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APPROXIMATION OF FUZZY NUMBERS BY MAX-PRODUCT OPERATORS

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Abstract. Here we study quantitatively the approximation of fuzzy numbers by fuzzy approximators generated by the Max-product operators of Bernstein type and Meyer-Köning and Zeller type.

1. Background

We need the following

Definition 1. *(see [7]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*
(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.
(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ (μ is called a convex fuzzy subset).
(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\ x_0 \in \mathbb{R}$ and $\forall\ \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.
(iv) The set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).
We call μ a fuzzy real number, or fuzzy number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.
E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$ ([4]). For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \forall\ r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [7]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$, then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$. For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\} \tag{1}$$
$$= \sup_{r \in [0,1]} \text{Hausdorff distance } ([u]^r, [v]^r),$$

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where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [7], [8], with the properties

$$\begin{gathered} D(u \oplus w, v \oplus w) = D(u,v), \ \ \forall\ u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) = |k|\, D(u,v), \ \ \forall\ u, v \in \mathbb{R}_{\mathcal{F}},\ \forall\ k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e),\ \forall\ u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{gathered} \tag{2}$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$" (or "$\preceq$"): $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ (or $u \preceq v$) iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\ r \in [0,1]$.

The zero element $\widetilde{0} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\widetilde{0} := \chi_{\{0\}}$, clearly it holds $\widetilde{0}_{\pm}^{(r)} = 0$, $\forall\ r \in [0,1]$.

We call $u \in \mathbb{R}_{\mathcal{F}}$ positive, iff $u \succeq \widetilde{0}$, iff $u_-^{(r)} \geq 0$ and $u_+^{(r)} \geq 0$, $\forall\ r \in [0,1]$.

From now on we denote $u_-^{(r)} := u^-(r)$ and $u_+^{(r)} := u^+(r)$, $\forall\ r \in [0,1]$. Actually we have that $u^-, u^+ : [0,1] \to \mathbb{R}$, furthermore if $u \in \mathbb{R}_{\mathcal{F}}$ is positive then we get that

$$u^-, u^+ : [0,1] \to \mathbb{R}_+.$$

We mention the important characterization.

Theorem 1. *(Goetschel and Voxman [4]) Let $u \in \mathbb{R}_{\mathcal{F}}$. Then*
(1) u^- is a bounded increasing function on $[0,1]$,
(2) u^+ is a bounded decreasing function on $[0,1]$.
(3) $u^-(1) \leq u^+(1)$,
(4) u^- and u^+ are left continuous on $(0,1]$ and right continuous at 0.
(5) If u^-, u^+ satisfy the above conditions (1)-(4), then there exists a unique $\upsilon \in \mathbb{R}_{\mathcal{F}}$ such that $\upsilon^-(r) = u^-(r)$ and $\upsilon^+(r) = u^+(r)$, $\forall\ r \in [0,1]$.

Theorem 1 says that a fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ is completely determined by the end points of the intervals $[u]^r = [u^-(r), u^+(r)]$, $\forall\ r \in [0,1]$. Therefore we can identify a fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ with its parametric representation $\{(u^-(r), u^+(r))\, |0 \leq r \leq 1\}$, and we can write $u = (u^-, u^+)$, and we call u^-, u^+ the level functions of u.

In this article we deal only with positive fuzzy numbers.

Define $C_+([0,1]) := \{f : [0,1] \to \mathbb{R}_+, \text{ continuous functions}\}$.

In [1], p. 10, the authors introduced the Max-product Bernstein operators

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x)\, f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad N \in \mathbb{N}, \tag{3}$$

where $\bigvee$ stands for maximum, and $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$, $f \in C_+([0,1])$, $\forall$ $x \in [0,1]$.

These are nonlinear and piecewise rational operators.

We notice that $B_N^M(1) = 1$, furthermore $B_N^{(M)}$ maps $C_+([0,1])$ into itself, $B_N^M(f)(x) \geq 0$, and satisfies

$$B_N^M(f)(0) = f(0), \quad B_N^M(f)(1) = f(1), \quad \forall\ N \in \mathbb{N}, \tag{4}$$

where $f \in C_+([0,1])$, see [1], p. 39.

Additionally we have (see [1], p. 40): if $f : [0,1] \to \mathbb{R}_+$ is nondecreasing, then $B_N^M(f)$ is nondecreasing, and if $f : [0,1] \to \mathbb{R}_+$ is nonincreasing, then $B_N^{(M)}(f)$ is nonincreasing, $\forall\ N \in \mathbb{N}$.

Next let $u \in \mathbb{R}_{\mathcal{F}}$ be a positive fuzzy number: $u = (u^-, u^+)$. We consider $B_N^{(M)}(u^-)$, $B_N^{(M)}(u^+)$ and since $B_N^{(M)}$ preserves the monotonicity it follows that $B_N^{(M)}(u^-)$ is increasing and $B_N^{(M)}(u^+)$ is decreasing over $[0,1]$. We assume that $u^\pm$ are continuous, thus $B_N^{(M)}(u^\pm)$ are continuous too.

We further have

$$B_N^{(M)}\left(u^\pm\right)(0) = u^\pm(0), \tag{5}$$

$$B_N^{(M)}\left(u^\pm\right)(1) = u^\pm(1), \text{ respectively, } \forall\ N \in \mathbb{N}.$$

Also we have that

$$B_N^{(M)}\left(u^-\right)(1) = u^-(1) \le u^+(1) = B_N^{(M)}\left(u^+\right)(1), \ \ \forall\ N \in \mathbb{N}. \tag{6}$$

In conclusion (by Theorem 1)

$$\overline{B}_N^{(M)}(u) := \left(B_N^{(M)}\left(u^-\right), B_N^{(M)}\left(u^+\right)\right), \tag{7}$$

defines a proper fuzzy number in $\mathbb{R}_{\mathcal{F}}$, $\forall\ N \in \mathbb{N}$.

We mention

Theorem 2. *(Bede-Coroianu-Gal,* [1], *p. 111) Let $u = (u^-, u^+)$ be a positive fuzzy number with the level functions u^- and u^+ continuous. Then, denoting $u_N := \left(u_N^-, u_N^+\right) = \overline{B}_N^{(M)}(u)$, we have*

$$D\left(\overline{B}_N^{(M)}(u), u\right) \le 12 \max\left\{\omega_1\left(u^-, \frac{1}{\sqrt{N+1}}\right), \omega_1\left(u^+, \frac{1}{\sqrt{N+1}}\right)\right\}, \tag{8}$$

$\forall\ N \in \mathbb{N}$, where for $f \in C_+([0,1])$:

$$\omega_1(f, \delta) := \sup_{\substack{x,y \in [0,1] \\ |x-y| \le \delta}} |f(x) - f(y)|, \ \ \delta > 0, \tag{9}$$

is the first modulus of continuity of f.

In this article we study further the approximation of positive fuzzy numbers by Max-product operators generated sequences of positive fuzzy numbers.

2. Main Results

In [1], p. 11, the authors mentioned the Max-product Meyer-Köning and Zeller operators defined by

$$Z_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{N,k}(x) f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{\infty} s_{N,k}(x)}, \ \ \forall\ N \in \mathbb{N},\ f \in C_+([0,1]), \tag{10}$$

$s_{N,k}(x) = \begin{pmatrix} N+k \\ k \end{pmatrix} x^k$, $x \in [0,1]$.

It holds $Z_N^{(M)}(1) = 1$, and $Z_N^{(M)}$ maps $C_+([0,1])$ into itself.

We mention

Theorem 3. *(*[1]*, p. 248) Let $f \in C_+([0,1])$. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \le 18\omega_1\left(f, \frac{(1-x)\sqrt{x}}{\sqrt{N}}\right), \tag{11}$$

$\forall\ N \in \mathbb{N}$, $N \ge 4$, $\forall\ x \in [0,1]$.

We need

Lemma 1. *([1], p. 257) For any bounded function $f:[0,1)\to\mathbb{R}_+$, the Max-product operator $Z_N^{(M)}(f)(x)$ is nonnegative, bounded, continuous on $[0,1)$ and satisfies $Z_N^{(M)}(f)(0)=f(0)$, $\forall\ N\in\mathbb{N}$. If, in addition, f is supposed to be defined and continuous on $[0,1]$, then $Z_N(f)(x)$ is continuous at $x=1$ too, and $Z_N^{(M)}(f)(1)=f(1)$, $\forall\ N\in\mathbb{N}$.*

We need

Theorem 4. *([1], p. 259) If $f:[0,1]\to\mathbb{R}_+$ is nondecreasing and continuous on $[0,1]$, then $Z_N^{(M)}(f)$ is nondecreasing and continuous on $[0,1]$.*

We need

Corollary 1. *([1], p. 259) If $f:[0,1]\to\mathbb{R}_+$ is continuous and nonincreasing on $[0,1]$, then $Z_N^{(M)}(f)$ is continuous and nonincreasing on $[0,1]$.*

Next let $u\in\mathbb{R}_{\mathcal{F}}$ be a positive fuzzy number: $u=(u^-,u^+)$. We consider $Z_N^{(M)}(u^-)$, $Z_N^{(M)}(u^+)$ and since $Z_N^{(M)}$ preserves the monotonicity, it follows that $Z_N^{(M)}(u^-)$ is increasing and $Z_N^{(M)}(u^+)$ is decreasing over $[0,1]$. We assume that $u^\pm$ are continuous, thus $Z_N^{(M)}(u^\pm)$ are continuous too.

We further have

$$Z_N^{(M)}\left(u^\pm\right)(0)=u^\pm(0), \tag{12}$$

$$Z_N^{(M)}\left(u^\pm\right)(1)=u^\pm(1)\text{, respectively, }\forall\ N\in\mathbb{N}.$$

Also we have that

$$Z_N^{(M)}\left(u^-\right)(1)=u^-(1)\le u^+(1)=Z_N^{(M)}\left(u^+\right)(1),\ \ \forall\ N\in\mathbb{N}. \tag{13}$$

In conclusion (by Theorem 1)

$$\overline{Z}_N^{(M)}(u):=\left(Z_N^{(M)}\left(u^-\right),Z_N^{(M)}\left(u^+\right)\right), \tag{14}$$

defines a proper fuzzy number in $\mathbb{R}_{\mathcal{F}}$, $\forall\ N\in\mathbb{N}$.

We present

Theorem 5. *Let $u=(u^-,u^+)$ be a positive fuzzy number with the level functions u^- and u^+ continuous. Then, denoting $u_N:=\left(u_N^-,u_N^+\right)=\overline{Z}_N^{(M)}(u)$, we have*

$$D\left(\overline{Z}_N^{(M)}(u),u\right)\le 18\max\left\{\omega_1\left(u^-,\frac{2}{3\sqrt{3N}}\right),\omega_1\left(u^+,\frac{2}{3\sqrt{3N}}\right)\right\}, \tag{15}$$

$\forall\ N\in\mathbb{N}:N\ge 4$.

Proof. We use (1) and (11). We notice the following: let $g(x)=(1-x)\sqrt{x}$, $x\in(0,1]$, then $g'(x)=-\sqrt{x}+(1-x)\frac{1}{2\sqrt{x}}$, setting $g'(x)=0$ we get the only critical number $x=\frac{1}{3}\in(0,1]$. Furthermore we have $g''(x)=-\left(\frac{1}{\sqrt{x}}+\frac{(1-x)}{4}x^{-\frac{3}{2}}\right)$, $x\in(0,1]$ and $g''\left(\frac{1}{3}\right)<0$. Therefore $g(x)$ has an absolute maximum over $(0,1]:g\left(\frac{1}{3}\right)=\frac{2}{3\sqrt{3}}$. ☐

We need

Definition 2. *(see also [1], pp. 20-21) The expected interval of a fuzzy number $u\in\mathbb{R}_{\mathcal{F}}$ was introduced by Dubois and Prade ([3]) and Heilpern ([6]). It is the real interval*

$$EI(u)=\left[EI_*(u),EI^*(u)\right]=\left[\int_0^1u^-(r)\,dr,\int_0^1u^+(r)\,dr\right]. \tag{16}$$

The expected value of u is given by

$$EV(u) = \frac{1}{2}\left(\int_0^1 u^-(r)\,dr + \int_0^1 u^+(r)\,dr\right).$$

A reducing function ([2]) is a nondecreasing continuous function $s : [0,1] \to [0,1]$ with the property that $s(0) = 0$ and $s(1) = 1$. Let $u \in \mathbb{R}_{\mathcal{F}}$, the ambiguity of u with respect to s is defined by

$$Amb_s(u) = \int_0^1 s(r)\left(u^+(r) - u^-(r)\right)dr, \tag{17}$$

and the value of u with respect to s is given by

$$Val_s(u) = \int_0^1 s(r)\left(u^+(r) + u^-(r)\right)dr. \tag{18}$$

If for fixed $k \in \mathbb{N}$ we have $s_k(r) = r^k$, $r \in [0,1]$, then we denote $Amb_{s_k}(u) = Amb_k(u)$ and $Val_{s_k}(u) = Val_k(u)$, i.e.

$$Amb_k(u) = \int_0^1 r^k\left(u^+(r) - u^-(r)\right)dr, \tag{19}$$

and

$$Val_k(u) = \int_0^1 r^k\left(u^+(r) + u^-(r)\right)dr. \tag{20}$$

The width or the non-specificity of $u \in \mathbb{R}_{\mathcal{F}}$ is given by

$$width(u) = \int_0^1 \left(u^+(r) - u^-(r)\right)dr. \tag{21}$$

We give

Theorem 6. *Same assumptions and notations as in Theorem 5. Then*

$$EI(u_N) \to EI(u), \tag{22}$$

$$width(u_N) \to width(u), \tag{23}$$

and

$$Amb_s(u_N) \to Amb_s(u), \tag{24}$$

$$Amb_k(u_N) \to Amb_k(u), \quad k \in \mathbb{N}, \tag{25}$$

where $s : [0,1] \to [0,1]$ is a reduction function.

Proof. Similar to [1], p. 112. Indeed for $u_N := \overline{Z}_N^{(M)}(u) = \left(u_N^-, u_N^+\right)$, in order to obtain the required convergence of the expected interval, width, ambiguity and of the expected value of u_N, it is enough to prove that

$$\lim_{N\to\infty} \int_0^1 s(r)\,u_N^-(r)\,dr = \int_0^1 s(r)\,u^-(r)\,dr, \tag{26}$$

and

$$\lim_{N\to\infty} \int_0^1 s(r)\,u_N^+(r)\,dr = \int_0^1 s(r)\,u^+(r)\,dr, \tag{27}$$

for any reducing function s and in particular for $s(r) = r^k$, $k \in \mathbb{N} \cup \{0\}$.

Indeed, taking $s(r) = r^0 = 1$, we easily get the convergence of the expected interval and of the width. For any $N \in \mathbb{N}$, we get

$$\left|\int_0^1 s(r)\,u_N^-(r)\,dr - \int_0^1 s(r)\,u^-(r)\,dr\right| \leq \tag{28}$$

$$s(1)\int_0^1 \left|u_N^{(r)}(r)-u^{(r)}(r)\right| dr \le D\left(\overline{Z}_N^{(M)}(u),u\right),$$

which by (15) implies that

$$\lim_{N\to\infty}\int_0^1 s(r)\,u_N^-(r)\,dr=\int_0^1 s(r)\,u^-(r)\,dr.$$

The proof of (27) as totally similar to (26) is omitted. □

We need

Theorem 7. *([1], p. 30) Let $f\in C_+([0,1])$. Then*

$$\left|B_N^{(M)}(f)(x)-f(x)\right|\le 12\omega_1\left(f,\frac{1}{\sqrt{N+1}}\right),\quad \forall\ N\in\mathbb{N},\ \forall\ x\in[0,1]. \tag{29}$$

We also need

Corollary 2. *([1], p. 36) Let $f\in C_+([0,1])$ which is concave. Then*

$$\left|B_N^{(M)}(f)(x)-f(x)\right|\le 2\omega_1\left(f,\frac{1}{N}\right),\quad \forall\ N\in\mathbb{N},\ \forall\ x\in[0,1]. \tag{30}$$

Let $u,\upsilon\in\mathbb{R}_{\mathcal{F}}$, $p\ge1$, an L_p-metric (see [5]) is given by

$$d_p(u,\upsilon)=\left(\int_0^1\left(\left|u^-(r)-\upsilon^-(r)\right|^p+\left|u^+(r)-\upsilon^+(r)\right|^p\right)dr\right)^{\frac{1}{p}}. \tag{31}$$

We have that

$$d_p\left(\overline{B}_N^{(M)}(u),u\right)=$$

$$\left(\int_0^1\left(\left|\left(\overline{B}_N^{(M)}(u^-)\right)(r)-u^-(r)\right|^p+\left|\left(\overline{B}_N^{(M)}(u^+)\right)(r)-u^+(r)\right|^p\right)dr\right)^{\frac{1}{p}}\overset{(29)}{\le} \tag{32}$$

$$12\left(\int_0^1\left(\left(\omega_1\left(u^-,\frac{1}{\sqrt{N+1}}\right)\right)^p+\left(\omega_1\left(u^+,\frac{1}{\sqrt{N+1}}\right)\right)^p\right)dr\right)^{\frac{1}{p}}=$$

$$12\left[\left(\omega_1\left(u^-,\frac{1}{\sqrt{N+1}}\right)\right)^p+\left(\omega_1\left(u^+,\frac{1}{\sqrt{N+1}}\right)\right)^p\right]^{\frac{1}{p}}.$$

We have proved:

Theorem 8. *Let $u=(u^-,u^+)$ be a positive fuzzy number with the level functions u^- and u^+ continuous. Then, denoting $u_N:=(u_N^-,u_N^+)=\overline{B}_N^{(M)}(u)$, we have*

$$d_p\left(\overline{B}_N^{(M)}(u),u\right)\le \tag{33}$$

$$12\left[\left(\omega_1\left(u^-,\frac{1}{\sqrt{N+1}}\right)\right)^p+\left(\omega_1\left(u^+,\frac{1}{\sqrt{N+1}}\right)\right)^p\right]^{\frac{1}{p}},$$

$p\ge1$, $\forall\ N\in\mathbb{N}$.

Similarly we get

Theorem 9. *All as in Theorem 8, plus u^-,u^+ are concave. Then*

$$d_p\left(\overline{B}_N^{(M)}(u),u\right)\le 2\left[\left(\omega_1\left(u^-,\frac{1}{N}\right)\right)^p+\left(\omega_1\left(u^+,\frac{1}{N}\right)\right)^p\right]^{\frac{1}{p}}, \tag{34}$$

$p\ge1$, $\forall\ N\in\mathbb{N}$.

We also obtain

Theorem 10. *All as in Theorem 5. Then*

$$d_p\left(\overline{Z}_N^{(M)}(u), u\right) \leq \tag{35}$$

$$18\left[\left(\omega_1\left(u^-, \frac{2}{3\sqrt{3N}}\right)\right)^p + \left(\omega_1\left(u^+, \frac{2}{3\sqrt{3N}}\right)\right)^p\right]^{\frac{1}{p}},$$

$p \geq 1$, $\forall\ N \in \mathbb{N} : N \geq 4$.

Finally we give

Theorem 11. *All as in Theorem 2. Additionally assume that* $u^{\pm}$ *are concave. Then*

$$D\left(\overline{B}_N^{(M)}(u), u\right) \leq 2\max\left\{\omega_1\left(u^-, \frac{1}{N}\right), \omega_1\left(u^+, \frac{1}{N}\right)\right\}, \tag{36}$$

$\forall\ N \in \mathbb{N}$.

Proof. Use of (1) and (30). □

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