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RESULTS ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY A DERIVATIVE OPERATOR

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ABSTRACT. In this paper, we introduce and study the classes $S^{\alpha,n,\beta}(m,l,q,\lambda)$ and $TS^{\alpha,n,\beta}(m,l,q,\lambda)$ defined by a generalised derivative operator $D^{\alpha,n}(m,l,q,\lambda)$. Coefficient inequalities are obtained for the classes $S^{\alpha,n,\beta}(m,l,q,\lambda)$ and $TS^{\alpha,n,\beta}(m,l,q,\lambda)$. Further, growth and distortion, extreme points, and inclusion are also given for the class $TS^{\alpha,n,\beta}(m,l,q,\lambda)$.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of normalised analytic univalent functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

where $z \in \mathbb{U} = \{z : |z| < 1\}$.

Let T denote the analytic functions with negative coefficients

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad (z \in \mathbb{U}). \tag{2}$$

For the function $f \in \mathcal{A}$ given by (1), we state a generalised derivative operator given by Eghbiq and Darus [2] as follows:

$$D^{\alpha,n}(m,l,q,\lambda)(f)(z) = z + \sum_{k=2}^{\infty} k^{\alpha} \left(\frac{q+\lambda(k-1)+l}{q+l}\right)^m c(n,k) a_k z^k, \tag{3}$$

where $n, \alpha \in \mathbb{N}_0 = \{0, 1, 2...\}, m \in \mathbb{Z}$, $\lambda, l, q \geq 0, l + q \neq 0$ and $c(n,k) = \frac{\prod_{j=1}^{k-1}(j+n)}{(k-1)!}$.
Here $D^{\alpha,n}(m,l,q,\lambda)f(z)$ can also be written as

$$\phi(z) := \left(\frac{l+q-\lambda}{l+q}\right) \frac{z}{1-z} + \left(\frac{\lambda}{l+q}\right) \frac{z}{(1-z)^2}, \quad (z \in \mathbb{U}),$$

in terms of convolution.

If $m = 0, 1, 2, ...,$ then

$$\begin{aligned} D^{\alpha,n}(m,l,q,\lambda)f(z) &= \underbrace{\phi(z) * ... * \phi(z)}_{(m)-times} * \left[\frac{z}{(1-z)^{n+1}}\right] * \sum_{k=1}^{\infty} k^{\alpha} z^k * f(z) \\ &= R^n * D^{\alpha}(m,l,q,\lambda)f(z), \end{aligned}$$

where $R^n = z + \sum_{k=2}^{\infty} c(n,k) z^k$, the Ruscheweyh operator and $D^{\alpha}(m,l,q,\lambda) = z + \sum_{k=2}^{\infty} k^{\alpha}(1 + \frac{k-1}{l+q}\lambda)^m$.

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If $m=-1,-2,...$, then

$$\begin{aligned} D^{\alpha,n}(m,l,q,\lambda)f(z) &= \underbrace{\phi(z)*...*\phi(z)}_{(m)-times}*\left[\frac{z}{(1-z)^{n+1}}\right]*\sum_{k=1}^{\infty}k^{\alpha}z^{k}*f(z) \\ &= R^{n}*D^{\alpha}(m,l,q,\lambda)f(z). \end{aligned}$$

Note that:

$$D^{0,0}(0,l,q,\lambda)f(z)=f(z), \quad and$$
$$D^{1,0}(0,l,q,\lambda)f(z) = zf'(z).$$

By specialising the parameters of $D^{\alpha,n}(m,l,q,\lambda)f(z)$, we get the following various operators.

• The operator introduced by Ruscheweyh [18];

$$D^{0,n}(0,l,q,\lambda);(n\in\mathbb{N}_0)\equiv R^{n}=z+\sum_{k=2}^{\infty}c(n,k)a_kz^k.$$

• The operator introduced by Sălăgean [5];

$$D^{n,0}(0,l,q,\lambda);(n\in\mathbb{N}_0)\equiv D^{n}=z+\sum_{k=2}^{\infty}k^na_kz^k.$$

• The generalised Sălăgean operator introduced by Al-Oboudi (see [4]);

$$D^{0,0}(n,1,0,\lambda);(n\in\mathbb{N}_0)\equiv D^{n}_{\lambda}=z+\sum_{k=2}^{\infty}(1+\lambda(k-1))^na_kz^k.$$

• The generalised Ruscheweyh operator given by Al-Shaqsi and Darus in [8];

$$D^{0,n}(1,1,0,\lambda);(n\in\mathbb{N}_0)\equiv R^{n}_{\lambda}=z+\sum_{k=2}^{\infty}(1+\lambda(k-1))c(n,k)a_kz^k.$$

• The generalised Ruscheweyh and Sălăgean operator introduced by Darus and Al-Shaqsi in (see [7]);

$$D^{0,\beta}(m,1,0,\lambda);(m\in\mathbb{N}_0)\equiv D^{m}_{\lambda,\beta}=z+\sum_{k=2}^{\infty}(1+\lambda(k-1))^mc(\beta,k)a_kz^k.$$

• The operator introduced by Catas (see [1]);

$$D^{0,\beta}(m,l,1,\lambda);(m\in\mathbb{N}_0)\equiv D^{m}(\lambda,\beta,l)=z+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m c(\beta,k)a_kz^k.$$

• The operator introduced by Uralegaddi and Somanatha (see [3]);

$$D^{0,0}(n,1,1,1)\equiv I^{n}=z+\sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^n a_kz^k.$$

• The multiplier transformations studied by Flett (see [19]);

$$D^{0,0}(n,1,\lambda,1)\equiv I^{n}_{\lambda}=z+\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_kz^k.$$

• The integral operator introduced by Cho and Kim (see [12]);

$$D^{0,0}(-n,1,\lambda,1) \equiv I_n^{\lambda} = z + \sum_{k=2}^{\infty} k \left(\frac{1+\lambda}{k+\lambda}\right)^n a_k z^k.$$

• The derivative operator introduced by Mustafa and Darus ((see [13]);

$$D^{\alpha,n}(m,1,q,\lambda)(f)(z) \equiv D^{\alpha,n}(m,q,\lambda)(f)(z) = z + \sum_{k=2}^{\infty} k^{\alpha}\left(1+\frac{k-1}{1+q}\lambda\right)^m c(n,k) a_k z^k,$$

In this paper, we study a new subclass of analytic functions in the open unit disc which is defined by a generalised operator $D^{\alpha,n}(m,l,q,\lambda)$. Several results related to coefficient estimates, growth and distortion, closure theorems, and extreme points for the subclass of analytic functions defined by the aforementioned operator will be given. The work here is motivated by [11] and many others, in terms of problems and method of proofs.

Using the operator $D^{\alpha,n}(m,l,q,\lambda)$, the definition for the subclasses $S^{\alpha,n,\beta}(m,l,q,\lambda)$ and $TS^{\alpha,n,\beta}(m,l,q,\lambda)$ are given as follows:

For $0 \le \beta < 1$, $n,\alpha \in \mathbb{N}_0 = \{0,1,2...\}, m \in \mathbb{Z}$, $\lambda, l, q \ge 0, l+q \ne 0$, let $S^{\alpha,n,\beta}(m,l,q,\lambda)$ be a subclass of $\mathcal{A}$ consisting of functions f satisfying

$$\Re\left\{\frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'}{D^{\alpha,n}(m,l,q,\lambda)f(z)}\right\} > \beta, \qquad (z \in \mathbb{U}). \tag{4}$$

We then define the class $TS^{\alpha,n,\beta}(m,l,q,\lambda)$ by

$$TS^{\alpha,n,\beta}(m,l,q,\lambda) = S^{\alpha,n,\beta}(m,l,q,\lambda) \cap T. \tag{5}$$

Also, we note that various subclass $TS^{\alpha,n,\beta}(m,l,q,\lambda)$ has been studied by several authors by suitably choosing the values of λ, m, n, and α. For example,

$$TS^{0,0,\beta}(0,0,q,\lambda) = TS^{0,0,\beta}(1,0,0,0) = S_{\tau}^{*}(\beta),$$

is starlike of order β with negative coefficients. While

$$TS^{0,0,\beta}(1,0,0,1) = TS^{0,1,\beta}(0,0,q,\lambda) = C_{\tau}(\beta),$$

is the class of convex function of order β with negative coefficients. The classes $S_{\tau}^{*}(\beta)$ and $C_{\tau}(\beta)$ were introduced and studied by Silverman (1975) (see [6]).

2. Coefficient inequalities

In this section, coefficient inequalities are obtained for the classes $S^{\alpha,n,\beta}(m,l,q,\lambda)$ and $TS^{\alpha,n,\beta}(m,l,q,\lambda)$.

Theorem 1. *For* $0 \le \beta < 1$, $n,\alpha \in \mathbb{N}_0 = \{0,1,2...\}, m \in \mathbb{Z}$, $\lambda, l, q \ge 0, l+q \ne 0$, *let the function* f *be defined by (1). If*

$$\sum_{k=2}^{\infty}(k-\beta)k^{\alpha}\left(1+\frac{k-1}{l+q}\lambda\right)^m c(n,k)|a_k| \le 1-\beta, \tag{6}$$

then $f \in S^{\alpha,n,\beta}(m,l,q,\lambda)$.

Proof. Assuming (6) holds. Then it is suffices to show that the values $\frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'}{D^{\alpha,n}(m,l,q,\lambda)f(z)}$ lie in a circle centred at $w = 1$ with radius $1-\beta$. We have

$$\left|\frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'}{D^{\alpha,n}(m,l,q,\lambda)f(z)} - 1\right| \le 1-\beta, \quad (z \in \mathbb{U}). \tag{7}$$

So, we can write

$$\begin{aligned}\left|\frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'}{D^{\alpha,n}(m,l,q,\lambda)f(z)}-1\right| &= \left|\frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'-D^{\alpha,n}(m,l,q,\lambda)f(z)}{D^{\alpha,n}(m,l,q,\lambda)f(z)}\right| \\ &= \left|\frac{\sum_{k=2}^{\infty}(k-1)k^{\alpha}(1+\frac{k-1}{l+q}\lambda)^{m}c(n,k)}{z+\sum_{k=n+1}^{\infty}k^{\alpha}(1+\frac{k-1}{l+q}\lambda)^{m}c(n,k)}\right| \\ &\leq \frac{\sum_{k=2}^{\infty}(k-1)k^{\alpha}(1\frac{k-1}{l+q}\lambda)^{m}c(n,k)|a_k|}{1-\sum_{k=2}^{\infty}k^{\alpha}(1+\frac{k-1}{l+q}\lambda)^{m}c(n,k)|a_k|}.\end{aligned}$$

This expression is bounded by $(1-\beta)$ if the following inequality which is equivalent to (7) holds.

$\sum_{k=2}^{\infty}(k-1)k^{\alpha}\left(1+\frac{k-1}{l+q}\lambda\right)^{m}c(n,k)|a_k| \leq (1-\beta)\left[1-\sum_{k=2}^{\infty}k^{\alpha}\left(1+\frac{k-1}{l+q}\lambda\right)^{m}c(n,k)|a_k|\right].$

This gives

$$\sum_{k=2}^{\infty}(k-\beta)k^{\alpha}\left(1+\frac{k-1}{l+q}\lambda\right)^{m}c(n,k)|a_k| \leq 1-\beta,$$

by(6). Hence $f \in S^{\alpha,n,\beta}(m,l,q,\lambda)$. □

Next,

Theorem 2. *For* $0 \leq \beta < 1$, $n,\alpha \in \mathbb{N}_0 = \{0,1,2...\}, m \in \mathbb{Z}$, $\lambda, l, q \geq 0, l+q \neq 0$, *let the function* f *be defined by (2). Then* $f \in TS^{\alpha,n,\beta}(m,l,q,\lambda)$ *if and only if (2.1) is satisfied.*

Proof. In view of Theorem 1, it suffices to show the only if part. Assume that

$$\Re\left\{\frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'}{D^{\alpha,n}(m,l,q,\lambda)f(z)}\right\}$$

$$=\Re\left\{\frac{z-\sum_{k=2}^{k\infty}k^{\alpha+1}(1+\frac{k-1}{l+q}\lambda)^{m}c(n,k)|a_k|z^k}{z-\sum_{k=2}^{\infty}k^{\alpha}(1+\frac{k-1}{l+q}\lambda)^{m}c(n,k)|a_k|z^k}\right\} > \beta, \quad (z \in \mathbb{U}).$$

Having the values of z on the real axis so that $\frac{z(D^{\alpha,n}(m,l,q,\lambda)f(z))'}{D^{\alpha,n}(m,l,q,\lambda)f(z)}$ is real, and letting $z \to 1^-$ through the real values, we get

$$1-\sum_{k=2}^{\infty}k^{\alpha+1}\left(1+\frac{k-1}{l+q}\lambda\right)^{m}c(n,k)|a_k| > \beta\left[1-\sum_{k=2}^{\infty}k^{\alpha}\left(1+\frac{k-1}{l+q}\lambda\right)^{m}c(n,k)|a_k|\right].$$

Thus we obtain

$$\sum_{k=2}^{\infty}(k-\beta)k^{\alpha}\left(1+\frac{k-1}{l+q}\lambda\right)^{m}c(n,k)|a_k| \leq 1-\beta,$$

which is (6). Hence we prove the theorem.
The result is sharp with the extremal function f given by

$$f(z) = z - \frac{(1-\beta)(l+q)^m}{(k-\beta)(k)^{\alpha}(l+q+(k-1)\lambda)^m c(n,k)}z^k, \quad (k \geq 2). \tag{8}$$

□

Corollary 1. *Let the function* f *be defined by (2) be in the class* $TS^{\alpha,n,\beta}(m,l,q,\lambda)$*. Then*

$$|a_k| \leq \frac{(1-\beta)(l+q)^m}{(k-\beta)(k)^{\alpha}(l+q+(k-1)\lambda)^m c(n,k)}, \quad (k \geq 2). \tag{9}$$

The equality is attained for the function f *given by (8).*

3. Growth and Distortion theorems

Here, growth and distortion theorems are considered.

Theorem 3. *Let the function f given by (2) be in the class $TS^{\alpha,n,\beta}(m,l,q,\lambda)$. Then for $0<|z|=r<1$,*

$$r-\frac{(1-\beta)(l+q)^m}{(2-\beta)(2)^\alpha(l+q+\lambda)^m c(n,2)}r^2 \le |f(z)| \le r+\frac{(1-\beta)(l+q)^m}{(2-\beta)(2)^\alpha(l+q+\lambda)^m c(n,2)}r^2,$$

and

$$1-\frac{2(1-\beta)(l+q)^m}{(2-\beta)2^\alpha(l+q+\lambda)^m c(n,2)}r \le |f'(z)| \le 1+\frac{2(1-\beta)(l+q)^m}{(2-\beta)2^\alpha(1+q+\lambda)^m c(n,2)}r, \quad (10)$$

where $0\le\beta<1$, $n,\alpha\in\mathbb{N}_0, m\in\mathbb{Z}$, $\lambda,q\ge 0$. where $0\le\beta<1$, $n,\alpha\in\mathbb{N}_0=\{0,1,2...\}, m\in\mathbb{Z}$, $\lambda,l,q\ge 0, l+q\ne 0$.

Proof. Since $f\in TS^{\alpha,n,\beta}(m,l,q,\lambda)$, by Theorem 2, we get

$$\sum_{k=2}^{\infty}(k-\beta)k^\alpha\left(1+\frac{k-1}{l+q}\lambda\right)^m c(n,k)|a_k| \le 1-\beta.$$

Now

$$\begin{aligned}(2-\beta)(2)^\alpha\left(1+\frac{1}{l+q}\lambda\right)^m c(n,2)\left(\sum_{k=2}^{\infty}|a_k|\right) &\le \sum_{k=2}^{\infty}(k-\beta)k^\alpha\left(1+\frac{k-1}{l+q}\lambda\right)^m c(n,k)|a_k| \\ &\le 1-\beta, \quad (k\ge 2).\end{aligned}$$

Therefore,

$$\sum_{k=2}^{\infty}|a_k| \le \frac{(1-\beta)(l+q)^m}{(2-\beta)(2)^\alpha(l+q+\lambda)^m c(n,2)}. \quad (11)$$

Since

$$f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k,$$

we have
$|f(z)|=\left|z-\sum_{k=2}^{\infty}a_kz^k\right| \le |f(z)| \le |z|+|z|^2\sum_{k=2}^{\infty}|a_k||z^{k-2}|$

$$\le r+r^2\sum_{k=2}^{\infty}|a_k| \le r+\frac{(1-\beta)(l+q)^m}{(2-\beta)(n+1)^\alpha(l+q+\lambda)^m c(n,2)}r^2.$$

So

$$|f(z)| \ge r-r^2\sum_{k=2}^{\infty}|a_k| \ge r-\frac{(1-\beta)(l+q)^m}{(2-\beta)(n+1)^\alpha(l+q+\lambda)^m c(n,2)}r^2.$$

Further

$$f'(z)=1-\sum_{k=2}^{\infty}k|a_k|z^{k-1},$$

then we have

$$1-|z|\sum_{k=2}^{\infty}k|a_k||z^{k-2}| \le |f'(z)| \le 1+|z|\sum_{k=2}^{\infty}k|a_|z^{k-2}|.$$

Therefore,

$$1-r\sum_{k=2}^{\infty}ka_k \le |f'(z)| \le 1+r\sum_{k=2}^{\infty}ka_k. \quad (12)$$

By putting (11) in (12), we have the inequality (10). The proof is complete. □

4. Extreme points

In this part, we discuss extreme points for functions f belong to the class $TS^{\alpha,n,\beta}(m,l,q,\lambda)$.

Theorem 4. *Let* $f_1(z)=z$ *and*

$$f_k(z)=z-\frac{(1-\beta)(l+q)^m z^k}{(k-\beta)k^\alpha(l+q+(k-1)\lambda)^m c(n,k)}, \qquad (k\geq 2),$$

$0\leq\beta<1$, $n,\alpha\in\mathbb{N}_0=\{0,1,2...\}, m\in\mathbb{Z}$, $\lambda,l,q\geq 0, l+q\neq 0$. *Then* $f\in TS^{\alpha,n,\beta}(m,l,q,\lambda)$ *if and only if it can be expressed in the form*

$$f(z)=\sum_{k=1}^{\infty}\omega_k f_k(z), \tag{13}$$

where $\omega_k\geq 0$ *and* $\sum_{k=1}^{\infty}\omega_k=1$.

Proof. Suppose that f can be expressed as in (13). We need to show that $f\in TS^{\alpha,n,\beta}(m,l,q,\lambda)$. By (13), we obtain

$$\begin{aligned}
f(z) &= \sum_{k=1}^{\infty}\omega_k f_k(z)\\
&= \omega_1 f_1(z)+\sum_{k=2}^{\infty}\omega_k f_k(z)\\
&= \omega_1 f_1(z)+\sum_{k=2}^{\infty}\omega_k\left(z-\frac{(1-\beta)(l+q)^m}{(k-\beta)k^\alpha(l+q+(k-1)\lambda)^m c(n,k)}z^k\right)\\
&= \omega_1 z+\sum_{k=2}^{\infty}\omega_k z-\sum_{k=2}^{\infty}\left(\frac{\omega_k(1-\beta)(l+q)^m}{(k-\beta)k^\alpha(l+q+(k-1)\lambda)^m c(n,k)}z^k\right)\\
&= \left(\sum_{k=1}^{\infty}\omega_k\right)z-\sum_{k=2}^{\infty}\left(\frac{\omega_k(1-\beta)(l+q)^m}{(k-\beta)k^\alpha(l+q+(k-1)\lambda)^m c(n,k)}z^k\right)\\
&= z-\sum_{k=2}^{\infty}\left(\frac{\omega_k(1-\beta)(l+q)^m}{(k-\beta)k^\alpha(l+q+(k-1)\lambda)^m c(n,k)}z^k\right).
\end{aligned}$$

Since

$$\sum_{k=2}^{\infty}\omega_k=1-\omega_k\leq 1,$$

then

$$\sum_{k=2}^{\infty}\omega_k=\sum_{k=2}^{\infty}\frac{\omega_k(1-\beta)(l+q)^m}{(k-\beta)k^\alpha(l+q+(k-1)\lambda)^m c(n,k)}\frac{(k-\beta)k^\alpha(l+q+(k-1)\lambda)^m c(n,k)}{(1-\beta)(l+q)^m}$$

Thus $f\in TS^{\alpha,n,\beta}(m,l,q,\lambda)$ by Theorem 2.
Conversely, suppose that $f\in TS^{\alpha,n,\beta}(m,l,q,\lambda)$. By using (9) we may set

$$\omega_k=\frac{(k-\beta)k^\alpha(l+q+(k-1)\lambda)^m c(n,k)}{(l+q)^m(1-\beta)}|a_k|,\ (k\geq 2) \quad\text{and}\quad \omega_1=1-\sum_{k=2}^{\infty}\omega_k.$$

Then

$$\begin{aligned}
f(z) &= z-\sum_{k=2}^{\infty}a_kz^k\\
&= z-\sum_{k=2}^{\infty}\frac{\omega_k(1-\beta)(l+q)^m}{(k-\beta)k^{\alpha}(l+q+(k-1)\lambda)^mc(n,k)}z^k\\
&= z-\sum_{k=2}^{\infty}\omega_k[z-f_k(z)]\\
&= z-\sum_{k=2}^{\infty}\omega_kz+\sum_{k=2}^{\infty}\omega_kf_k(z)\\
&= \left(1-\sum_{k=2}^{\infty}\omega_k\right)z+\sum_{k=2}^{\infty}\omega_kf_k(z)\\
&= \omega_1z+\sum_{k=2}^{\infty}\omega_kf_k(z)\\
&= \omega_1f_1(z)+\sum_{k=2}^{\infty}\omega_kf_k(z)=\sum_{k=1}^{\infty}\omega_kf_k(z).
\end{aligned}$$

Thus the proof is complete. □

Corollary 2. *The extreme points of* $TS^{\alpha,n,\beta}(m,l,q,\lambda)$ *are the functions*

$$f_1(z)=z \quad and$$

$$f_k(z)=z-\frac{(1-\beta)(l+q)^m}{(k+1-\beta)(k+1)^{\alpha}(l+q+k\lambda)^mc(n,k+1)}z^k, \qquad (k\geq 2).$$

5. Closure theorem

We will need the following definition to prove the following closure theorem for the subclass $TS^{\alpha,n,\beta}(m,l,q,\lambda)$.

Definition 1. *Let the functions* $f_j(z)(j=1,2)$ *be defined by*

$$f_j(z)=z-\sum_{k=2}^{\infty}a_{k,j}z^k, \quad for \quad all \quad (a_{k,j}\geq \quad 0, z\in\mathbb{U}). \tag{14}$$

The modified Hadamard product of $f_1(z)$ *and* $f_2(z)$ *is defined by*

$$(f_1*f_2)(z)=z-\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k.$$

Theorem 5. *Let the function* $f_j(j=1,2,...,p)$ *be defined by (14), belongs to the subclass* $f\in TS^{\alpha,n,\beta}(m,q,\lambda)$, *and let* $c_j\geq 0,(j=1,2,...,p)$ *such that* $\sum_{j=1}^{p}c_j=1$. *Then the function* h *defined by*

$$h=\sum_{j=1}^{p}c_jf_j,$$

is also in the subclass $TS^{\alpha,n,\beta}(m,l,q,\lambda)$.

Proof. In virtue of the definition of h, we can write

$$h=\sum_{j=1}^{p}c_j(z-\sum_{k=2}^{\infty}a_{k,j}z^k)=(\sum_{j=1}^{p}c_j)z-\sum_{k=2}^{\infty}(\sum_{j=1}^{p}c_ja_{k,j})z^k=z-\sum_{k=2}^{\infty}(\sum_{j=1}^{p}c_ja_{k,j})z^k.$$

Since the functions f_j are in $TS^{\alpha,n,\beta}(m,l,q,\lambda)$, for every $j=1,2,3,..r$, we have

$$\sum_{k=2}^{\infty}\frac{(k-\beta)k^{\alpha}(l+q+(k-1)\lambda)^m c(n,k)}{(l+q)^m}a_{k,j}\leq(1-\beta).$$

Hence we get

$$\sum_{k=2}^{\infty}\frac{(k-\beta)k^{\alpha}(l+q+(k-1)\lambda)^m c(n,k)}{(l+q)^m}(\sum_{j=1}^{p}c_ja_{k,j})$$
$$=\sum_{j=1}^{p}c_j\sum_{k=2}^{\infty}\frac{(k-\beta)k^{\alpha}(l+q+(k-1)\lambda)^m c(n,k)}{(l+q)^m}a_{k,j}$$
$$\leq\sum_{J=1}^{p}c_j(1-\beta)=(1-\beta),$$

which implies that h is in the subclass $TS^{\alpha,n,\beta}(m,l,q,\lambda)$. Thus the proof is complete. □

Other work regarding differential operators for various problems can be found in [7],[9], [10], [14], [15], [16], [17], [20].

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