COEFFICIENT PROPERTIES INVOLVING THE GENERALIZED
$K$–MITTAG-LEFFLER FUNCTIONS

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Abstract. In this article we investigate the Fekete-Szeg"{o} problem for the integral
operator associated with the most generalized $K$– Mittag-Leffler function. Our results
will focus on some of the subclasses of starlike and convex functions.

1. Introduction

Let $N$ be the family of all normalized analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} t_n z^n$$

which are analytic in the unit disk

$$T = \{ z \in \mathbb{C} : |z| < 1 \}.$$ 

Suppose $S$ be the subclass of $N$ consisting of functions that are univalent in $T$. A classical
theorem of Fekete-Szeg"{o} \cite{1} states that for $f \in N$ given by (1),

$$|t_3 - \mu t_2^2| \leq 1 + 2 \exp \left( \frac{-2\mu}{1 - \mu} \right),$$

for $0 \leq \mu < 1$ and that this inequality is sharp.

Definition 1. For a function $\{f, g\} \in N$ given by (1) and $g(z) = z + \sum_{n=2}^{\infty} d_n z^n$, we define
the Hadamard product (or convolution) of $f$ and $g$ by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} t_n d_n z^n, \quad (t_n \geq 0, z \in T).$$

The one-parameter Mittag-Leffler function $E_{\alpha}(z) : (z) \in \mathbb{C}$ see \cite{6} and \cite{7},

$$E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha + 1)} =: E_{\alpha, 1}(z),$$

and its two-parameters extension $E_{\alpha, \beta}(z)$ was studied by Wiman \cite{8},

$$E_{\alpha, \beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\alpha \geq 0),$$

where $\alpha, \beta \in \mathbb{C}, \mathcal{R}(\alpha) > 0$ and $\mathcal{R}(\beta) > 0$.
Lemma 1. Let $q$ be a positive integer and $(\gamma)_n$ be an analytic function of class $\mathcal{S}$, Srivastava and his co-authors [3] have considered some normalization on $E_{nq}$ type of generalized $K$–Mittag-Leffler function which is of the form

$$E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma(k(\alpha n + \beta))(\delta)_n} z^n,$$

the imposition on parameters are $\alpha, \beta, \gamma \in \mathbb{C}, R(\alpha) > 0, R(\beta) > 0$, $k \in \mathbb{R}, \delta$ is non-negative real number, $nq$ is a positive integer and $(\gamma)_n$ is the Pochhammer symbol:

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}, \quad n = 0, 1, \ldots, (\gamma)_n \in \mathbb{N}.$$ 

Note that

$$(x)_n = x(x+1)\ldots(x+n-1), \quad n \in \mathbb{N},$$

where $(\gamma)_{nq,k}$ is the $k$–Pochhammer symbol defined as:

$$(\gamma)_{nq,k} = (\gamma + k)(\gamma + k + 2q)\ldots(\gamma + (nq - 1)k) \quad (\gamma \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N}). \quad (7)$$

Since the Mittag-Leffler function in (5) does not belong to the class $\mathcal{N}$ therefore, Srivastava and his co-authors [3] have considered some normalization on $E_{nq}$ and made it an analytic function of class $\mathcal{N}$. Similarly we consider some normalization on the most generalized Mittag-Leffler function $E^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ defined in [6]

$$Q^{\gamma,q,\delta}_{k,\alpha,\beta}(z) = \frac{k^\delta}{(\gamma)_k} z \left( E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) \right), \quad (z \in \mathbb{T})$$

$$= z + \sum_{n=1}^{\infty} \frac{(\gamma)_{nq,k} k^\delta \Gamma(\delta + 1)}{(\gamma)_n k \Gamma(\alpha n + \beta) \Gamma(\delta + n)} z^n + 1. \quad (8)$$

$$Q^{\gamma,q,\delta}_{k,\alpha,\beta,\delta}(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k} k^\delta \Gamma(\delta + 1)}{(\gamma)_{nq,k} \Gamma(\alpha(n - 1) + \beta) \Gamma(\delta + n - 1)} z^n. \quad (9)$$

Let $f(z) \in \mathcal{N}$. Denote $L^{\gamma,q,\delta}_{k,\alpha,\beta,\delta}(f)(z) : \mathcal{N} \rightarrow \mathcal{N}$ the operator is defined by

$$L^{\gamma,q,\delta}_{k,\alpha,\beta,\delta}(f)(z) = Q^{\gamma,q,\delta}_{k,\alpha,\beta}(z) * f(z),$$

and now by convolution or Hadamard product $(\ast)$ the operator $L^{\gamma,q,\delta}_{k,\alpha,\beta,\delta}(f)(z)$ becomes

$$L^{\gamma,q,\delta}_{k,\alpha,\beta,\delta}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k} k^\delta \Gamma(\delta + 1)}{(\gamma)_{nq,k} \Gamma(\alpha(n - 1) + \beta) \Gamma(\delta + n - 1)} t_n z^n. \quad (10)$$

where $\alpha, \beta, \gamma \in \mathbb{C}, R(\alpha) > 0, R(\beta) > 0, k \in \mathbb{R}, \delta$ is non-negative real number, $nq$ is a positive integer and $q \in (0,1) \cup \mathbb{N}$.

Some properties and relation of this integral operator are given in the next Lemma [7]

2. Preliminary Results

Lemma 1. Let $f(z) \in \mathcal{N}$. Then

i) $L^{1,1}_{1,0,1,1}(f)(z) = z + \sum_{n=2}^{\infty} t_n z^n = f(z)$

ii) $L^{1,1}_{1,0,0,0}(f)(z) = z + \sum_{n=2}^{\infty} nt_n z^n = zf'(z)$
iii) \( L_{1,0,1,1}^2(f)(z) = \frac{z}{2} \sum_{n=2}^{\infty} \left( \frac{n+1}{2} \right) t_n z^n = \frac{1}{2} [f(z) + zf'(z)] \)

iv) \( L_{1,0,1,2}^2(f)(z) = \frac{2}{z} \int_0^z f(t) dt = z + \sum_{n=2}^{\infty} \left( \frac{2}{n+1} \right) t_n z^n = \left( -\frac{2\log(1-z)}{z} - 2 \right) \)

v) \( L_{1,0,1,2}^2(f)(z) = z + \sum_{n=2}^{\infty} (n+1)t_n z^n = f(z) + zf'(z). \)

Remark 1. Note that in above Lemma part (iv) is the type of Bernardi Integral [15] and is the special case of studied by Libera [18] and Livingston [19].

Definition 2. Let \( f(z) \in \mathcal{N}. \) Then \( f(z) \in S_{k,\alpha,\beta,\delta}^q(\lambda) \) if and only if

\[
\Re \left\{ \frac{z}{2} \left[ L_{k,\alpha,\beta,\delta}^{q, q,f}(z) \right] \right\} > \lambda, \quad 0 \leq \lambda < 1, \quad z \in \mathbb{T}.
\]

Definition 3. Let \( f(z) \in \mathcal{N}. \) Then \( f(z) \in C_{k,\alpha,\beta,\delta}^q(\lambda) \) if and only if

\[
\Re \left\{ \frac{z}{2} \left[ L_{k,\alpha,\beta,\delta}^{q, q,f}(z) \right] \right\} > \lambda, \quad 0 \leq \lambda < 1, \quad z \in \mathbb{T}.
\]

Now we discuss the general properties and distortion theorems for the function \( f(z) \in \mathcal{N} \) belonging to the classes \( S_{k,\alpha,\beta,\delta}^q(\lambda) \) and \( C_{k,\alpha,\beta,\delta}^q(\lambda) \) by obtaining the coefficient bounds. In context to obtain sharp upper bounds of \( |t_2| \) and of the Fekete-Szegö functional \( |t_3 - \mu t_2^2| \) for the classes \( S_{k,\alpha,\beta,\delta}^q(\lambda) \) and \( C_{k,\alpha,\beta,\delta}^q(\lambda) \), we need to state the following result due to Duren [16].

Lemma 2. Let \( h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \in \mathcal{P}, \) such that \( h \) be analytic in \( \mathbb{T}, \) and \( \mathcal{P} \) be the class of all analytic functions, and \( \Re \{h(z)\} > 0 \) for \( z \in \mathbb{T}. \) Then

(i) \( |h_2 - \frac{h_1^2}{2}| \leq 2 - |\frac{h_1^2}{2}|, \)

(ii) \( |h_n| \leq \frac{1}{2} \) for all \( n \in \mathbb{N}. \)

Theorem 1. Let \( f(z) \in \mathcal{N}. \) If for \( k, \beta, \gamma \geq 1 \) and \( \{\alpha, q, \delta\} \geq 0 \)

\[
\sum_{n=2}^{\infty} (n-\lambda)|t_n| \left| \Psi_{k,\alpha,\beta,\delta}^{q, q} \right| \leq 1 - \lambda, \quad 0 \leq \lambda < 1, \quad (11)
\]

then \( f(z) \in S_{k,\alpha,\beta,\delta}^q(\lambda). \) The result \( (11) \) is sharp, where

\[
\Psi_{k,\alpha,\beta,\delta}^{q, q} = \frac{(\gamma)_{n,q,k}\Gamma(\alpha + \beta)\Gamma(\delta + 1)}{(\gamma)_{q,k}\Gamma(\beta)\Gamma(\delta + n)}
\]

Proof. Let us say that \( (11) \) holds true. Since

\[
1 - \lambda \geq \sum_{n=2}^{\infty} (n-\lambda)|t_n| \left| \Psi_{k,\alpha,\beta,\delta}^{q, q} \right|
\]

\[
\geq \sum_{n=2}^{\infty} \lambda|t_n| \left| \Psi_{k,\alpha,\beta,\delta}^{q, q} \right| - \sum_{n=2}^{\infty} n|t_n| \left| \Psi_{k,\alpha,\beta,\delta}^{q, q} \right|
\]
and this means that
\[
1 + \sum_{n=2}^{\infty} n|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} > \lambda,
\]
\[
1 + \sum_{n=2}^{\infty} |t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\}
\]
hence
\[
\mathcal{R}\left\{\frac{z \left( L_{k,\alpha,\beta,\delta,\lambda}^{\gamma,q} f(z) \right)'}{L_{k,\alpha,\beta,\delta,\lambda}^{\gamma,q} f(z)} \right\} > \lambda.
\]
We also note that the assertion (11) is sharp and the extremal function is given by
\[
f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\lambda)}{(n-\lambda)} \left\{\frac{1}{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}} \right\} z^n.
\]

**Corollary 1.** Let the assumption of Theorem 1 is true. Then for \(0 \leq \lambda < 1\)
\[
|t_n| \leq \frac{1-\lambda}{n-\lambda} \left\{\frac{1}{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}} \right\}, \quad \forall n \geq 2. \tag{12}
\]

**Corollary 2.** Let the assumption of Theorem 1 is true, and if we put \(\alpha = \lambda = 0\) and \(\gamma = q = k = \beta = \delta = 1\) then we obtain
\[
|t_n| \leq \frac{1}{n}, \quad \forall n \geq 2. \tag{13}
\]

In a similar method we can verify our results for convex function \(C_{k,\alpha,\beta,\gamma,\delta}^{\gamma,q}(\lambda)\) that:

**Theorem 2.** Let \(f(z) \in \mathcal{N}\). If for \(\{k,\beta,\gamma\} \geq 1\) and \(\{\alpha,\gamma,\delta\} \geq 0\)
\[
\sum_{n=2}^{\infty} n(n-\lambda)|t_n|\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} \leq 1 - \lambda, \quad 0 \leq \lambda < 1, \tag{14}
\]
then \(f(z) \in C_{k,\alpha,\beta,\gamma,\delta}^{\gamma,q}(\lambda)\). The result (14) is sharp, where
\[
\Psi_{k,\alpha,\beta,\gamma,\delta}^{\gamma,q} = \frac{(\gamma)_n k (\alpha + \beta) \Gamma(\delta + 1)}{(\gamma)_{k+1} \Gamma(\alpha n + \beta) \Gamma(\delta + n)}.
\]

**Corollary 3.** Let the assumption of Theorem 2 is true. Then for \(0 \leq \lambda < 1\)
\[
|t_n| \leq \frac{1-\lambda}{n(n-\lambda)} \left\{\frac{1}{\Psi_{k,\alpha,\beta,\gamma,\delta}^{\gamma,q}} \right\}, \quad \forall n \geq 2. \tag{15}
\]

**Theorem 3.** Let the hypotheses of Theorem 1 holds. Then for \(z \in \mathbb{T}\) and \(0 \leq \lambda < 1\)
\[
|L_{k,\alpha,\beta,\gamma,\delta}^{\gamma,q} f(z)| \geq |z| \left| \frac{1-\lambda}{2-\lambda} |z|^2 \right|
\]
and
\[
|L_{k,\alpha,\beta,\gamma,\delta}^{\gamma,q} f(z)| \leq |z| \left| \frac{1-\lambda}{2-\lambda} |z|^2 \right|.
\]
Proof. One can easily understand from Theorem 1 that

\[(2 - \lambda) \sum_{n=2}^{\infty} |t_n| \{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} \]
\[\leq \sum_{n=2}^{\infty} (n - \lambda) |t_n| \{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} \]
\[\leq 1 - \lambda \]
so,
\[\sum_{n=2}^{\infty} |t_n| \{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} \leq \frac{1 - \lambda}{2 - \lambda}.\]

Thus we have
\[|L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z)| \leq |z| + \sum_{n=2}^{\infty} |t_n| \{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} |z|^n \]
\[\leq |z| + \sum_{n=2}^{\infty} |t_n| \{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} |z|^2 \]
\[\leq |z| + \left(1 - \frac{\lambda}{2 - \lambda}\right) |z|^2.\]

Now for second assertion we proceed as follows
\[|L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z)| = |z + \sum_{n=2}^{\infty} t_n \{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} z^n| \]
\[\geq |z| - \sum_{n=2}^{\infty} |t_n| \{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} |z|^n \]
\[\geq |z| - \sum_{n=2}^{\infty} (n - \lambda) \left(\frac{1}{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}}\right) |t_n| |z|^2 \]
\[\geq |z| - \left(1 - \frac{\lambda}{2 - \lambda}\right) |z|^2,\]
and hence both parts of our proof are completed. \(\Box\)

Theorem 4. Let the hypotheses of Theorem 3 be satisfied. Then for \(z \in T\) and \(0 \leq \lambda < 1\)
\[|L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z)| \geq |z| - \frac{1 - \lambda}{2(2 - \lambda)} |z|^2\]
and
\[|L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z)| \leq |z| + \frac{1 - \lambda}{2(2 - \lambda)} |z|^2.\]

Theorem 5. Let the hypotheses of Theorem 2 be satisfied. Then for all \(\{n \geq 2 : n \in \mathbb{N}\}\), \(0 \leq \lambda < 1\), and \((n - \lambda)\{\Psi_{k,\alpha,\beta,\delta}^{\gamma,q}\} \geq 1\), implies
\[|f(z)| \geq |z| - (1 - \lambda) |z|^2\]
and
\[|f(z)| \leq |z| + (1 - \lambda) |z|^2.\]
Proof. By using Theorem 1, we get
\[ \sum_{n=2}^{\infty} |t_n| \leq \sum_{n=2}^{\infty} (n - \lambda) \{ \Psi_{j,k,\alpha,\beta,\delta}^{\gamma,q} \} |t_n| \leq (1 - \lambda) \]

then
\[ \sum_{n=2}^{\infty} |t_n| \leq (1 - \lambda) \]

So, we obtain
\[ |f(z)| = \left| z + \sum_{n=2}^{\infty} t_n z^n \right| \leq |z| + \sum_{n=2}^{\infty} |t_n||z|^2 \leq |z| + (1 - \lambda)|z|^2 \]

To prove the second assertion we adopt the following steps
\[ |f(z)| \geq \left| z - \sum_{n=2}^{\infty} t_n z^n \right| \geq |z| - \sum_{n=2}^{\infty} |t_n||z|^2 \geq |z| - (1 - \lambda)|z|^2 \]

and hence the proof. \( \square \)

**Theorem 6.** Let the hypotheses of Theorem 2 be satisfied. Then for all \( \{ n \geq 2 : n \in \mathbb{N} \} \), \( 0 \leq \lambda < 1 \), and \( n(n - \lambda) \{ \Psi_{j,k,\alpha,\beta,\delta}^{\gamma,q} \} \geq 1 \), poses
\[ |f(z)| \geq |z| - \frac{1 - \lambda}{2}|z|^2 \]

and
\[ |f(z)| \leq |z| + \frac{1 - \lambda}{2}|z|^2. \]

**Definition 4.** Let \( \phi(z) \) be univalent starlike function with respect to 1 which maps the unit disk \( \mathbb{T} \) onto a region in the right half plane which is symmetric about the real axis, \( \phi(0) = 1 \) and \( \phi'(0) > 0 \). A function \( f \in \mathcal{N} \) is in the class \( L_{k,\alpha,\beta,\delta}^{\gamma,q} (\phi) \) if
\[ z \left( L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z) \right)' L_{k,\alpha,\beta,\delta}^{\gamma,q} f(z) < \phi(z). \]  

(16)

**Remark 2.** It can be seen if we put \( \gamma = q = k = \beta = \delta = 1 \) and \( \alpha = 0 \) then the class \( L_{k,\alpha,\beta,\delta}^{\gamma,q} \phi(z) \) becomes the class of starlike function \( S^*(\phi) \).

**Lemma 3.** (see [12]) If \( h_1 = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \) is an analytic function with positive real part in the unit disk \( \mathbb{T} \), then
\[ |c_2 - vc_1^2| \leq 2 \max \{ 1, |2v - 1| \} \]  

(17)

and the result is sharp for the functions given by
\[ h(z) = \frac{1 + z}{1 - z}, \quad h(z) = \frac{1 + z^2}{1 - z^2}. \]  

(18)
3. FEKETE-SZEGÖ FOR THE CLASSES $S^γ,q_{k,α,β,δ}(λ)$ AND $C^γ,q_{k,α,β,δ}(λ)$

Now we determine the sharp upper bound for $|t_2|$ for the class $S^γ,q_{k,α,β,δ}(λ)$ and $C^γ,q_{k,α,β,δ}(λ)$, and calculate the Fekete-Szegö function of $|t_3 - µt_2^2|$ for these classes.

**Theorem 7.** Let the hypotheses of Theorem 1 be satisfied. Then

$$|t_2| \leq \frac{2(1 - λ)(γ)_{2q,k}Γk(2α + β)(δ + 1)}{(1 + λ)(γ)_{2q,k}Γk(α + β)}, \quad 0 ≤ λ < 1,$$

and for all $µ ∈ ℂ$ the following bound is sharp

$$|t_3 - µt_2^2| \leq \frac{1 - λ}{(2 + λ)B} \max \left\{ 1, \frac{1 - λ}{1 + λ} B_1^2 - \frac{1 - λ}{1 + λ} B_1^2 - \frac{1 - λ}{(1 + λ)^2 A^2} B_1^2 \right\},$$

where values of $A$ and $B$ are given by equation (21), $k, β, γ ≥ 1$ and $α, q, δ ≥ 0$.

**Proof.** Since $f ∈ S^γ,q_{k,α,β,δ}(λ)$ then the condition

$$R \left\{ \frac{z \left( L^γ,q_{k,α,β,δ}f(z) \right)^{'}}{L^γ,q_{k,α,β,δ}f(z)} \right\} > λ, \quad 0 ≤ λ < 1,$$

is equivalent to

$$z \left( L^γ,q_{k,α,β,δ}f(z) \right)^{'} = (1 - λ)h(z)L^γ,q_{k,α,β,δ}f(z), \quad z ∈ ℂ,$$

for some $h ∈ ℙ$. Equating the coefficients we have the values of

$$t_2 = \frac{(1 - λ)h_1}{(1 + λ)A}$$

and

$$t_3 = \frac{1 - λ}{(2 + λ)B} \left( \frac{1 - λ}{1 + λ} h_1^2 + h_2 \right),$$

and hence by using Lemma 2 and equation (19) we achieve the required result of $|t_2|$.

where

$$A = \frac{(γ)_{2q,k}Γk(α + β)}{(γ)_{2q,k}Γk(α + β)(δ + 1)} \quad \text{and} \quad B = \frac{(γ)_{3q,k}Γk(α + β)}{(γ)_{3q,k}Γk(3α + β)(δ + 1)(δ + 2)}. \quad (21)$$

For the Fekete-Szegö function $|t_3 - µt_2^2|$, consider $φ(z) = 1 + B_1z + B_2z^2 + ...$ if $f(z)$ is given by (11) belongs to $L^γ,q_{k,α,β,δ}(φ)$, then

$$h(z) = φ \left( \frac{h_1(z) - 1}{h_1(z) + 1} \right). \quad (22)$$

Since $φ(z)$ is univalent and $h(z) ≺ φ$, then the function below, is analytic and has a positive real part in $ℂ$.

$$h_1(z) = \frac{1 + φ^{-1}(h(z))}{1 - φ^{-1}(h(z))} = 1 + c_1z + c_2z^2 + ...$$

and then from (22) and (23) we obtain the values of $h_1(z)$ and $h_2(z)$

$$h_1 = \frac{1}{2} B_1c_1$$

and

$$h_2 = \frac{1}{2} \left( c_2 - \frac{1}{2} c_1^2 \right) + \frac{1}{4} B_2c_1,$$
so, by using Lemma 2 we find

\[ |t_3 - \mu t_2^2| \leq \frac{2(1-\lambda)}{(2 + \lambda)B} + \left\{ \frac{(1-\lambda)^2}{(1+\lambda)(2 + \lambda)} - \mu \frac{(1-\lambda)^2}{(1+\lambda)^2 A^2} \right\} |h_1|^2 \]

\[ \leq \mathcal{H}(x) = \frac{2(1-\lambda)}{(2 + \lambda)B} + \left\{ K - \frac{(1-\lambda)}{2(2 + \lambda)B} \right\} x^2, \quad x := |h_1|^2. \tag{24} \]

As a result we obtain

\[ |t_3 - \mu t_2^2| \leq \begin{cases} \mathcal{H}(0) = \frac{2(1-\lambda)}{(2 + \lambda)B} \quad \text{if } K \leq \frac{(1-\lambda)^2}{2(2 + \lambda)B} \\ \mathcal{H}(2) = 4K \quad \text{if } K > \frac{(1-\lambda)^2}{2(2 + \lambda)B} \end{cases} \]

where

\[ K := \frac{(1-\lambda)}{2(2 + \lambda)B} + \frac{1}{(1+\lambda)(2 + \lambda)} - \mu \frac{(1-\lambda)^2}{(1+\lambda)^2 A^2}. \]

Equality is attained for the functions given by

\[ z \left[ \frac{L_{k,\alpha,\beta,\delta}^q(z)}{L_{k,\alpha,\beta,\delta}^q(z)} \right]' = 1 + z(1-2\lambda) \]

\[ z \left[ \frac{L_{k,\alpha,\beta,\delta}^q(z)}{L_{k,\alpha,\beta,\delta}^q(z)} \right]' = 1 + z^2(1-2\lambda), \]

Also we have

\[ t_3 - \mu t_2^2 = \frac{1-\lambda}{2(2+\lambda)B} \left[ v_2 - v c_2^2 \right], \tag{25} \]

where values of \( A \) and \( B \) are given by (21), and

\[ v = \frac{1}{2} \left( 1 - B_2 - \frac{1-\lambda}{1+\lambda} B_1^2 + \mu \frac{(1-\lambda)(2 + \lambda)B}{(1+\lambda)^2 A^2} B_1^2 \right), \]

hence by applying Lemma 3 the result of sharp bound \( |t_4 - \mu t_2^2| \) is proved. \( \square \)

**Corollary 4.** Let the assumption of Theorem 7 is true. Then for \( \lambda = 0 \)

\[ |t_2| \leq \frac{2(\gamma)^q \Gamma k(2\alpha + \beta)(\delta + 1)}{(\gamma)_{2q,k} \Gamma k(\alpha + \beta)} \]

and

\[ |t_3 - \mu t_2^2| \leq \frac{1}{2B} \max \left\{ 1, \left| B_2 + B_1^2 - \frac{B}{A^2} B_1^2 \right| \right\} \]

Now we prove the result for the class of \( C_{k,\alpha,\beta,\delta}(\lambda) \).

**Theorem 8.** Let the hypotheses of Theorem 2 hold. Then

\[ |t_2| \leq \frac{(1-\lambda)(\gamma)^q \Gamma k(2\alpha + \beta)(\delta + 1)}{(\gamma)_{2q,k} \Gamma k(\alpha + \beta)} \]

and for all \( \mu \in \mathbb{C} \) the following bound is sharp

\[ |t_3 - \mu t_2^2| \leq \frac{1-\lambda}{3(2+\lambda)B} \max \left\{ 1, \left| B_2 + \frac{1-\lambda}{1+\lambda} B_1^2 - \mu \frac{3(1-\lambda)(2 + \lambda)B}{2(1+\lambda)^2 A^2} B_1^2 \right| \right\} \]
Proof. Since \( f \in C_{k,\alpha,\beta,\delta}^{\gamma,q}(\lambda) \) then the condition
\[
\mathcal{R} \left\{ \left. \begin{array}{c}
\left. \frac{z}{L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z)} \right|^{'} \\
\left. \frac{L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z)}{z} \right|^{'}
\end{array} \right\} > \lambda, \quad 0 \leq \lambda < 1,
\]
is equivalent to
\[
\left( L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z) \right)' + z \left( L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z) \right)' = (1 - \lambda)h(z) \left( L_{k,\alpha,\beta,\delta}^{\gamma,q}f(z) \right)', \quad z \in \mathbb{T},
\]
for some \( h \in \mathcal{P} \). Equating the coefficients we have the values of
\[
t_2 = \frac{(1 - \lambda)h_1}{2(1 + \lambda)A} \quad (26)
\]
and
\[
t_3 = \frac{1 - \lambda}{3(2 + \lambda)B} \left( \frac{1 - \lambda}{1 + \lambda}h_1^2 + h_2 \right), \quad (27)
\]
and hence by using Lemma 2 and equation (26) we achieve the required result of \( |t_2| \).

Further we obtain,
\[
t_3 - \mu t_2^2 = \frac{1 - \lambda}{6(2 + \lambda)B} \left[ c_2 - vc_2^2 \right], \quad (28)
\]
where values of \( A \) and \( B \) are given by (21), and
\[
v = \frac{1}{2} \left( 1 - B_2 - \frac{1 - \lambda}{1 + \lambda}B_2^2 + \mu \frac{3(1 - \lambda)(2 + \lambda)B}{2(1 + \lambda)^2A^2}B_2^2 \right),
\]

Corollary 5. Let the assumption of Theorem 5 is true. Then for \( \lambda = 0 \)
\[
|t_2| \leq \frac{(\gamma)_{\alpha,k}^{\mu,k}(2\alpha + \beta)(\delta + 1)}{(\gamma)_{\alpha,k}^{\mu,k}(\alpha + \beta)}
\]
and
\[
|t_3 - \mu t_2^2| \leq \frac{1}{6B} \max \left\{ 1, \left| B_2 + B_1^2 - \mu \frac{3\beta B}{A^2} \right| \right\}.
\]

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