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EXISTENCE, INTERVAL OF EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR IMPLICIT CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we use the contraction mapping principle to obtain the existence, interval of existence and uniqueness of solutions for nonlinear implicit Caputo fractional differential equations. We also use the generalization of Gronwall's inequality to show the estimate of the solutions. The results obtained here extend the work of Dong, Feng and Jiang in [6].

1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[6], [8]–[14] and the references therein.

In this paper, we are interested in the analysis of qualitative theory of the problems of the existence, interval of existence and uniqueness of solutions to implicit fractional differential equations. Inspired and motivated by the references in this paper, we concentrate on the existence, interval of existence and uniqueness of solutions for the nonlinear implicit fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f\left(t, x(t), {}^{C}D^{\alpha}x(t)\right), \\ x(0) + g(x) = x_0, \end{cases} \tag{1}$$

where $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : C([0,T], \mathbb{R}) \to \mathbb{R}$ are nonlinear continuous functions and ${}^{C}D^{\alpha}$ denotes the Caputo derivative of order $0 < \alpha < 1$. In passing, we note that the application of nonlinear condition $x(0) + g(x) = x_0$ in physical problems yields better effect than the initial condition $x(0) = x_0$ (see [2]). To show the existence, interval of existence and uniqueness of solutions, we transform (1) into an integral equation and then use the contraction mapping principle. Further, by the generalization of Gronwall's inequality we obtain the estimate of solutions of (1). In the special case $g(x) = 0$, Dong, Feng and Jiang in [6] show that (1) has a unique solution by using the contraction mapping theorem. Then, the results obtained here extend the work of Dong, Feng and Jiang in [6].

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2. Preliminaries

In this section we present some basic definitions, notations and results of fractional calculus [5, 8, 12] which are used throughout this paper.

Definition 1 ([8]). *The fractional integral of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2 ([8]). *The Caputo fractional derivative of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$${}^C D^\alpha x(t) = I^{n-\alpha} x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 1 ([8]). *Let $\Re(\alpha) > 0$. Suppose $x \in C^{n-1}[0, +\infty)$ and $x^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$\left(I^\alpha \, {}^C D^\alpha x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

In particular, when $0 < \Re(\alpha) < 1$, $\left(I^\alpha \, {}^C D^\alpha x\right)(t) = x(t) - x(0)$.

Lemma 2 ([8]). *For all $\alpha, \beta \in [0, \infty)$, Then*

$$\int_0^t (t-s)^{\beta-1} s^{\alpha-1} ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}.$$

The following generalization of Gronwall's lemma for singular kernels plays an important role in obtaining our main results.

Lemma 3 ([7]). *Let $x : [0, T] \to [0, \infty)$ be a real function and w is a nonnegative locally integrable function on $[0, T]$. Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$*

$$x(t) \le w(t) + a \int_0^t (t-s)^{\alpha-1} x(s) \, ds.$$

Then, there exist a constant $K = K(\alpha)$ such that

$$x(t) \le w(t) + Ka \int_0^t (t-s)^{\alpha-1} w(s) \, ds,$$

for every $t \in [0, T]$.

3. Main results

In this section, we give the equivalence of the initial value problem (1) and prove the existence, interval of existence, uniqueness and estimate of solutions of (1).

The proof of the following lemma is close to the proof of Lemma 6.2 given in [5].

Lemma 4. *If the functions $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous, then the initial value problem (1) is equivalent to nonlinear fractional Volterra integro-differential equation*

$$x(t) = x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), {}^C D^\alpha x(s)\right) ds, \ t \in [0, T].$$

Theorem 1. *Let $T > 0$. Assume $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : C([0,T], \mathbb{R}) \to \mathbb{R}$ satisfy the following condition*

(H1) There exist $K_1 \in \mathbb{R}^+$, $K_2, K_3 \in (0,1)$ such that

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \leq K_1 |u - \tilde{u}| + K_2 |v - \tilde{v}|,$$

and

$$|g(x) - g(\tilde{x})| \leq K_3 \|x - \tilde{x}\|.$$

Let

$$b < \min\left\{T, \left(\frac{(1-K_3)(1-K_2)\Gamma(\alpha+1)}{K_1}\right)^{\frac{1}{\alpha}}\right\},$$

then (1) has a unique solution $x \in C([0,b], \mathbb{R})$.

Proof. Let

$${}^C D^\alpha x(t) = z_x(t), \; x(0) + g(x) = x_0,$$

then by Lemma 4,

$$x(t) = x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s)\, ds, \; t \in [0,T],$$

where

$$z_x(t) = f(t, x_0 - g(x) + I^\alpha z_x(t), z_x(t)).$$

That is $x(t) = x_0 - g(x) + I^\alpha z_x(t)$. Define the mapping $P : C([0,b], \mathbb{R}) \to C([0,b], \mathbb{R})$ as follows

$$(Px)(t) = x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s)\, ds.$$

It is clear that the fixed points of P are solutions of (1). Let $x, y \in C([0,b], \mathbb{R})$, then we have

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&\leq |g(x) - g(y)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_x(s) - z_y(s)|\, ds \\
&\leq K_3 \|x - y\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_x(s) - z_y(s)|\, ds, \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
|z_x(t) - z_y(t)| &\leq |f(t, x(t), z_x(t)) - f(t, x(t), z_y(t))| \\
&\leq K_1 |x(t) - y(t)| + K_2 |z_x(t) - z_y(t)| \\
&\leq \frac{K_1}{1-K_2} |x(t) - y(t)|. \qquad (3)
\end{aligned}$$

By replacing (3) in the inequality (2), we get

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&\leq K_3 \|x - y\| + \frac{1}{\Gamma(\alpha)} \frac{K_1}{1-K_2} \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)|\, ds \\
&\leq K_3 \|x - y\| + \frac{1}{\Gamma(\alpha)} \frac{K_1}{1-K_2} \left(\int_0^t (t-s)^{\alpha-1} ds\right) \|x - y\| \\
&\leq \left(K_3 + \frac{K_1}{1-K_2} \frac{t^\alpha}{\Gamma(\alpha+1)}\right) \|x - y\|.
\end{aligned}$$

Since $t \in [0,b]$, then

$$\|Px - Py\| \leq \beta \|x - y\|, \ 0 < \beta < 1,$$

where

$$\beta = K_3 + \frac{K_1}{1 - K_2} \frac{b^\alpha}{\Gamma(\alpha+1)}.$$

That is to say the mapping P is a contraction in $C([0,b],\mathbb{R})$. Hence P has a unique fixed point $x \in C([0,b],\mathbb{R})$. Therefore, (1) has a unique solution. □

Theorem 2. *Assume that $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : C([0,T],\mathbb{R}) \to \mathbb{R}$ satisfy (H1). If x is a solution of (1), then*

$$\|x\| \leq \frac{(1-K_2)(1-K_3)\Gamma(\alpha+1) + K_1 K T^\alpha}{(1-K_2)(1-K_3)^2\Gamma(\alpha+1)} \left(|x_0| + Q_1 + \frac{Q_2 T^\alpha}{(1-K_2)\Gamma(\alpha+1)} \right),$$

where $Q_1 = |g(0)|$, $Q_2 = \sup_{t\in[0,T]} |f(t,0,0)|$ and $K \in \mathbb{R}^+$ is a constant.

Proof. Let

$${}^cD^\alpha x(t) = z_x(t), \ x(0) + g(x) = x_0.$$

By Lemma 4, $x(t) = x_0 - g(x) + I^\alpha z_x(t)$. Then by hypothesis (H1), for any $t \in [0,T]$ we have

$$\begin{aligned} |x(t)| &\leq |x_0| + |g(x)| + I^\alpha |z_x(t)| \\ &\leq |x_0| + |g(x) - g(0)| + |g(0)| + I^\alpha |z_x(t)| \\ &\leq |x_0| + Q_1 + K_3 \|x\| + I^\alpha |z_x(t)|, \end{aligned}$$

where $Q_1 = |g(0)|$. On the other hand, for any $t \in [0,T]$ we get

$$\begin{aligned} |z_x(t)| &= |f(t, x(t), z_x(t))| \\ &\leq |f(t, x(t), z_x(t)) - f(t,0,0)| + |f(t,0,0)| \\ &\leq K_1 |x(t)| + K_2 |z_x(t)| + |f(t,0,0)| \\ &\leq \frac{K_1}{1-K_2} \|x\| + \frac{Q_2}{1-K_2}, \end{aligned}$$

where $Q_2 = \sup_{t\in[0,T]} |f(t,0,0)|$. Therefore

$$|x(t)| \leq |x_0| + Q_1 + K_3 \|x\| + I^\alpha \left(\frac{Q_2}{1-K_2} + \frac{K_1}{1-K_2} \|x\| \right).$$

Thus

$$\begin{aligned} (1-K_3)\|x\| &\leq |x_0| + Q_1 + \frac{Q_2 T^\alpha}{(1-K_2)\Gamma(\alpha+1)} \\ &\quad + \frac{K_1}{(1-K_2)(1-K_3)} I^\alpha \{(1-K_3)\|x\|\}. \end{aligned}$$

By Lemma 3, there is a constant $K = K(\alpha)$ such that

$$\begin{aligned} &(1-K_3)\|x\| \\ &\leq |x_0| + Q_1 + \frac{Q_2 T^\alpha}{(1-K_2)\Gamma(\alpha+1)} \\ &+ \frac{K_1 K T^\alpha}{(1-K_2)(1-K_3)\Gamma(\alpha+1)} \left(|x_0| + Q_1 + \frac{Q_2 T^\alpha}{(1-K_2)\Gamma(\alpha+1)} \right) \\ &\leq \frac{(1-K_2)(1-K_3)\Gamma(\alpha+1) + K_1 K T^\alpha}{(1-K_2)(1-K_3)\Gamma(\alpha+1)} \left(|x_0| + Q_1 + \frac{Q_2 T^\alpha}{(1-K_2)\Gamma(\alpha+1)} \right). \end{aligned}$$

Hence

$$\|x\| \leq \frac{(1-K_2)(1-K_3)\Gamma(\alpha+1)+K_1KT^\alpha}{(1-K_2)(1-K_3)^2\Gamma(\alpha+1)}\left(|x_0|+Q_1+\frac{Q_2T^\alpha}{(1-K_2)\Gamma(\alpha+1)}\right).$$

This completes the proof. □

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