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GLOBAL ASYMPTOTIC STABILITY OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. This paper is mainly concerned the global asymptotic stability of the zero solution of a class of nonlinear neutral differential equations in C^1. By converting the nonlinear neutral differential equation into an equivalent integral equation, our main results are obtained via the Banach contraction mapping principle. Finally, an example is given to illustrate our results.

1. Introduction

In 1892 Lyapunov published a major work on stability of ordinary differential equations based on positive definite functions and the chain rule. Lyapunov's work has been the foundation of stability and instability theory as we know today for a wide variety of ordinary, functional, partial differential and integro-differential equations. Nevertheless, the application of this method to problems of stability in differential and integro-differential equations with delays has encountered serious obstacles if the delays are unbounded or if the equation has unbounded terms [11]–[13]. Maybe this is due to the pointwise aspect of the method while real world-problems asks for averaging conditions. So, it does seem that the times are ripe to try other avenues. In recent years, several investigators in this field have tried stability by using a new technique. Particularly, Burton, Furumochi, Becker, Zhang and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1]–[26], [28]–[30]). The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's. The conditions of the former are often averages but those of the latter are usually pointwise (see [11]). Moreover, recent works have shown that the fixed point technique can be applied to problems perturbed by stochastic terms and still yielding stability (see for example [24]). This is another important feature for applications to real-world problems.

In this paper, we consider the nonlinear neutral differential equation with infinite delay

$$x'(t) = -a(t)x(t) + c(t)x'(t-\tau(t)) + \int_{-\infty}^{t} g(t,s) f(x(s))\,ds, \tag{1}$$

with the initial condition

$$x(t) = \varphi(t) \text{ for } t \in (-\infty, t_0],$$

for each $t_0 \geq 0$.

We introduce the following hypotheses.

(H_1) $a, c \in C([0,\infty), \mathbb{R})$, $g \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R}, \mathbb{R})$ and $\tau \in C([0,\infty), (0,\infty))$ with $t - \tau(t) \to \infty$ as $t \to \infty$.

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(H_2) $f(0)=0$, and there exist $L \in C([0,\infty),(0,\infty))$ and a positive constant E such that

$$\int_{-\infty}^{t} |g(t,s)|\, ds \le L(t) \text{ and } |f(x)-f(y)| \le E|x-y|,$$

for any $x, y \in \mathbb{R}$.

(H_3) a is bounded on $[0,\infty)$, and $\liminf_{t\to\infty} \int_0^t a(s)\, ds > -\infty$.

(H_4) There exists $\alpha \in (0,1)$ such that for $t \in [0,\infty)$,

$$\int_0^t e^{-\int_u^t a(s)ds} [|c(u)| + EL(u)]\, du \le \alpha,$$

and

$$|a(t)| \int_0^t e^{-\int_u^t a(s)ds} [|c(u)| + EL(u)]\, du + |c(t)| + EL(t) \le \alpha.$$

For each $t_0 \in [0,\infty)$, denote $C_{t_0}^1 = C^1((-\infty,t_0],\mathbb{R})$ with the norm defined by

$$|x|_{t_0} = \max_{t\in(-\infty,t_0]} \{|x(t)|, |x'(t)|\},$$

for $x \in C_{t_0}^1$. In addition, denote

$$\Phi_{t_0} = \{\varphi \in C_{t_0}^1 : \varphi'_-(t_0) = -a(t_0)\varphi(t_0) + c(t_0)\varphi'(t_0-\tau(t_0)) + \int_{-\infty}^{t_0} g(t_0,s) f(\varphi(s))\, ds\}.$$

For each $t_0 \in [0,\infty)$, we always assume that the initial function for (1) is of the type $\varphi \in \Phi_{t_0}$. For convenience of stating our main result, we shall give the following definitions.

Definition 1. *For each $(t_0,\varphi) \in [0,\infty) \times \Phi_{t_0}$, x is said to be a solution of (1) through (t_0,φ) if $x \in C^1(\mathbb{R})$ satisfies (1) on $[0,\infty)$ and $x(t) = \varphi(t)$ for $t \in (-\infty,t_0]$. We denote such a solution by $x(t) = x(t,t_0,\varphi)$.*

Definition 2. (i) *The zero solution of (1) is said to be stable in C^1 if, for any $t_0 \in [0,\infty)$, $\epsilon > 0$, there is a $\delta = \delta(\epsilon,t_0)$ such that $\varphi \in \Phi_{t_0}$ and $|\varphi|_{t_0} < \delta$ implies*

$$\max_{s\in(-\infty,t]} \{|x(s)|, |x'(s)|\} < \epsilon,$$

for $t \ge t_0$.

(ii) *The zero solution of (1) is said to be globally asymptotically stable in C^1 if it is stable in C^1, and for any $t_0 \in [0,\infty)$, $\varphi \in \Phi_{t_0}$ implies*

$$\lim_{t\to\infty} x(t,t_0,\varphi) = \lim_{t\to\infty} x'(t,t_0,\varphi) = 0.$$

In view of the definition of solution of (1), it is clear that the conditions imposed on the initial functions are very natural. From the above assumptions, it is easy to see that for each $(t_0,\varphi) \in [0,\infty) \times \Phi_{t_0}$, there exists a unique solution $x(t) = x(t,t_0,\varphi)$ of (1) defined on $\mathbb{R}$. By (H_2), (1) has the zero solution.

Our results are obtained with no need of further assumptions on the differentiable of the neutral coefficient c and the twice differentiable of τ with $\tau'(t) \ne 1$ for $t \in [0,\infty)$, so that for a given initial function $\varphi \in \Phi_{t_0}$ a mapping P for (1) is constructed in such a way to map a, carefully chosen, closed convex nonempty subset D of a Banach space X into itself on which P is a contraction mapping possessing a fixed point. This procedure will enable us to establish and prove by means of the contraction mapping theorem ([27], p. 2) the global asymptotic stability in C^1 for the zero solution of (1) with a less restrictive conditions. An example is also given to illustrate our results.

2. Global asymptotic stability

In this section, we shall give the global asymptotic stability in C^1 of the zero solution to (1).

Theorem 1. *Assume that $(H_1)-(H_4)$ hold. Then the zero solution of (1) is globally asymptotically stable in C^1 if and only if*

$$\int_0^\infty a(s)\,ds = \infty. \tag{2}$$

Proof. (*i*) Suppose that (2) holds. For any $t_0 \in [0,\infty)$, let

$$X = \left\{x \in C^1(\mathbb{R}) : \lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0\right\},$$

with the norm defined by

$$\|x\|_{t_0} = \max_{t\in\mathbb{R}} \{|x(t)|, |x'(t)|\},$$

for $x \in X$. Then X is a Banach space. For any $\varphi \in \Phi_{t_0}$, let

$$D = \{x \in X : x(t) = \varphi(t) \text{ for } t \in (-\infty, t_0]\}.$$

It is easy to see that D is a nonempty, closed convex subset of X.

Multiply both sides of (1) by $e^{\int_{t_0}^t a(s)ds}$ and then integrate from t_0 to t to obtain

$$\begin{aligned} x(t) &= \varphi(t_0)\,e^{-\int_{t_0}^t a(s)ds} \\ &+ \int_{t_0}^t e^{-\int_u^t a(s)ds}\left[c(u)\,x'(u-\tau(u)) + \int_{-\infty}^u g(u,s)\,f(x(s))\,ds\right]du. \end{aligned} \tag{3}$$

Use (3) to define the operator $P : D \to C(\mathbb{R})$ by $(Px)(t) = \varphi(t)$ for $t \in (-\infty, t_0]$ and

$$\begin{aligned} (Px)(t) &= \varphi(t_0)\,e^{-\int_{t_0}^t a(s)ds} \\ &+ \int_{t_0}^t e^{-\int_u^t a(s)ds}\left[c(u)\,x'(u-\tau(u)) + \int_{-\infty}^u g(u,s)\,f(x(s))\,ds\right]du, \end{aligned} \tag{4}$$

for $t \in [t_0, \infty)$.

Firstly, we prove $P : D \to D$. From (4), for $t > t_0$,

$$\begin{aligned} (Px)'(t) &= -\varphi(t_0)\,a(t)\,e^{-\int_{t_0}^t a(s)ds} + c(t)\,x'(t-\tau(t)) + \int_{-\infty}^t g(t,s)\,f(x(s))\,ds \\ &- a(t)\int_{t_0}^t e^{-\int_u^t a(s)ds}\left[c(u)\,x'(u-\tau(u)) + \int_{-\infty}^u g(u,s)\,f(x(s))\,ds\right]du \\ &= -a(t)\,(Px)(t) + c(t)\,x'(t-\tau(t)) + \int_{-\infty}^t g(t,s)\,f(x(s))\,ds. \end{aligned} \tag{5}$$

By the definition of Φ_{t_0}, (5) yields

$$(Px)'_+(t_0) = -a(t_0)\,\varphi(t_0) + c(t_0)\,\varphi'(t_0-\tau(t_0)) + \int_{-\infty}^{t_0} g(t_0,s)\,f(\varphi(s))\,ds = \varphi'_-(t_0).$$

Hence, $Px \in C^1(\mathbb{R})$ for $x \in D$.

For $x \in D$, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$. Note that $\lim_{t\to\infty} t-\tau(t) = \infty$. Therefore, for any $\epsilon > 0$, there exists $T > 0$ such that for $t \geq T$,

$$\max\{|x(t)|, |x(t-\tau(t))|, |x'(t-\tau(t))|\} < \epsilon. \tag{6}$$

It follows from (4), (6) and (H_2) and (H_4) that for $t \geq T$ and $x \in D$,

$$\begin{aligned}
\left|(Px)(t)\right| &= \left|\varphi(t_0)\right| e^{-\int_{t_0}^{t} a(s)ds} \\
&+ \int_{t_0}^{T} e^{-\int_{u}^{t} a(s)ds} \left| c(u)\, x'(u-\tau(u)) + \int_{-\infty}^{u} g(u,s) f(x(s))\, ds \right| du \\
&+ \int_{T}^{t} e^{-\int_{u}^{t} a(s)ds} \left| c(u)\, x'(u-\tau(u)) + \int_{-\infty}^{u} g(u,s) f(x(s))\, ds \right| du \\
&\leq e^{-\int_{t_0}^{t} a(s)ds} \left[\left|\varphi(t_0)\right| + \int_{t_0}^{T} e^{\int_{t_0}^{u} a(s)ds} \left| c(u)\, x'(u-\tau(u)) \right.\right. \\
&\left.\left. + \int_{-\infty}^{u} g(u,s) f(x(s))\, ds \right| du \right] \\
&+ \int_{T}^{t} e^{-\int_{u}^{t} a(s)ds} \left[\left|c(u)\right| \left|x'(u-\tau(u))\right| + E \int_{-\infty}^{u} \left|g(u,s)\right| \left|x(s)\right| ds \right] du \\
&\leq e^{-\int_{t_0}^{t} a(s)ds} \left[\left|\varphi(t_0)\right| + \int_{t_0}^{T} e^{\int_{t_0}^{u} a(s)ds} \left| c(u)\, x'(u-\tau(u)) \right.\right. \\
&\left.\left. + \int_{-\infty}^{u} g(u,s) f(x(s))\, ds \right| du \right] \\
&+ \epsilon \int_{T}^{t} e^{-\int_{u}^{t} a(s)ds} \left[\left|c(u)\right| + EL(u) \right] du \\
&\leq e^{-\int_{t_0}^{t} a(s)ds} \left[\left|\varphi(t_0)\right| + \int_{t_0}^{T} e^{\int_{t_0}^{u} a(s)ds} \left| c(u)\, x'(u-\tau(u)) \right.\right. \\
&\left.\left. + \int_{-\infty}^{u} g(u,s) f(x(s))\, ds \right| du \right] + \alpha\epsilon.
\end{aligned}$$

From (2), there exists $T_1 > T$ such that for $t > T_1$,

$$e^{-\int_{t_0}^{t} a(s)ds} \left[\left|\varphi(t_0)\right| + \int_{t_0}^{T} e^{\int_{t_0}^{u} a(s)ds} \left| c(u)\, x'(u-\tau(u)) + \int_{-\infty}^{u} g(u,s) f(x(s))\, ds \right| du \right] < \epsilon.$$

Hence, $\lim_{t\to\infty} (Px)(t) = 0$ for $x \in D$. In addition, it follows from (5) and (H_2) that

$$\begin{aligned}
\left|(Px)'(t)\right| &\leq \left|a(t)(Px)(t)\right| + \left|c(t)\, x'(t-\tau(t))\right| + \left| \int_{-\infty}^{t} g(t,s) f(x(s))\, ds \right| \\
&\leq \left|a(t)(Px)(t)\right| + \left|c(t)\right| \left|x'(t-\tau(t))\right| + E \int_{-\infty}^{t} \left|g(t,s)\right| \left|x(s)\right| ds.
\end{aligned}$$

This, together with (H_3) and (H_4), yields $\lim_{t\to\infty} (Px)'(t) = 0$ for $x \in D$. Therefore, $Px \in D$ for $x \in D$, i.e. $P : D \to D$.

Secondly, we show that $P : D \to D$ is a contraction mapping. For any $x, y \in D$, it follows from (4), (H_2) and (H_4) that for $t \in [t_0, \infty)$,

$$\begin{aligned}
&\left|(Px)(t) - (Py)(t)\right| \\
&\leq \int_{t_0}^{t} e^{-\int_u^t a(s)ds} \left[\left|c(u)\right| \left|x'(u - \tau(u)) - y'(u - \tau(u))\right| \right. \\
&\left. + \int_{-\infty}^{u} \left|g(u,s)\right| \left|f(x(s)) - f(y(s))\right| ds \right] du \\
&\leq \|x - y\|_{t_0} \int_{t_0}^{t} e^{-\int_u^t a(s)ds} \left[\left|c(u)\right| + EL(u)\right] du \\
&\leq \alpha \|x - y\|_{t_0}.
\end{aligned} \tag{7}$$

In addition, it follows from (5), (7), (H_2) and (H_4) that for $t \in [t_0, \infty)$,

$$\begin{aligned}
&\left|(Px)'(t) - (Py)'(t)\right| \\
&\leq \left|a(t)\right| \left|(Px)(t) - (Py)(t)\right| + \left|c(t)\right| \left|x'(t - \tau(t)) - y'(t - \tau(t))\right| \\
&+ \int_{-\infty}^{t} \left|g(t,s)\right| \left|f(x(s)) - f(y(s))\right| ds \\
&\leq \|x - y\|_{t_0} \left\{ \left|a(t)\right| \int_{t_0}^{t} e^{-\int_u^t a(s)ds} \left[\left|c(u)\right| + EL(u)\right] du + \left|c(t)\right| + EL(t) \right\} \\
&\leq \alpha \|x - y\|_{t_0}.
\end{aligned} \tag{8}$$

From (7) and (8), $P : D \to D$ is a contraction mapping. By the contraction mapping principle, P has a unique fixed point x in D, which is a unique solution of (1) through (t_0, φ) and satisfies

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0. \tag{9}$$

Finally, we show that the zero solution of (1) is stable in C^1. Let

$$K = \sup_{t\in[t_0,\infty)} \left\{ e^{-\int_{t_0}^{t} a(s)ds} \right\} \text{ and } A = \sup_{t\in[t_0,\infty)} \left\{\left|a(t)\right|\right\}.$$

From (2) and (H_3), $K, A \in (0, \infty)$. For any $\epsilon > 0$, let $\delta > 0$ such that

$$\delta < \epsilon \min\left\{1, \frac{1-\alpha}{K}, \frac{1-\alpha}{KA}\right\}.$$

If $x(t) = x(t, t_0, \varphi)$ is a solution of (1) with $|\varphi|_{t_0} < \delta$, then $x(t) = (Px)(t)$ on $[t_0, \infty)$. We claim that $\|x\|_{t_0} < \epsilon$. Otherwise, there exists $t_1 > t_0$ such that

$$\max\left\{\left|x(t_1)\right|, \left|x'(t_1)\right|\right\} = \epsilon,$$

and

$$\max\left\{\left|x(t)\right|, \left|x'(t)\right|\right\} < \epsilon,$$

for $t \in (-\infty, t_1]$. If $|x(t_1)| = \epsilon$, then it follows from (4), (H_2) and (H_4) that

$$\begin{aligned}
\left|x(t_1)\right| &\leq \left|\varphi(t_0)\right| e^{-\int_{t_0}^{t_1} a(s)ds} \\
&+ \int_{t_0}^{t_1} e^{-\int_u^{t_1} a(s)ds} \left| c(u)\, x'(u - \tau(u)) + \int_{-\infty}^{u} g(u,s)\, f(x(s))\, ds \right| du \\
&\leq K\delta + \epsilon \int_{t_0}^{t_1} e^{-\int_u^{t_1} a(s)ds} \left[\left|c(u)\right| + EL(u)\right] du \\
&\leq K\delta + \alpha\epsilon < \epsilon.
\end{aligned}$$

This is a contradiction. If $|x'(t_1)| = \epsilon$, then it follows from (5), (H_2) and (H_4) that

$$\begin{aligned}|x'(t_1)| &\le |\varphi(t_0) a(t_1)| e^{-\int_{t_0}^{t_1} a(s)ds} + |c(t_1) x'(t_1 - \tau(t_1))| + \left|\int_{-\infty}^{t_1} g(t_1, s) f(x(s)) ds\right| \\ &\quad + |a(t_1)| \int_{t_0}^{t_1} e^{-\int_u^{t_1} a(s)ds} \left|c(u) x'(u - \tau(u)) + \int_{-\infty}^{u} g(u, s) f(x(s)) ds\right| du \\ &\le KA\delta + \epsilon \left\{|a(t_1)| \int_{t_0}^{t_1} e^{-\int_u^{t_1} a(s)ds} [|c(u)| + EL(u)] du + |c(t_1)| + EL(t_1)\right\} \\ &\le KA\delta + \alpha\epsilon < \epsilon.\end{aligned}$$

This is also a contradiction. Hence, the zero solution of (1) is stable in C^1. This, together with (9), implies that the zero solution of (1) is globally asymptotically stable in C^1.

(ii) Assume that the zero solution of (1) is globally asymptotically stable in C^1. Now we prove that (2) holds. Otherwise, set

$$l = \liminf_{t\to\infty} \int_0^t a(s) ds, \; K_0 = \sup_{t\in[0,\infty)} \left\{e^{-\int_0^t a(s)ds}\right\} \text{ and } A_0 = \sup_{t\in[0,\infty)} \{|a(t)|\},$$

thus it follows from (H_3) that $l \in (-\infty, \infty)$, $K_0 \in (0, \infty)$, $A_0 \in [0, \infty)$. Hence, there exists an increasing sequence $\{t_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} \int_0^{t_n} a(s) ds = l. \tag{10}$$

Denote

$$I_n = \int_0^{t_n} e^{\int_0^u a(s)ds} [|c(u)| + EL(u)] du, \; n = 1, 2, ...$$

From (H_4), it follows that

$$I_n = e^{\int_0^{t_n} a(s)ds} \int_0^{t_n} e^{-\int_u^{t_n} a(s)ds} [|c(u)| + EL(u)] du \le \alpha e^{\int_0^{t_n} a(s)ds}.$$

This, together with (10), implies that the sequence $\{I_n\}$ is bounded. Further, there exists a convergent subsequence. For brevity of notation, we may assume that $\{I_n\}$ is convergent. Therefore, there exists a positive integer m such that for any integer $n > m$,

$$\int_{t_m}^{t_n} e^{\int_0^u a(s)ds} [|c(u)| + EL(u)] du < \frac{1-\alpha}{8B(e^{-l} + 1)}, \tag{11}$$

and

$$e^{-\int_{t_m}^{t_n} a(s)ds} > \frac{1}{2}, \; e^{-\int_0^{t_n} a(s)ds} < e^{-l} + 1, \; e^{\int_0^{t_m} a(s)ds} < e^l + 1, \tag{12}$$

where $B = \max\left\{K_0\left(e^l + 1\right), K_0 A_0\left(e^l + 1\right), 1\right\}$.

For any $\delta > 0$, consider the solution $x(t) = x(t, t_m, \varphi)$ of (1) with $|\varphi|_{t_m} < \delta$ and $|\varphi(t_m)| > \delta/2$. It follows from (4), (5), (12), (H_2) and (H_4) that for $t \in [t_m, \infty)$,

$$\begin{aligned}
|x(t)| &\le |\varphi(t_m)| e^{-\int_{t_m}^t a(s)ds} \\
&\quad + \int_{t_m}^t e^{-\int_u^t a(s)ds} \left| c(u) x'(u - \tau(u)) + \int_{-\infty}^u g(u,s) f(x(s)) ds \right| du \\
&\le |\varphi(t_m)| e^{-\int_0^t a(s)ds} e^{\int_0^{t_m} a(s)ds} + \|x\|_{t_m} \int_{t_m}^t e^{-\int_u^t a(s)ds} [|c(u)| + EL(u)] du \\
&\le K_0 \left(e^l + 1\right) \delta + \|x\|_{t_m} \int_0^t e^{-\int_u^t a(s)ds} [|c(u)| + EL(u)] du \\
&\le B\delta + \alpha \|x\|_{t_m},
\end{aligned}$$

and

$$\begin{aligned}
|x'(t)| &\le |\varphi(t_m) a(t)| e^{-\int_{t_m}^t a(s)ds} + |c(t) x'(t - \tau(t))| + \left| \int_{-\infty}^t g(t,s) f(x(s)) ds \right| \\
&\quad + |a(t)| \int_{t_m}^t e^{-\int_u^t a(s)ds} \left| c(u) x'(u - \tau(u)) + \int_{-\infty}^u g(u,s) f(x(s)) ds \right| du \\
&\le K_0 A_0 \left(e^l + 1\right) \delta \\
&\quad + \|x\|_{t_m} \left\{ |a(t)| \int_{t_m}^t e^{-\int_u^t a(s)ds} [|c(u)| + EL(u)] du + |c(t)| + EL(t) \right\} \\
&\le B\delta + \alpha \|x\|_{t_m}.
\end{aligned}$$

Hence,

$$\|x\|_{t_m} \le B\delta + \alpha \|x\|_{t_m}, \text{ i.e. } \|x\|_{t_m} \le \frac{B}{1-\alpha}\delta. \tag{13}$$

It follows from (4), (11)–(13) and (H_2) that for any $n > m$,

$$\begin{aligned}
|x(t_n)| &\ge |\varphi(t_m)| e^{-\int_{t_m}^{t_n} a(s)ds} \\
&\quad - e^{-\int_0^{t_n} a(s)ds} \int_{t_m}^{t_n} e^{\int_0^u a(s)ds} \left| c(u) x'(u - \tau(u)) + \int_{-\infty}^u g(u,s) f(x(s)) ds \right| du \\
&\ge |\varphi(t_m)| e^{-\int_{t_m}^{t_n} a(s)ds} - \|x\|_{t_m} e^{-\int_0^{t_n} a(s)ds} \int_{t_m}^{t_n} e^{\int_0^u a(s)ds} [|c(u)| + EL(u)] du \\
&> \frac{1}{4}\delta - \frac{B}{1-\alpha}\delta \left(e^{-l} + 1\right) \frac{1-\alpha}{8B\left(e^{-l}+1\right)} = \frac{1}{8}\delta.
\end{aligned}$$

This contradicts the fact that $\lim_{n\to\infty} t_n = \infty$ and the zero solution of (1) is globally asymptotically stable in C^1. The proof is complete. □

Example 1. *In (1), let* $a(t) = 1/(1+t)$, $c(t) = 1/[8(1+t)]$, $\tau(t) = 2 + \sin t$, $g(t,s) = 2e^{s-t}/(1+t)$ *and* $f(x) = \ln\left(1 + \frac{|x|}{8}\right)$. *Hence* $a, c \in C([0,\infty), \mathbb{R})$, $g \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ *and* $\tau \in C([0,\infty), (0,\infty))$ *with* $t - \tau(t) \to \infty$ *as* $t \to \infty$. *A straightforward calculation shows that for* $t \in [0,\infty)$, $|a(t)| \le 1$, $\int_0^\infty a(s) ds = \infty$. *Let* $E = 1/8$ *and* $L(t) = 2/(1+t)$, *then* (H_2) *holds. In addition, let* $\alpha = 3/4$, *then for* $t \in [0,\infty)$,

$$\int_0^t e^{-\int_u^t a(s)ds} [|c(u)| + EL(u)] du = \int_0^t \frac{1+u}{1+t} \frac{3}{8(1+u)} du = \frac{3}{8} \frac{t}{1+t} \le \alpha,$$

and

$$|a(t)| \int_0^t e^{-\int_u^t a(s)ds} [|c(u)| + EL(u)] \, du + |c(t)| + EL(t) \leq \frac{3}{8} + \frac{3}{8(1+t)} \leq \alpha.$$

Hence, $(H_1) - (H_4)$ *hold. From Theorem 1, the zero solution of (1) is globally asymptotically stable in* C^1*.*

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