c\rho u^2(4-c^2)(\lambda+1) + 3c^2v(\lambda+2)\right\} \\ &= H(c,\rho), \quad \text{for} \quad 0 \leq \rho = |x| \leq 1. \end{aligned} \tag{38}$$

We assume that the upper bound for (36) attains at the interior point of $\rho \in [0,1]$ and $c \in [0,2]$. Next, we maximize the function $H(c,\rho)$ on the closed square $[0,2]\times[0,1]$. Since

$$\frac{\partial H}{\partial \rho} = \frac{1}{12u^2v(\lambda+1)(\lambda+2)}\Big\{4cu^2(4-c^2)(\lambda+1)\Big\}.$$

with elementary calculus, we can show that $\frac{\partial H}{\partial \rho} > 0$. Thus, $H(c,\rho)$ is an increasing function of ρ and hence it cannot have a maximum in the interior of the closed region $[0,2]\times[0,1]$. Moreover, for fixed $c \in [0,2]$ we have

$$\max_{0\le\rho\le1} H(c,\rho) = H(c,1). \tag{39}$$

Therefore, by substituting $\rho = 1$ in (38), upon simplification we obtain

$$H(c,1) = \frac{1}{12u^2v(\lambda+1)(\lambda+2)}\Big\{4c^3u^2(\lambda+1) + 4cu^2(4-c^2)(\lambda+1) + 3c^2v(\lambda+2)\Big\} \tag{40}$$

then

$$\begin{aligned} H'(c,1) &= \frac{1}{12u^2v(\lambda+1)(\lambda+2)}\Big\{12c^2u^2(\lambda+1) + 4u^2(4-c^2)(\lambda+1) \\ &\quad -8c^2u^2(\lambda+1) + 6cv(\lambda+2)\Big\} \end{aligned} \tag{41}$$

From (41), we note that $H'(c,1) \le 0$ for every $c \in [0,2]$. Therefore, $H(c,1)$ is a decreasing function of c in the interval $c \in [0,2]$, whose maximum values occurs at points of H must be on the boundary of $c \in [0,2]$. However, $H(0,1) \ge H(2,1)$ and thus the local maximum at $H(0,1)$.
The upper bound for (36) corresponds to $\rho = 1$ and $c = 0$, in which case

$$|a_3 - a_2^2| \le \frac{4}{3v(\lambda+1)(\lambda+2)}. \tag{42}$$

By substituting $v = 3^m$. We have the upper bound

$$|a_3 - a_2^2| \le \frac{4}{3^{m+1}(\lambda+1)(\lambda+2)}. \tag{43}$$

By setting $c_1 = 0$ and choosing $x = |1|$ in (11) and (12), we find that $c_2 = 2$ and $c_3 = 0$. Substitute these values in (42), the equality is attained, which shows that our result is sharp. This concludes the proof of our theorem. □

Remark 3. *For the choice of $m = 0$ and $\lambda = 0$ into Theorem 3, we will obtained the result coincides with Babalola and Opoola* [6] *which show $|a_3 - a_2^2| \le 2/3$.*

It is well known from Macgregor [24]. If f in the form of (1), then $|a_n| \le 2/n, (n = 2,3,...)$. Using these coefficient bounds together with Theorem 1, 2 and 3, we obtained

$$\begin{aligned} |H_{3,1}(f)| &\le |a_3||(a_2a_4 - a_3^2)| - |a_4||(a_4 - a_2a_3)| + |a_5||(a_3 - a_2^2)|. \\ &\le \frac{2}{3}\left(\frac{16}{9^{m+1}(\lambda+1)^2(\lambda+2)^2}\right) - \frac{2}{4}\left(\frac{3}{4^m(\lambda+2)(\lambda+3)}\right) \\ &\quad + \frac{2}{5}\left(\frac{4}{3^{m+1}(\lambda+1)(\lambda+2)}\right). \end{aligned} \tag{44}$$

Thus, we state that:

Theorem 4. *Let the function f given by (1) be in the class $\mathfrak{D}_{\lambda}^{m} f(z)$. Then*

$$|H_{3,1}(f)| \leq \frac{32}{3^{2m+3}(\lambda+1)^2(\lambda+2)^2} - \frac{6}{4^{m+1}(\lambda+2)(\lambda+3)} + \frac{8}{5\cdot 3^{m+1}(\lambda+1)(\lambda+2)}. \tag{45}$$

The result obtained is sharp and the equality holds for the function

$$f'(z) = \frac{1+z^2}{1-z^2}.$$

Remark 4. *By taking $m = 0$ and $\lambda = 0$ into Theorem 4, we will obtained the result coincides with Bansal et. al* [8] *which* $|H_{3,1}(f)| \leq 439/540$.

4. Acknowledgement

The work here is supported by Universiti Kebangsaan Malaysia grant: GUP-2017-064.

References

[1] Al-Refai, O., Darus, M., *Second Hankel Determinant for certain class of analytic function defined by a fractional operator,* European Journal of Scientific Research, **28** (2) (2009), 234–241.

[2] Abubaker,A., Darus, M., *Hankel determinant for a class of analytic functions involving a generalized linear differential operator,*International Journal of Pure and Applied Mathematics, **69** (4) (2011), 429–435.

[3] Al-Abbadi, M.H., Darus, M., *Hankel Determinant for certain class of analytic function defined by generalized derivative operator,* Tamkang Journal of Mathematics, **43** (4) (1933), 445–453.

[4] Al-Shaqsi,K., Darus, M., *An operator defined by convolution involving the polylogarithms functions,* Journal of Mathematics and Statistics, **4** (1) (2008), 46–50.

[5] Babalola, K.O., *On third order Hankel determinant for some classes of univalent functions,* Inequality Theory and Applications, **6** (2010), 1–7.

[6] Babalola, K.O., Opoola, T.O., *On the coefficients of certain analytic and univalent functions,* Advances in Inequality for Series, (Edited by S.S. Dragomir and A. Sofo) Nova Science Publishers (2008), 5–17.

[7] Bansal, D., *Upper bound of second Hankel determinant for a new class of analytic functions* , Applied Mathematics Letters, **26** (1) (2013), 103–107.

[8] Bansal, D., Maharana, S., Prajapat, J. K., *Third order Hankel determinant for certain univalent functions,* Journal of the Korean Mathematical Society, **52** (6) (2015), 1139–1148.

[9] Cantor, D.G., *Power series with integral coefficients* , Bull. Am. Math. Soc., **26** (1963), 362–366.

[10] Duren, P.L.,*Univalent functions,* Grundlehren der Mathematischen Wissenschaften, (Band 259) (1983), NY:Springer-Verlag.

[11] Ehrenborg, R., *The Hankel determinant of exponential polynomials* , The American Mathematical Monthly, **107** (6) (2000), 557–560.

[12] Fekete, M., Szegö, G., *Eine Bemerkung uber ungerade schlichte funktionen,* J. London Math. Soc., **8** (1933), 85–89.

[13] Grenander, U., Szegö, G., *Toeplitz Forms and their Application,* Univ. of California Press, Berkeley and Los Angeles, (1958).

[14] Hayman, W.K., *On the second Hankel determinant of mean univalent functions,* Proc. Lond. Math. Soc., **18** (1968), 77–94.

[15] Janteng,A., Halim, S.A., Darus, M.,*Coefficient Inequality for a Function whose Derivative has a Positive Real Part,* J. Ineq. Pure and Appl. Math., **7** (2) art. 50(2006).

[16] Janteng,A., Halim, S.A., Darus, M., *Hankel Determinant for Starlike and Convex Functions,* Int. J. Math. Anal., Ruse **1** (2007), 619–625.

[17] Keogh, F.R., Merkes,E.P., *A coefficient inequality for certain classes of analytic functions,* Proc. Amer. Math. Soc., **20** (1969), 8–12.

[18] Koepf, W., *On the Fekete-Szego problem for close-to-convex functions,* Proc. Amer. Math. Soc., **101** (1987), 89–95.

[19] Krishna, D.V., Venkateswarlu, B., RamReddy, T., *Third Hankel determinant for bounded turning functions of order alpha,* Journal of the Nigerian Mathematical Society, **34** (2) (2015), 121–127.

[20] Layman, J.W., *The Hankel transform and some of its properties*, J. Integer Seq., **4** (1) (2001), 1–11.
[21] Libera, R.J., Zlotkiewicz, E.J.,*Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc., **85** (2) (1982), 225–230.
[22] ———, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc., **87** (2) (1983), 251–289.
[23] London, R.R., *Fekete-Szegö inequalities for close-to-convex functions,* Proc. .American. Math. Soc., **117** (4) (1993), 947–950.
[24] MacGregor, T.H., *Functions whose derivative has a positive real part,* Trans. Amer. Math. Soc., **104** (1962), 532–537.
[25] Noonan, J.W., Thomas, D.K.,*On the second Hankel determinant of areally mean and p-valent function,* Trans. Amer. Math. Soc., **223** (2) (1976), 337–346.
[26] Noor,K.I., *Hankel determinant problem for the class of functions with bounded boundary rotation,* Revue Roumaine de Mathèmatiques Pures et Appliquèes, **28** (8) (1983), 731–739.
[27] Noor, K.I., Al-Bany, S.A., *On Bazilevic functions,* International Journal of Mathematics and Mathematical Sciences, **10** (1) (1987), 79–88.
[28] Prajapat, J.K., Bansal, D., Singh, A., Mishra, A.K., *Bounds on third Hankel determinant for close-to-convex functions,* Acta Universitatis Sapientiae, Mathematica, **7** (2) (2015), 210–219.
[29] Pommerenke, C.H., *On the Hankel determinant of univalent functions,* Mathematika, **14** (1966), 108–112.
[30] Pommerenke, C.H., *Univalent Functions,* Vandenhoeck and Ruprecht, Gottingen, (1975).
[31] Shanmugam, T.N., Stephen, B.A., Babalola, K.O., *Third Hankel determinant for α-starlike functions,* Gulf Journal of Mathematics, **2** (2) (2014), 107–113.
[32] Srivastava, H.M., Mishra, A.K., Das, M.K., *The Fekete-Szego problem for a subclass of close-to-convex functions,* Complex Variables and Elliptic Equations, **44** (2) (2001), 145–163.
[33] Wilson, R., *Determinantal criteria for meromorphic functions,* Proc. Lond. Math, **4** (1954), 357–374.
[34] Zaprawa,P., *Third Hankel Determinants for Subclasses of Univalent Functions,* Mediterranean Journal of Mathematics, 14:19 (2017), 10 pages.

School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600 Selangor D.E., Malaysia
E-mail address: nazran@unisel.edu.my

School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600 Selangor D.E., Malaysia
E-mail address: maslina@ukm.edu.my

Department of Science and Biotechnology,
Faculty of Engineering and Life Sciences,
Universiti Selangor, Batang Berjuntai,
Bestari Jaya 45600, Selangor D.E., Malaysia
E-mail address: saibahmath@yahoo.com