SOME PROPERTIES OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH GENERALIZED DIFFERENTIAL OPERATOR INVOLVING MITTAG-LEFFLER FUNCTION

SUHILA ELHADDAD AND MASLINA DARUS

Abstract. A subclass in the open unit disc of analytic functions is introduced in this paper. This subclass $S_{m, \delta}^{\alpha, \beta}(\lambda, \alpha, \beta, A, B)$ is mainly defined by generalized differential Operator. Some interesting properties of this class are also obtained.

1. Introduction

We begin by letting $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane and $\mathcal{H} = \mathcal{H}(U)$ denote the class of analytic functions. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + anz^n + an+1z^{n+1} + \ldots, \quad (a \in \mathbb{C}).$$

And, let $A$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad a_k \geq 1.$$  \hspace{1cm} (1)

A function $f$ belonging to the class $A$ is said to be convex of order $\alpha$ $(0 \leq \alpha < 1)$ if and only if $\Re\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha$, $z \in U$. This class is denoted by $C(\alpha)$. On the other hand, a function $f \in A$ is said to be starlike of order $\alpha$ $(0 \leq \alpha < 1)$ if and only if $\Re\{zf'(z)\} > \alpha$, $z \in U$. This class is denoted by $S^*(\alpha)$.

The Hadamard product for two analytic functions $f$ defined in (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad b_k \geq 1,$

is given by

$$f(z) \ast g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Given two analytic functions $f$ and $g$ in $U$, the subordination between them is written as $f \prec g$ or $f(z) \prec g(z)$. In addition, we say $f(z)$ is subordinate to $g(z)$ if there is a Schwarz function $w$ with $w(z) = 0$, $|w(z)| < 1$, $(z \in U)$ such that $f(z) = g(w(z))$ for all $z \in U$. Furthermore, if $g(z)$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

2010 Mathematics Subject Classification. 30C45, 30C50.

Key words and phrases. analytic functions, differential operator, Mittag-Leffler function, starlike and convex functions, Differential subordination.
The following defines the familiar Mittag-Leffler function \( E_{\alpha}(z) \) introduced by Mittag-Leffler \([7]\) and \([8]\) and its generalization \( E_{\alpha,\beta}(z) \) introduced by Wiman \([15]\)

\[
E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},
\]

and

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},
\]

where \( \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0 \) and \( \Re(\beta) > 0 \).

As a result, a lot of useful work have been made by many researchers in attempt to explain Mittag-Leffler function and its generalization, see for example \([13]\), \([2]\), \([9]\), \([11]\), and \([14]\).

Now, we define the function \( Q_{\alpha,\beta}(z) \) by

\[
Q_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1) + \beta)}a_k z^k.
\]

Corresponding to \( Q_{\alpha,\beta}(z) \), Elhaddad et al.\([3]\) defined the differential operator \( D^m_\lambda(\alpha,\beta)f: \mathcal{A} \rightarrow \mathcal{A} \) as follows:

\[
D^m_\lambda(\alpha,\beta)f(z) = z + \sum_{k=2}^{\infty} \frac{[1 + (k-1)\lambda]^m\Gamma(\beta)}{\Gamma(\alpha(k-1) + \beta)}a_k z^k,
\]

where \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0 \).

We can easily verify from \([2]\) that

\[
D^{m+1}_\lambda(\alpha,\beta)f(z) = (1 - \lambda)D^m_\lambda(\alpha,\beta)f(z) + \lambda z(D^m_\lambda(\alpha,\beta)f(z))',
\]

Note that

- for \( \alpha = 0 \) and \( \beta = 1 \) we get Al-Oboudi operator \([4]\);
- for \( \alpha = 0, \beta = 1, \lambda = 1 \) we get Sălăgean operator \([10]\);
- for \( m = 0 \) we get \( E_{\alpha,\beta}(z) \) \([3]\).

Next, as a result of full utilization of the differential operator \( D^m_\lambda(\alpha,\beta)f(z) \) and the principle of subordination, we define and study the class \( S^{m,\delta}_{\alpha,\beta}(\lambda,\alpha,\beta,A,B) \) as follows:

**Definition 1.** A function \( f \) belonging to the class \( \mathcal{A} \) is said to be in the class \( S^{m,\delta}_{\alpha,\beta}(\lambda,\alpha,\beta,A,B) \) if it satisfies the following subordination condition

\[
(1 - \vartheta) \left( \frac{D^m_\lambda(\alpha,\beta)f(z)}{z} \right)^\delta + \vartheta \left( \frac{D^{m+1}_\lambda(\alpha,\beta)f(z)}{D^m_\lambda(\alpha,\beta)f(z)} \right) \left( \frac{D^m_\lambda(\alpha,\beta)f(z)}{z} \right)^\delta < \frac{1 + Az}{1 + Bz},
\]

where \( \vartheta \in \mathbb{C}, \Re(\delta) > 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0, -1 \leq B \leq 1 \) and \( A \neq B \in \mathbb{N}_0 \).

Note that, if we set \( m = 0, \lambda = 1, \alpha = 0 \) and \( \beta = 1 \) in the class \( S^{m,\delta}_{\alpha,\beta}(\lambda,\alpha,\beta,A,B) \), then we get the class studied by Liu \([4]\).

In order to prove and validate above results, following preliminary results are required.
2. Preliminary Results

Lemma 1. (see [5]). Let the function $h$ be analytic and univalent (convex) in $U$ with $h(0) = 1$. Suppose also that the function $\varphi$ given by

$$\varphi(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ..., \quad (\text{see [5].})$$

is analytic in $U$. If

$$\varphi(z) + \frac{z \varphi'(z)}{c} < h(z) \quad (\text{Re}(z) \geq 0; c \neq 0, z \in U),$$

then

$$\varphi(z) \prec \phi(z) = \frac{z}{h(z)} \int_0^z t^{n-1} h(t) dt < h(z) \quad (z \in U),$$

and $\phi(z)$ is the best dominant.

Lemma 2. (see [12]). Let $q(z)$ be a convex univalent function in $U$ and let $\sigma, \eta \in \mathbb{C}$ with

$$\max \left\{ 0, -\text{Re}\left( \frac{\sigma}{\eta} \right) \right\}.$$ 

If the function $p$ is analytic in $U$ and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),$$

then

$$p(z) \prec q(z),$$

and $q(z)$ is the best dominant.

Lemma 3. (see [6]). Let $q$ be convex univalent in $U$ and $k \in \mathbb{C}$. Further assume that $\text{Re}(k) > 0$, if

$$p(z) \in \mathcal{H}[q(0), 1] \cap Q,$$

and $p(z) + k z p'(z)$ is univalent in $U$, then

$$q(z) + k z q'(z) \prec p(z) + k z p'(z)$$

Implies $q(z) \prec p(z)$ and $q(z)$ is the best subdominant

3. Main Result

In what follows we focus to study a set of interesting characteristics of the class $S^m_{\lambda}(\lambda, \alpha, \beta, A, B)$

Theorem 1. Let $f \in S^m_{\lambda}(\lambda, \alpha, \beta, A, B)$. Then

$$\left( \frac{D^m_{\lambda}(\alpha, \beta) f(z)}{z} \right)^{\delta} \prec \frac{\delta}{\lambda \nu \theta} \int_0^1 \frac{1 + A z u^{m \lambda - 1} du}{1 + B z u},$$

where $\theta \in \mathbb{C}, \text{Re}(\theta) > 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0, -1 \leq B \leq 1$ and $A \neq B \in \mathbb{N}_0$.

Proof. Define the function $p(z)$ by

$$p(z) = \left( \frac{D^m_{\lambda}(\alpha, \beta) f(z)}{z} \right)^{\delta} \quad (z \in U).$$
Then the function \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \). Therefore, differentiating (4) logarithmically with respect to \( z \), and using the identity (3), we get

\[
(1 - \vartheta) \left( \frac{D^m_\lambda (\alpha, \beta) f(z)}{z} \right) + \vartheta \left( \frac{D^{m+1}_\lambda (\alpha, \beta) f(z)}{D^m_\lambda (\alpha, \beta) f(z)} \right) \left( \frac{D^n_\lambda (\alpha, \beta) f(z)}{z} \right) ^\delta < p(z) + \frac{\lambda \vartheta}{\delta} z p'(z) < \frac{1 + A z}{1 + B z} \quad (z \in U). \tag{5}
\]

By applying Lemma 1 in the last equation, we have

\[
\left( \frac{D^n_\lambda (\alpha, \beta) f(z)}{z} \right) ^\delta < \frac{\delta}{\lambda \vartheta} z \int_0^z t^{\frac{\delta}{\lambda \vartheta} - 1} \frac{1 + A t}{1 + B t} \, dt = \frac{\varsigma}{n} \int_0^1 u^{\frac{\delta}{\lambda \vartheta} - 1} \frac{1 + A zu}{1 + B zu} \, du < \frac{1 + A z}{1 + B z} \quad (z \in U),
\]

where \( \varsigma = \frac{\delta}{\lambda \vartheta} \).

The proof of Theorem 1 is complete. \( \square \)

Theorem 2. Let \( q(z) \) be univalent in \( U \). Suppose also that \( q(z) \) satisfies

\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{\delta}{\lambda \vartheta} \right) \right\}. \tag{6}
\]

If \( f(z) \in A \) satisfying the following subordination

\[
(1 - \vartheta) \left( \frac{D^m_\lambda (\alpha, \beta) f(z)}{z} \right) + \vartheta \left( \frac{D^{m+1}_\lambda (\alpha, \beta) f(z)}{D^m_\lambda (\alpha, \beta) f(z)} \right) \left( \frac{D^n_\lambda (\alpha, \beta) f(z)}{z} \right) ^\delta < q(z) + \frac{\lambda \vartheta}{\delta} z q'(z), \tag{7}
\]

then

\[
\left( \frac{D^n_\lambda (\alpha, \beta) f(z)}{z} \right) ^\delta < q(z),
\]

and \( q(z) \) is the best dominant.

Proof. Let \( p(z) \) be defined by (4). We know that (5) is true. Combining (5) and (7), we see that

\[
p(z) + \frac{\lambda \vartheta}{\delta} z p'(z) < q(z) + \frac{\lambda \vartheta}{\delta} z q'(z). \tag{8}
\]

By using Lemma 2 and (8), we get the assertion of Theorem 2. \( \square \)

Taking \( q(z) = \frac{1 + A z}{1 + B z} \) in Theorem 2, we get the following result.

Corollary 1. Let \( \vartheta \in \mathbb{C} \) and \( -1 \leq B < A \leq 1 \). Suppose also that \( \frac{1 + A z}{1 + B z} \) satisfies the condition (6). If \( f(z) \in A \) satisfying the following subordination

\[
(1 - \vartheta) \left( \frac{D^m_\lambda (\alpha, \beta) f(z)}{z} \right) + \vartheta \left( \frac{D^{m+1}_\lambda (\alpha, \beta) f(z)}{D^m_\lambda (\alpha, \beta) f(z)} \right) \left( \frac{D^n_\lambda (\alpha, \beta) f(z)}{z} \right) ^\delta < \frac{1 + A z}{1 + B z} + \frac{\lambda \vartheta (A - B) z}{\delta(1 + B z)^2},
\]
then
\[ \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta \prec \frac{1 + Az}{1 + Bz}, \]
and \( \frac{1 + Az}{1 + Bz} \) is the best dominant.

**Theorem 3.** Let \( q(z) \) be convex univalent in \( U \), \( \theta \in C \), with \( \Re(\theta) > 0 \). Also let
\[ \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta \in \mathcal{H}[q(0), 1] \cap Q \]
be univalent in \( U \). If
\[
q(z) + \frac{\lambda \theta}{\delta} z q'(z) \prec (1 - \theta) \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta \nabla \theta \left( \frac{D^{m+1}_\alpha(\alpha, \beta)f(z)}{D^m_\alpha(\alpha, \beta)f(z)} \right) \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta,
\]
then
\[ q(z) \prec \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta, \]
and \( q(z) \) is the best subdominant.

**Proof.** Let \( p(z) \) be defined by (4). Then
\[
q(z) + \frac{\lambda \theta}{\delta} z q'(z) \prec (1 - \theta) \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta \nabla \theta \left( \frac{D^{m+1}_\alpha(\alpha, \beta)f(z)}{D^m_\alpha(\alpha, \beta)f(z)} \right) \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta = p(z) + \frac{\lambda \theta}{\delta} z p'(z).
\]
An application of Lemma 3 yields the assertion of Theorem 3. \( \square \)

**Corollary 2.** Let \( q(z) \) be convex univalent in \( U \) and \( -1 < B < A < 1 \), \( \theta \in C \) with \( \Re(\theta) > 0 \). Also let
\[ \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta \in \mathcal{H}[q(0), 1] \cap Q \]
be univalent in \( U \). If
\[
\frac{1 + Az}{1 + Bz} + \frac{\lambda \theta(A - B)z}{\delta(1 + Bz)^2} \prec (1 - \theta) \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta \nabla \theta \left( \frac{D^{m+1}_\alpha(\alpha, \beta)f(z)}{D^m_\alpha(\alpha, \beta)f(z)} \right) \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta,
\]
then
\[ \frac{1 + Az}{1 + Bz} \prec \left( \frac{D^m_\alpha(\alpha, \beta)f(z)}{z} \right)^\delta, \]
and \( \frac{1 + Az}{1 + Bz} \) is the best subdominant.
Theorem 4. Let \( f(z) \in S_\theta^{m,\delta}(\lambda, \alpha, \beta, A, B) \), then
\[
\inf_{z \in \mathbb{U}} \Re \left\{ \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du \right\} < \Re \left\{ \left( \frac{D^m_\lambda(\alpha, \beta)f(z)}{z} \right)^\delta \right\} < \sup_{z \in \mathbb{U}} \Re \left\{ \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du \right\},
\]
where \( \theta \in \mathbb{C}, \Re(\delta) > 0, m \in \mathbb{N}_0, \lambda \geq 0, -1 \leq B \leq 1 \) and \( A \neq B \in \mathbb{N}_0 \).

Proof. Suppose that \( f(z) \in S_\theta^{m,\delta}(\lambda, \alpha, \beta, A, B) \), then from Theorem 1 we know that
\[
\left( \frac{D^m_\lambda(\alpha, \beta)f(z)}{z} \right)^\delta < \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du.
\]
Thus, from the definition of the subordination, we get
\[
\Re \left\{ \frac{D^m_\lambda(\alpha, \beta)f(z)}{z} \right\} > \inf_{z \in \mathbb{U}} \Re \left\{ \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du \right\}
\]
and
\[
\Re \left\{ \frac{D^m_\lambda(\alpha, \beta)f(z)}{z} \right\} < \sup_{z \in \mathbb{U}} \Re \left\{ \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du \right\}.
\]

Corollary 3. Let \( \theta \in \mathbb{C}, \Re(\delta) > 0, m \in \mathbb{N}_0, \lambda \geq 0 \) and \( -1 \leq B < A \leq 1 \). If \( f(z) \in S_\theta^{m,\delta}(\lambda, \alpha, \beta, A, B) \), then
\[
\frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{\delta}{\lambda n \theta} - 1} du < \Re \left\{ \frac{D^m_\lambda(\alpha, \beta)f(z)}{z} \right\} < \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du \quad (z \in \mathbb{U}).
\]

Proof. Suppose that \( f(z) \in S_\theta^{m,\delta}(\lambda, \alpha, \beta, A, B) \), then from Theorem 1 we know that
\[
\left( \frac{D^m_\lambda(\alpha, \beta)f(z)}{z} \right)^\delta < \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du.
\]

Thus, from the definition of the subordination, we get
\[
\Re \left\{ \frac{D^m_\lambda(\alpha, \beta)f(z)}{z} \right\} > \inf_{z \in \mathbb{U}} \Re \left\{ \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du \right\}
\]
\[
> \frac{\delta}{\lambda n \theta} \int_0^1 \inf_{z \in \mathbb{U}} \left\{ \frac{1 + A zu}{1 + B zu} u^{\frac{\delta}{\lambda n \theta} - 1} du \right\}
\]
\[
> \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{\delta}{\lambda n \theta} - 1} du.
\]
and

\[
\Re \left\{ \left( \frac{D^n_{\lambda} (\alpha, \beta) f(z)}{z} \right)^{\delta} \right\} \leq \sup_{z \in U} \Re \left\{ \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{1}{1+n \theta} - 1} du \right\}
\]

\[
\leq \frac{\delta}{\lambda n \theta} \int_0^1 \sup_{z \in U} \left( \frac{1 + Au}{1 + Bu} \right)^{1/n \theta} du
\]

\[
< \frac{\delta}{\lambda n \theta} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{1}{1+n \theta} - 1} du,
\]

which proves the result. □

**Acknowledgement**: The work here is supported by Universiti Kebangsaan Malaysia grant: GUP-2017-064.

**REFERENCES**


**Universiti Kebangsaan Malaysia**

**School of Mathematical Sciences**

**Faculty of Science and Technology, 43600, Bangi, Malaysia**

E-mail address: [suhila.e@yahoo.com](mailto:suhila.e@yahoo.com)

**Universiti Kebangsaan Malaysia**

**School of Mathematical Sciences**

**Faculty of Science and Technology, 43600, Bangi, Malaysia**

E-mail address: [maslina@ukm.edu.my](mailto:maslina@ukm.edu.my)