TJMM
9 (2017), No. 2, 155-164

COEFFICIENT PROPERTIES INVOLVING THE GENERALIZED $\mathcal{K}-$MITTAG-LEFFLER FUNCTIONS

HAMEED UR REHMAN, MASLINA DARUS, AND JAMAL SALAH

Abstract. In this article we investigate the Fekete-Szegö problem for the integral operator associated with the most generalized $\mathcal{K}-$ Mittag-Leffler function. Our results will focus on some of the subclasses of starlike and convex functions.

1. Introduction

Let $\mathcal{N}$ be the family of all normalized analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} t_n z^n \tag{1}$$

which are analytic in the unit disk

$$\mathbb{T} = \{z \in \mathbb{C} : |z| < 1\}.$$

Suppose $\mathbb{S}$ be the subclass of $\mathcal{N}$ consisting of functions that are univalent in $\mathbb{T}$. A classical theorem of Fekete-Szegö [1] states that for $f \in \mathcal{N}$ given by (1),

$$|t_3 - \mu t_2^2| \leq 1 + 2exp\left(\frac{-2\mu}{1-\mu}\right), \tag{2}$$

for $0 \leq \mu < 1$ and that this inequality is sharp.

Definition 1. *For a function $\{f, g\} \in \mathcal{N}$ given by (1) and $g(z) = z + \sum_{n=2}^{\infty} d_n z^n$, we define the Hadamard product (or convolution) of f and g by*

$$(f * g)(z) = z + \sum_{n=2}^{\infty} t_n\, d_n\, z^n, \quad (t_n \geq 0, z \in \mathbb{T}). \tag{3}$$

The one-parameter Mittag-Leffler function $E_\alpha(z) : (z) \in \mathbb{C}$ see [6] and [7],

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha+1)} =: E_{\alpha,1}(z), \tag{4}$$

and its two-parameters extension $E_{\alpha,\beta}(z)$ was studied by Wiman [8],

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \qquad (\alpha \geq 0), \tag{5}$$

where $\alpha, \beta \in \mathbb{C}, \mathcal{R}(\alpha) > 0$ and $\mathcal{R}(\beta) > 0$.

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Mittag-Leffler function, Fekete-Szegö problem, Hadamard product, integral operator, starlike and convex functions.

Consequently many researchers have worked on the generalization of Mittag-Leffer function see for example [2], [9], [10],[13] and [11]. Fekete-Szegö problem has been addressed by many authors with amalgamation of certain integral or differential operator, see for example [4], [5], [17] and [14]. However to achieve our results we use the Salim [11] type of generalized $\mathcal{K}-$ Mittag-Leffler function which is of the form

$$E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma k(\alpha n+\beta)(\delta)_n} z^n, \tag{6}$$

the imposition on parameters are $\alpha, \beta, \gamma \in \mathbb{C}, \mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$, $k \in \mathbb{R}$, δ is non-negative real number, nq is a positive integer and $(\gamma)_n$ is the Pochhammar symbol:

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1, & n=0 \\ \gamma(\gamma+1)...(\gamma+n-1). \end{cases}$$

Note that

$$(x)_n = x(x+1)_{n-1}, \qquad n \in \mathbb{N},$$

where $(\gamma)_{nq,k}$ is the $k-$Pochhammer symbol defined as:

$$(\gamma)_{nq,k} = \gamma(\gamma+k)(\gamma+2k)...(\gamma+(nq-1)k) \qquad (\gamma \in \mathbb{C}, k \in \mathcal{R}, n \in \mathbb{N}). \tag{7}$$

Since the Mittag-Leffler function in (5) does not belong to the class $\mathcal{N}$ therefore, Srivastava and his co-authors [3] have considered some normalization on $E_{\alpha,\beta}$ and made it an analytic function of class $\mathcal{N}$. Similarly we consider some normalization on the most generalized Mittag-Leffler function $E^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ defined in (6)

$$\begin{aligned} Q^{\gamma,q,\delta}_{k,\alpha,\beta}(z) &= \frac{\Gamma k(\beta)}{(\gamma)_k} z \left(E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) \right), \qquad (z \in \mathbb{T}) \\ &= z + \sum_{n=1}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} z^{n+1}. \end{aligned} \tag{8}$$

$$Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{(n-1)q,k}\Gamma k(\alpha+\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha(n-1)+\beta)\Gamma(\delta+n-1)} z^n. \tag{9}$$

Let $f(z) \in \mathcal{N}$. Denote $L^{\gamma,q}_{k,\alpha\beta,\delta}(f)(z) : \mathcal{N} \longrightarrow \mathcal{N}$ the operator is defined by

$$L^{\gamma,q}_{k,\alpha,\beta,\delta}(f)(z) = Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z) * f(z),$$

and now by convolution or Hadamard product (*) the operator $L^{\gamma,q}_{k,\alpha,\beta,\delta}(f)(z)$ becomes

$$L^{\gamma,q}_{k,\alpha,\beta,\delta}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\alpha+\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} t_n z^n. \tag{10}$$

where $\alpha, \beta, \gamma \in \mathbb{C}, \mathcal{R}(\alpha) > 0, \mathcal{R}(\beta) > 0, k \in \mathbb{R}, \delta$ is non-negative real number, nq is a positive integer and $q \in (0,1) \cup \mathbb{N}$.

Some properties and relation of this integral operator are given in the next Lemma 1.

2. Preliminary Results

Lemma 1. *Let $f(z) \in \mathcal{N}$. Then*

i) $L^{1,1}_{1,0,1,1}(f)(z) = z + \sum_{n=2}^{\infty} t_n z^n = f(z)$

ii) $L^{1,1}_{1,0,1,0}(f)(z) = z + \sum_{n=2}^{\infty} n t_n z^n = zf'(z)$

iii) $L^{2,1}_{1,0,1,1}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right) t_n z^n = \frac{1}{2}\left[f(z) + zf'(z)\right]$

iv) $L^{1,1}_{1,0,1,2}(f)(z) = \frac{2}{z}\int_0^z f(t)dt = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right) t_n z^n = \left(-\frac{2log(1-z)}{z} - 2\right)$

v) $L^{2,1}_{1,0,1,2}(f)(z) = z + \sum_{n=2}^{\infty}(n+1)t_n z^n = f(z) + zf'(z).$

Remark 1. *Note that in above Lemma part (iv) is the type of Bernardi Integral* [15] *and is the special case of studied by Libera* [18] *and Livingston* [19].

Definition 2. *Let* $f(z) \in \mathcal{N}$. *Then* $f(z) \in \mathbf{S}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ *if and only if*

$$\mathcal{R}\left\{\frac{z\left[L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right]'}{L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)}\right\} > \lambda, \qquad 0 \le \lambda < 1, \quad z \in \mathbb{T}.$$

Definition 3. *Let* $f(z) \in \mathcal{N}$. *Then* $f(z) \in \mathbf{C}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ *if and only if*

$$\mathcal{R}\left\{\frac{\left[z(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z))'\right]'}{(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z))'}\right\} > \lambda, \qquad 0 \le \lambda < 1, \quad z \in \mathbb{T}.$$

Now we discuss the general properties and distortion theorems for the function $f(z) \in \mathcal{N}$ belonging to the classes $\mathbf{S}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ and $\mathbf{C}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ by obtaining the coefficient bounds. In context to obtain sharp upper bounds of $|t_2|$ and of the Fekete-Szegö functional $|t_3 - \mu t_2^2|$ for the classes $\mathbf{S}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ and $\mathbf{C}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$, we need to state the following result due to Duren [16].

Lemma 2. *Let* $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \in \mathcal{P}$, *such that* h *be analytic in* $\mathbb{T}$, *and* $\mathcal{P}$ *be the class of all analytic functions, and* $\mathcal{R}\{h(z)\} > 0$ *for* $z \in \mathbb{T}$. *Then*

(i) $|h_2 - \frac{h_1^2}{2}| \le 2 - \frac{|h_1^2|}{2}$,

(ii) $|h_n| \le 2$ *for all* $n \in \mathbb{N}$.

Theorem 1. *Let* $f(z) \in \mathcal{N}$. *If for* $\{k, \beta, \gamma\} \ge 1$ *and* $\{\alpha, q, \delta\} \ge 0$

$$\sum_{n=2}^{\infty}(n-\lambda)|t_n|\{\Psi^{\gamma,q}_{k,\alpha,\beta,\delta}\} \le 1 - \lambda, \qquad 0 \le \lambda < 1, \tag{11}$$

then $f(z) \in \mathbf{S}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$. *The result (11) is sharp, where*

$$\Psi^{\gamma,q}_{k,\alpha,\beta,\delta} = \frac{(\gamma)_{nq,k}\Gamma k(\alpha+\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)}$$

Proof. Let us say that (11) holds true. Since

$$\begin{aligned} 1 - \lambda &\ge \sum_{n=2}^{\infty}(n-\lambda)|t_n|\{\Psi^{\gamma,q}_{k,\alpha,\beta,\delta}\} \\ &\ge \sum_{n=2}^{\infty}\lambda|t_n|\{\Psi^{\gamma,q}_{k,\alpha,\beta,\delta}\} - \sum_{n=2}^{\infty} n|t_n|\{\Psi^{\gamma,q}_{k,\alpha,\beta,\delta}\} \end{aligned}$$

and this means that

$$\frac{1+\sum_{n=2}^{\infty} n|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}}{1+\sum_{n=2}^{\infty} |t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}} > \lambda,$$

hence

$$\mathcal{R}\left\{\frac{z\left(L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z)\right)'}{L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z)}\right\} > \lambda.$$

We also note that the assertion (11) is sharp and the extremal function is given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\lambda)}{(n-\lambda)}\left\{\frac{1}{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}}\right\} z^n.$$

□

Corollary 1. *Let the assumption of Theorem 1 is true. Then for* $0 \leq \lambda < 1$

$$|t_n| \leq \frac{1-\lambda}{n-\lambda}\left\{\frac{1}{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}}\right\}, \quad \forall n \geq 2. \tag{12}$$

Corollary 2. *Let the assumption of Theorem 1 is true, and if we put* $\alpha = \lambda = 0$ *and* $\gamma = q = k = \beta = \delta = 1$ *then we obtain*

$$|t_n| \leq \frac{1}{n}, \qquad \forall n \geq 2. \tag{13}$$

In a similar method we can verify our results for convex function $\mathbf{C}_{k,\alpha,\beta,\delta}^{\gamma,q}(\lambda)$ that:

Theorem 2. *Let* $f(z) \in \mathcal{N}$. *If for* $\{k, \beta, \gamma\} \geq 1$ *and* $\{\alpha, q, \delta\} \geq 0$

$$\sum_{n=2}^{\infty} n(n-\lambda)|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} \leq 1-\lambda, \qquad 0 \leq \lambda < 1, \tag{14}$$

then $f(z) \in \mathbf{C}_{k,\alpha,\beta,\delta}^{\gamma,q}(\lambda)$. *The result (14) is sharp, where*

$$\Psi_{k,\alpha,\beta,\delta}^{\gamma,q} = \frac{(\gamma)_{nq,k}\Gamma k(\alpha+\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)}.$$

Corollary 3. *Let the assumption of Theorem 2 is true. Then for* $0 \leq \lambda < 1$

$$|t_n| \leq \frac{1-\lambda}{n(n-\lambda)}\left\{\frac{1}{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}}\right\}, \quad \forall n \geq 2. \tag{15}$$

Theorem 3. *Let the hypotheses of Theorem 1 holds. Then for* $z \in \mathbb{T}$ *and* $0 \leq \lambda < 1$

$$\left|L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z)\right| \geq |z| - \frac{1-\lambda}{2-\lambda}|z|^2$$

and

$$\left|L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z)\right| \leq |z| + \frac{1-\lambda}{2-\lambda}|z|^2.$$

Proof. One can easily understand from Theorem 1 that

$$\begin{aligned}&(2-\lambda)\sum_{n=2}^{\infty}|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}\\&\leq\sum_{n=2}^{\infty}(n-\lambda)|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}\\&\leq 1-\lambda\end{aligned}$$

so,

$$\sum_{n=2}^{\infty}|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}\leq\frac{1-\lambda}{2-\lambda}.$$

Thus we have

$$\begin{aligned}\left|L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z)\right|&\leq|z|+\sum_{n=2}^{\infty}|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}|z|^n\\&\leq|z|+\sum_{n=2}^{\infty}|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}|z|^2\\&\leq|z|+\left(\frac{1-\lambda}{2-\lambda}\right)|z|^2\end{aligned}$$

Now for second assertion we proceed as follows

$$\begin{aligned}\left|L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z)\right|&=\left|z+\sum_{n=2}^{\infty}t_n\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}z^n\right|\\&\geq|z|-\sum_{n=2}^{\infty}|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}|z|^n\\&\geq|z|-\sum_{n=2}^{\infty}(n-\lambda)\left(\frac{1}{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}}\right)|t_n|z^2\\&\geq|z|-\left(\frac{1-\lambda}{2-\lambda}\right)|z|^2,\end{aligned}$$

and hence both parts of our proof are completed. □

Theorem 4. *Let the hypotheses of Theorem 2 be satisfied. Then for $z\in\mathbb{T}$ and $0\leq\lambda<1$*

$$\left|L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z)\right|\geq|z|-\frac{1-\lambda}{2(2-\lambda)}|z|^2$$

and

$$\left|L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z)\right|\leq|z|+\frac{1-\lambda}{2(2-\lambda)}|z|^2.$$

Theorem 5. *Let the hypotheses of Theorem 1 be satisfied. Then for all $\{n\geq 2:n\in\mathbb{N}\}$, $0\leq\lambda<1$, and $(n-\lambda)\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}\geq 1$, implies*

$$|f(z)|\geq|z|-(1-\lambda)|z|^2$$

and

$$|f(z)|\leq|z|+(1-\lambda)|z|^2.$$

Proof. By using Theorem 1, we get

$$\sum_{n=2}^{\infty}|t_n| \leq \sum_{n=2}^{\infty}(n-\lambda)\{\Psi^{\gamma,q}_{k,\alpha,\beta,\delta}\}|t_n| \leq (1-\lambda)$$

then

$$\sum_{n=2}^{\infty}|t_n| \leq (1-\lambda).$$

So, we obtain

$$\begin{aligned}|f(z)| &= \left|z+\sum_{n=2}^{\infty}t_n z^n\right| \\ &\leq |z| + \sum_{n=2}^{\infty}|t_n||z|^2 \\ &\leq |z| + (1-\lambda)|z|^2\end{aligned}$$

To prove the second assertion we adopt the following steps

$$\begin{aligned}|f(z)| &\geq \left|z-\sum_{n=2}^{\infty}t_n z^n\right| \\ &\geq |z| - \sum_{n=2}^{\infty}|t_n||z|^2 \\ &\geq |z| - (1-\lambda)|z|^2\end{aligned}$$

and hence the proof. □

Theorem 6. *Let the hypotheses of Theorem 2 be satisfied. Then for all* $\{n \geq 2 : n \in \mathbb{N}\}$, $0 \leq \lambda < 1$, *and* $n(n-\lambda)\{\Psi^{\gamma,q}_{k,\alpha,\beta,\delta}\} \geq 1$, *poses*

$$|f(z)| \geq |z| - \frac{1-\lambda}{2}|z|^2$$

and

$$|f(z)| \leq |z| + \frac{1-\lambda}{2}|z|^2.$$

Definition 4. *Let* $\phi(z)$ *be univalent starlike function with respect to* 1 *which maps the unit disk* $\mathbb{T}$ *onto a region in the right half plane which is symmetric about the real axis,* $\phi(0) = 1$ *and* $\phi'(0) > 0$. *A function* $f \in \mathcal{N}$ *is in the class* $L^{\gamma,q}_{k,\alpha,\beta,\delta}(\phi)$ *if*

$$\frac{z\left(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right)'}{L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)} \prec \phi(z). \tag{16}$$

Remark 2. *It can be seen if we put* $\gamma = q = k = \beta = \delta = 1$ *and* $\alpha = 0$ *then the class* $L^{\gamma,q}_{k,\alpha,\beta,\delta}\phi(z)$ *becomes the class of starlike function* $S^*(\phi)$.

Lemma 3. *(see* [12]*) If* $h_1 = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots$ *is an analytic function with positive real part in the unit disk* $\mathbb{T}$, *then*

$$|c_2 - vc_1^2| \leq 2\,max\,\{1, |2v-1|\} \tag{17}$$

and the result is sharp for the functions given by

$$h(z) = \frac{1+z}{1-z}, \quad h(z) = \frac{1+z^2}{1-z^2}. \tag{18}$$

3. Fekete-Szegö for the Classes $\mathbf{S}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ and $\mathbf{C}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$

Now we determine the sharp upper bound for $|t_2|$ for the class $\mathbf{S}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ and $\mathbf{C}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$, and calculate the Fekete-Szegö function of $|t_3 - \mu t_2^2|$ for these classes.

Theorem 7. *Let the hypotheses of Theorem 1 be satisfied. Then*

$$|t_2| \leq \frac{2(1-\lambda)(\gamma)_{q,k}\Gamma k(2\alpha+\beta)(\delta+1)}{(1+\lambda)(\gamma)_{2q,k}\Gamma k(\alpha+\beta)}, \qquad 0 \leq \lambda < 1,$$

and for all $\mu \in \mathbb{C}$ the following bound is sharp

$$\left|t_3 - \mu t_2^2\right| \leq \frac{1-\lambda}{(2+\lambda)B}\, max\left\{1, \left|B_2 + \frac{1-\lambda}{1+\lambda}B_1^2 - \mu\frac{(1-\lambda)(2+\lambda)B}{(1+\lambda)^2A^2}B_1^2\right|\right\},$$

where values of A and B are given by equation (21), $\{k,\beta,\gamma\} \geq 1$ and $\{\alpha,q,\delta\} \geq 0$.

Proof. Since $f \in \mathbf{S}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ then the condition

$$\mathcal{R}\left\{\frac{z\left(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right)'}{L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)}\right\} > \lambda, \qquad 0 \leq \lambda < 1,$$

is equivalent to

$$z\left[L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right]' = (1-\lambda)h(z)L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z), \qquad z \in \mathbb{T},$$

for some $h \in \mathcal{P}$. Equating the coefficients we have the values of

$$t_2 = \frac{(1-\lambda)h_1}{(1+\lambda)A} \tag{19}$$

and

$$t_3 = \frac{1-\lambda}{(2+\lambda)B}\left(\frac{1-\lambda}{1+\lambda}h_1^2 + h_2\right), \tag{20}$$

and hence by using Lemma 2 and equation (19) we achieve the required result of $|t_2|$. where

$$A = \frac{(\gamma)_{2q,k}\Gamma k(\alpha+\beta)}{(\gamma)_{q,k}\Gamma k(2\alpha+\beta)(\delta+1)} \quad \text{and} \quad B = \frac{(\gamma)_{3q,k}\Gamma k(\alpha+\beta)}{(\gamma)_{q,k}\Gamma k(3\alpha+\beta)(\delta+1)(\delta+2)}. \tag{21}$$

For the Fekete-Szegö function $\left|t_3 - \mu t_2^2\right|$, consider $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ if $f(z)$ is given by (1) belongs to $L^{\gamma,q}_{k,\alpha,\beta,\delta}(\phi)$, then

$$h(z) = \phi\left(\frac{h_1(z)-1}{h_1(z)+1}\right). \tag{22}$$

Since $\phi(z)$ is univalent and $h(z) \prec \phi$, then the function below, is analytic and has a positive real part in $\mathbb{T}$.

$$h_1(z) = \frac{1+\phi^{-1}(h(z))}{1-\phi^{-1}(h(z))} = 1 + c_1 z + c_2 z^2 + ... \tag{23}$$

and then from (22) and (23) we obtain the values of $h_1(z)$ and $h_2(z)$

$$h_1 = \frac{1}{2}B_1 c_1$$

and

$$h_2 = \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2,$$

so, by using Lemma 2 we find

$$\begin{aligned}\left|t_3-\mu t_2^2\right| &\leq \frac{2(1-\lambda)}{(2+\lambda)B}+\left\{\frac{(1-\lambda)^2}{(1+\lambda)(2+\lambda)}-\mu\frac{(1-\lambda)^2}{(1+\lambda)^2A^2}\right\}|h_1|^2\\ &\leq \mathcal{H}(x)=\frac{2(1-\lambda)}{(2+\lambda)B}+\left\{K-\frac{(1-\lambda)}{2(2+\lambda)B}\right\}x^2, \qquad x:=|h_1|^2. \end{aligned} \tag{24}$$

As a result we obtain

$$\left|t_3-\mu t_2^2\right|\leq\begin{cases}\mathcal{H}(0)= \dfrac{2(1-\lambda)}{(2+\lambda)B} & if\, K \leq\dfrac{(1-\lambda)}{2(2+\lambda)B}\\ \mathcal{H}(2)= 4K & if\, K >\dfrac{(1-\lambda)}{2(2+\lambda)B},\end{cases}$$

where

$$K:=\frac{(1-\lambda)}{2(2+\lambda)B}+\frac{(1-\lambda)^2}{(1+\lambda)(2+\lambda)}-\mu\frac{(1-\lambda)^2}{(1+\lambda)^2A^2}.$$

Equality is attained for the functions given by

$$\frac{z\left[L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right]'}{L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)}=\frac{1+z(1-2\lambda)}{1-z},$$

$$\frac{z\left[L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right]'}{L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)}=\frac{1+z^2(1-2\lambda)}{1-z^2},$$

Also we have

$$t_3-\mu t_2^2=\frac{1-\lambda}{2(2+\lambda)B}\left[c_2-vc_1^2\right], \tag{25}$$

where values of A and B are given by (21), and

$$v=\frac{1}{2}\left(1-B_2-\frac{1-\lambda}{1+\lambda}B_1^2+\mu\frac{(1-\lambda)(2+\lambda)B}{(1+\lambda)^2A^2}B_1^2\right),$$

hence by applying Lemma 3 the result of sharp bound $\left|t_3-\mu t_2^2\right|$ is proved. □

Corollary 4. *Let the assumption of Theorem 7 is true. Then for* $\lambda=0$

$$|t_2|\leq\frac{2(\gamma)_{q,k}\Gamma k(2\alpha+\beta)(\delta+1)}{(\gamma)_{2q,k}\Gamma k(\alpha+\beta)}$$

and

$$\left|t_3-\mu t_2^2\right|\leq\frac{1}{2B}\,max\left\{1,\left|B_2+B_1^2-\mu\frac{2B}{A^2}B_1^2\right|\right\}$$

Now we prove the result for the class of $\mathbf{C}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$.

Theorem 8. *Let the hypotheses of Theorem 2 hold. Then*

$$|t_2|\leq\frac{(1-\lambda)(\gamma)_{q,k}\Gamma k(2\alpha+\beta)(\delta+1)}{(1+\lambda)(\gamma)_{2q,k}\Gamma k(\alpha+\beta)}$$

and for all $\mu\in\mathbb{C}$ *the following bound is sharp*

$$\left|t_3-\mu t_2^2\right|\leq\frac{1-\lambda}{3(2+\lambda)B}max\left\{1,\left|B_2+\frac{1-\lambda}{1+\lambda}B_1^2-\mu\frac{3(1-\lambda)(2+\lambda)B}{2(1+\lambda)^2A^2}B_1^2\right|\right\}$$

Proof. Since $f \in \mathbf{C}^{\gamma,q}_{k,\alpha,\beta,\delta}(\lambda)$ then the condition

$$\mathcal{R}\left\{\frac{\left[z\left(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right)'\right]'}{\left(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right)'}\right\} > \lambda, \qquad 0 \le \lambda < 1,$$

is equivalent to

$$\left(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right)' + z\left[\left(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right)'\right]' = (1-\lambda)h(z)\left(L^{\gamma,q}_{k,\alpha,\beta,\delta}f(z)\right)', \qquad z \in \mathbb{T},$$

for some $h \in \mathcal{P}$. Equating the coefficients we have the values of

$$t_2 = \frac{(1-\lambda)h_1}{2(1+\lambda)A} \tag{26}$$

and

$$t_3 = \frac{1-\lambda}{3(2+\lambda)B}\left(\frac{1-\lambda}{1+\lambda}h_1^2 + h_2\right), \tag{27}$$

and hence by using Lemma 2 and equation (26) we achieve the required result of $|t_2|$. Further we obtain,

$$t_3 - \mu t_2^2 = \frac{1-\lambda}{6(2+\lambda)B}\left[c_2 - vc_1^2\right], \tag{28}$$

where values of A and B are given by (21), and

$$v = \frac{1}{2}\left(1 - B_2 - \frac{1-\lambda}{1+\lambda}B_1^2 + \mu\frac{3(1-\lambda)(2+\lambda)B}{2(1+\lambda)^2A^2}B_1^2\right),$$

hence by applying Lemma 3 the result of sharp bound $\left|t_3 - \mu t_2^2\right|$ is proved. □

Corollary 5. *Let the assumption of Theorem 8 is true. Then for $\lambda = 0$*

$$|t_2| \le \frac{(\gamma)_{q,k}\Gamma k(2\alpha+\beta)(\delta+1)}{(\gamma)_{2q,k}\Gamma k(\alpha+\beta)}$$

and

$$\left|t_3 - \mu t_2^2\right| \le \frac{1}{6B}max\left\{1, \left|B_2 + B_1^2 - \mu\frac{3B}{A^2}B_1^2\right|\right\}.$$

Acknowledgment

The work here is supported by UKM grant: GUP-2017-064.

Conflict of interest: The authors declare that there is no conflict of interests. All the authors agreed with the contents of the manuscript.

References

[1] Fekete, M. and Szegö, G., *Eine Bermerkung uber ungerade schlichet functionen*, J. Lond. Math. Soc., **8** (1933), 85–89.

[2] Nisar, K.S., Purohit, S.D., Abouzaid, M.S., Qurashi, M. and Baleanu, D., *Generalized k-Mittag-Leffler function and its composition with pathway integral operators*, J. Nonlinear Sci. Appl **9** (6), 3519–3526, 2016.

[3] Srivastava, H.M., Frasin, B.A. and Pescar, V., *Univalence of Integral Operators Involving Mittag-Leffler functions*, Appl. Math. Inf. Sci. **11** (3), 635–641, 2017.

[4] Darus, M. and Ibrahim, R.W., *On classes of analytic functions containing generalization of integral operator*, Journal of the Indonesian Mathematical Society **17** (1), 29–38, 2012.

[5] Ramachandran, C., Kavitha, D. and Annamalai, S., *Fekete-Szegö Coefficient for the Komatu Integral Operator in Unit Disk*, International Journal of Mathematical Analysis **9** (19), 935–945, 2015.

[6] Mittag-Leffler, GM, *Sur la nouvelle fonction $E_\alpha(x)$*, CR Acad. Sci. Paris **137** (2), 554–558, 1903.

[7] Mittag-Leffler, Gosta, *Sur la representation analytique d'une branche uniforme d'une fonction monogene*, Acta Mathematica **29** (1), 101–181, 1905.

[8] Wiman, Adders, *Über den Fundamentalsatz in der Teorie der FunktionenE a (x)*, Acta Mathematica **29** (1), 191–201, 1905.

[9] Salah, J. and Darus, M., *A note on Generalized Mittag-Leffler function and Application*, Far East Journal of Mathematical Sciences(FJMS) **48** (1), 33–46, 2011.

[10] Shukla, A.K. and Prajapti, J.C., *On a generalization of Mittag-Leffler function and its properties*, Journal of Mathematical Analysis and Applications **336** (2), 797–811, 2007.

[11] Salim, T.O., *Some properties relating to the generalized Mittag-Leffler function*, Adv. Appl. Math. Anal **4** (1), 21–30, 2009.

[12] Thomas, D.K. and others, *On the Fekete-Szegö theorem for close-to-convex functions*, Mathematica japonicae **47** (1), 125–132, 1998.

[13] Gehlot, Kuldeep Singh, *The generalized $k-$Mittag-Leffler function*, Int. J. Contemp. Math. Sciences **7** (45), 2213–2219, 2012.

[14] Salah, J., *Fekete-Szegö Problems Involving Certain Integral Operator*, International Journal of Mathematical Trends and Technology (IJMTT) **7** (1), 54–60, 2014.

[15] Bernardi. S.D., *Convex and starlike univalent functions*, Transactions of the American Mathematical Society **135**, 429–446,1969.

[16] Duren, P.L., *Univalent functions*, Springer-Verlag, New York, 1983.

[17] Al-Shaqsi, K. and Darus, M., *On Fekete-Szegö problems for certain subclass of analytic functions*, Applied Mathematical Sciences **2** (9), 431–441, 2008.

[18] Libera. R.J., *Some classes of regular univalent functions*, Proceedings of the American Mathematical Society **16** (4)(1965), 755–758.

[19] Livingston, A.E., *On the radius of univalence of certain analytic functions*, Proceedings of the American Mathematical Society, **17** (2) (1966), 352–357.

University of Kebangsaan Malaysia
School of Mathematical Sciences, Faculty of Science and Technology
Bangi 43600, Selangor Darul Ihsan, Malaysia
E-mail address: hameedktk09@gmail.com

University of Kebangsaan Malaysia
School of Mathematical Sciences, Faculty of Science and Technology
Bangi 43600, Selangor Darul Ihsan, Malaysia
E-mail address: maslina@ukm.edu.my

A' Sharqiyah University
College of Applied Sciences
Post Box No. 42, Post Code No. 400 Ibra, Sultanate of Oman
E-mail address: drjamals@asu.edu.om