

ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED
 BY MULTIPLIER TRANSFORMATION AND RUSCHEWEYH
 DERIVATIVE

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ABSTRACT. In the present paper we define a new operator using the multiplier transformation and Ruscheweyh derivative. Denote by $RI_{m,\lambda,l}^\alpha$ the operator given by $RI_{m,\lambda,l}^\alpha : \mathcal{A}_n \rightarrow \mathcal{A}_n$, $RI_{m,\lambda,l}^\alpha f(z) = (1 - \alpha)R^m f(z) + \alpha I(m, \lambda, l) f(z)$, for $z \in U$, where $R^m f(z)$ denote the Ruscheweyh derivative, $I(m, \lambda, l) f(z)$ is the multiplier transformation and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$ is the class of normalized analytic functions. A certain subclass, denoted by $\mathcal{RT}_m(\delta, \lambda, l, \alpha)$, of analytic functions in the open unit disc is introduced by means of the new operator. By making use of the concept of differential subordination we will derive various properties and characteristics of the class $\mathcal{RT}_m(\delta, \lambda, l, \alpha)$. Also, several differential subordinations are established regarding the operator $RI_{m,\lambda,l}^\alpha$.

1. INTRODUCTION

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U .

Let

$$\mathcal{A}(p, n) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, z \in U\},$$

with $\mathcal{A}(1, n) = \mathcal{A}_n$ and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in U\},$$

where $p, n \in \mathbb{N}$, $a \in \mathbb{C}$.

Denote by $K = \left\{f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U\right\}$, the class of normalized convex functions in U .

If f and g are analytic functions in U , we say that f is subordinate to g , written $f \prec g$, if there is a function w analytic in U , with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and h an univalent function in U . If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad \text{for } z \in U, \quad (1)$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1).

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A dominant \tilde{q} that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U .

Definition 1. (Ruscheweyh [11]) For $f \in \mathcal{A}_n$, $n, m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \rightarrow \mathcal{A}_n$,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\dots \\ (m+1) R^{m+1} f(z) &= z (R^m f(z))' + m R^m f(z), \quad \text{for } z \in U. \end{aligned}$$

Remark 1. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $R^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m a_j z^j$, for $z \in U$.

Definition 2. [7] For $f \in \mathcal{A}(p, n)$, $p, n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I_p(m, \lambda, l) f(z)$ is defined by the following infinite series

$$I_p(m, \lambda, l) f(z) := z^p + \sum_{j=p+n}^{\infty} \left(\frac{p + \lambda(j-1) + l}{p+l} \right)^m a_j z^j.$$

Remark 2. It follows from the above definition that

$$\begin{aligned} I_p(0, \lambda, l) f(z) &= f(z), \\ (p+l) I_p(m+1, \lambda, l) f(z) &= [p(1-\lambda) + l] I_p(m, \lambda, l) f(z) + \lambda z (I_p(m, \lambda, l) f(z))', \\ &\text{for } z \in U. \end{aligned}$$

Remark 3. If $p = 1$, we have $I_1(m, \lambda, l) f(z) = I(m, \lambda, l)$ and

$$(l+1) I(m+1, \lambda, l) f(z) = [l+1-\lambda] I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))',$$

for $z \in U$.

Remark 4. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^m a_j z^j,$$

for $z \in U$.

Remark 5. For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [6], which reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ [12] for $\lambda = 1$. The operator $I_l^m = I(m, 1, l)$ was studied recently by Cho and Srivastava [8] and Cho and Kim [9]. The operator $I_m = I(m, 1, 1)$ was studied by Uralegaddi and Somanatha [13], the operator $D_\lambda^\delta = I(\delta, \lambda, 0)$, with $\delta \in \mathbb{R}$, $\delta \geq 0$, was introduced by Acu and Owa [1].

Lemma 1. (Hallenbeck and Ruscheweyh [10, Th. 3.1.6, p. 71]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \setminus \{0\}$ be a complex number with $\text{Re } \gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \quad \text{for } z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \quad \text{for } z \in U,$$

where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$, for $z \in U$.

Lemma 2. (Miller and Mocanu [10]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, for $z \in U$, where $\alpha > 0$ and n is a positive integer.

If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots$, for $z \in U$ is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z), \quad \text{for } z \in U,$$

then

$$p(z) \prec g(z), \quad \text{for } z \in U,$$

and this result is sharp.

2. MAIN RESULTS

Definition 3. Let $\alpha, \lambda, l \geq 0, n, m \in \mathbb{N}$. Denote by $RI_{m,\lambda,l}^\alpha$ the operator given by $RI_{m,\lambda,l}^\alpha : \mathcal{A}_n \rightarrow \mathcal{A}_n$,

$$RI_{m,\lambda,l}^\alpha f(z) = (1 - \alpha)R^m f(z) + \alpha I(m, \lambda, l) f(z), \quad \text{for } z \in U.$$

Remark 6. If $f \in \mathcal{A}_n, f(z) = z + \sum_{j=n+1}^\infty a_j z^j$, then

$$RI_{m,\lambda,l}^\alpha f(z) = z + \sum_{j=n+1}^\infty \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1 - \alpha) C_{m+j-1}^m \right\} a_j z^j, \quad \text{for } z \in U.$$

Remark 7. For $\alpha = 0, RI_{m,\lambda,l}^0 f(z) = R^m f(z)$, where $z \in U$ and for $\alpha = 1, RI_{m,\lambda,l}^1 f(z) = I(m, \lambda, l) f(z)$, where $z \in U$, which was studied in [4]. For $l = 0$, we obtain $RI_{m,\lambda,0}^\alpha f(z) = RD_{1,\alpha}^m f(z)$ which was studied in [5] and for $l = 0$ and $\lambda = 1$, we obtain $RI_{m,1,0}^\alpha f(z) = L_\alpha^m f(z)$ which was studied in [2], [3].

For $m = 0, RI_{0,\lambda,l}^\alpha f(z) = (1 - \alpha)R^0 f(z) + \alpha I(0, \lambda, l) f(z) = f(z) = R^0 f(z) = I(0, \lambda, l) f(z)$, where $z \in U$.

Definition 4. Let $\delta \in [0, 1), \alpha, \lambda, l \geq 0$ and $n, m \in \mathbb{N}$. A function $f \in \mathcal{A}_n$ is said to be in the class $\mathcal{RI}_m(\delta, \lambda, l, \alpha)$ if it satisfies the inequality

$$Re (RI_{m,\lambda,l}^\alpha f(z))' > \delta, \quad \text{for } z \in U. \tag{2}$$

Theorem 1. The set $\mathcal{RI}_m(\delta, \lambda, l, \alpha)$ is convex.

Proof. Let the functions

$$f_j(z) = z + \sum_{j=n+1}^\infty a_{jk} z^j, \quad \text{for } k = 1, 2, \quad z \in U,$$

be in the class $\mathcal{RI}_m(\delta, \lambda, l, \alpha)$. It is sufficient to show that the function

$$h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$$

is in the class $\mathcal{RI}_m(\delta, \lambda, l, \alpha)$ with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

Since $h(z) = z + \sum_{j=n+1}^\infty (\eta_1 a_{j1} + \eta_2 a_{j2}) z^j$, for $z \in U$, then

$$RI_{m,\lambda,l}^\alpha h(z) = z + \sum_{j=n+1}^\infty \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1 - \alpha) C_{m+j-1}^m \right\} (\eta_1 a_{j1} + \eta_2 a_{j2}) z^j, \tag{3}$$

for $z \in U$.

Differentiating (3) we obtain

$$\left(RI_{m,\lambda,l}^\alpha h(z) \right)' = 1 + \sum_{j=n+1}^\infty \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1 - \alpha) C_{m+j-1}^m \right\} (\eta_1 a_{j1} + \eta_2 a_{j2}) j z^{j-1}$$

for $z \in U$.

Hence

$$Re (RI_{m,\lambda,l}^\alpha h(z))' =$$

$$1 + \operatorname{Re} \left(\eta_1 \sum_{j=n+1}^{\infty} j \left\{ \alpha \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^m + (1 - \alpha) C_{m+j-1}^m \right\} a_{j1} z^{j-1} \right) \quad (4)$$

$$+ \operatorname{Re} \left(\eta_2 \sum_{j=n+1}^{\infty} j \left\{ \alpha \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^m + (1 - \alpha) C_{m+j-1}^m \right\} a_{j2} z^{j-1} \right).$$

Taking into account that $f_1, f_2 \in \mathcal{RT}_m(\delta, \lambda, l, \alpha)$ we deduce

$$\operatorname{Re} \left(\eta_k \sum_{j=n+1}^{\infty} j \left\{ \alpha \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^m + (1 - \alpha) C_{m+j-1}^m \right\} a_{jk} z^{j-1} \right) > \eta_k (\delta - 1), \quad (5)$$

for $k = 1, 2$.

Using (5) we get from (4)

$$\operatorname{Re} (RI_{m,\lambda,l}^\alpha h(z))' > 1 + \eta_1 (\delta - 1) + \eta_2 (\delta - 1) = \delta, \quad \text{for } z \in U,$$

which is equivalent that $\mathcal{RT}_m(\delta, \lambda, l, \alpha)$ is convex. \square

Theorem 2. Let g be a convex function in U and let $h(z) = g(z) + \frac{1}{c+2} z g'(z)$, where $z \in U$, $c > 0$.

If $f \in \mathcal{RT}_m(\delta, \lambda, l, \alpha)$ and $F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt$, for $z \in U$, then

$$(RI_{m,\lambda,l}^\alpha f(z))' \prec h(z), \quad \text{for } z \in U, \quad (6)$$

implies

$$(RI_{m,\lambda,l}^\alpha F(z))' \prec g(z), \quad \text{for } z \in U,$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1} F(z) = (c+2) \int_0^z t^c f(t) dt. \quad (7)$$

Differentiating (7), with respect to z , we have $(c+1)F(z) + zF'(z) = (c+2)f(z)$ and

$$(c+1)RI_{m,\lambda,l}^\alpha F(z) + z(RI_{m,\lambda,l}^\alpha F(z))' = (c+2)RI_{m,\lambda,l}^\alpha f(z), \quad \text{for } z \in U. \quad (8)$$

Differentiating (8) we have

$$(RI_{m,\lambda,l}^\alpha F(z))' + \frac{1}{c+2} z (RI_{m,\lambda,l}^\alpha F(z))'' = (RI_{m,\lambda,l}^\alpha f(z))', \quad \text{for } z \in U. \quad (9)$$

Using (9), the differential subordination (6) becomes

$$(RI_{m,\lambda,l}^\alpha F(z))' + \frac{1}{c+2} z (RI_{m,\lambda,l}^\alpha F(z))'' \prec g(z) + \frac{1}{c+2} z g'(z). \quad (10)$$

If we denote

$$p(z) = (RI_{m,\lambda,l}^\alpha F(z))', \quad \text{for } z \in U, \quad (11)$$

then $p \in \mathcal{H}[1, n]$.

Replacing (11) in (10) we obtain

$$p(z) + \frac{1}{c+2} z p'(z) \prec g(z) + \frac{1}{c+2} z g'(z), \quad \text{for } z \in U.$$

Using Lemma 2 we have

$$p(z) \prec g(z), \quad \text{for } z \in U, \quad \text{i.e. } (RI_{m,\lambda,l}^\alpha F(z))' \prec g(z), \quad \text{for } z \in U,$$

and g is the best dominant. \square

Theorem 3. Let $h(z) = \frac{1+(2\delta-1)z}{1+z}$, where $\delta \in [0, 1]$ and $c > 0$.

If $\alpha, \lambda, l \geq 0, n, m \in \mathbb{N}$ and $I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt$, for $z \in U$, then

$$I_c[\mathcal{RI}_m(\delta, \lambda, l, \alpha)] \subset \mathcal{RI}_m(\delta^*, \lambda, l, \alpha), \tag{12}$$

where $\delta^* = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n} \beta\left(\frac{c+2}{n} - 2\right)$ and $\beta(x) = \int_0^1 \frac{t^{x+1}}{t+1} dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2 we get from the hypothesis of Theorem 3 that

$$p(z) + \frac{1}{c+2} zp'(z) \prec h(z),$$

where $p(z)$ is defined in (11).

Using Lemma 1 we deduce that

$$p(z) \prec g(z) \prec h(z), \quad \text{i.e.} \quad (RI_{m,\lambda,l}^\alpha F(z))' \prec g(z) \prec h(z),$$

where

$$g(z) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z t^{\frac{c+2}{n}-1} \frac{1+(2\delta-1)t}{1+t} dt = (2\delta-1) + \frac{(c+2)(2-2\delta)}{nz^{\frac{c+2}{n}}} \int_0^z \frac{t^{\frac{c+2}{n}-1}}{1+t} dt.$$

Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$Re(RD_{\lambda,\alpha}^m F(z))' \geq \min_{|z|=1} Reg(z) = Reg(1) = \delta^* = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n} \beta\left(\frac{c+2}{n} - 2\right) \tag{13}$$

From (13) we deduce inclusion (12). □

Theorem 4. Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$.

If $\alpha, \lambda, l \geq 0, n, m \in \mathbb{N}, f \in \mathcal{A}_n$ and satisfies the differential subordination

$$(RI_{m,\lambda,l}^\alpha f(z))' \prec h(z), \quad \text{for } z \in U, \tag{14}$$

then

$$\frac{RI_{m,\lambda,l}^\alpha f(z)}{z} \prec g(z), \quad \text{for } z \in U,$$

and this result is sharp.

Proof. By using the properties of operator $RI_{m,\lambda,l}^\alpha$, we have

$$RI_{m,\lambda,l}^\alpha f(z) = z + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C_{m+j-1}^m \right\} a_j z^j, \quad \text{for } z \in U.$$

Consider $p(z) = \frac{RI_{m,\lambda,l}^\alpha f(z)}{z} = \frac{z + \sum_{j=n+1}^{\infty} \{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C_{m+j-1}^m \} a_j z^j}{z} = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$, for $z \in U$.

We deduce that $p \in \mathcal{H}[1, n]$.

Let $RI_{m,\lambda,l}^\alpha f(z) = zp(z)$, for $z \in U$. Differentiating we obtain $(RI_{m,\lambda,l}^\alpha f(z))' = p(z) + zp'(z)$, for $z \in U$.

Then (14) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z), \quad \text{for } z \in U.$$

By using Lemma 2, we have

$$p(z) \prec g(z), \quad \text{for } z \in U, \quad \text{i.e.} \quad \frac{RI_{m,\lambda,l}^\alpha f(z)}{z} \prec g(z), \quad \text{for } z \in U.$$

□

Theorem 5. Let h be an holomorphic function which satisfies the inequality $-\frac{1}{2} < \operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right)$, for $z \in U$, and $h(0) = 1$.

If $\alpha, \lambda, l \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$(RI_{m,\lambda,l}^\alpha f(z))' \prec h(z), \quad \text{for } z \in U, \tag{15}$$

then

$$\frac{RI_{m,\lambda,l}^\alpha f(z)}{z} \prec q(z), \quad \text{for } z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best dominant.

Proof. Let

$$\begin{aligned} p(z) &= \frac{RI_{m,\lambda,l}^\alpha f(z)}{z} = \frac{z + \sum_{j=n+1}^\infty \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C_{m+j-1}^m \right\} a_j z^j}{z} \\ &= 1 + \sum_{j=n+1}^\infty \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C_{m+j-1}^m \right\} a_j z^{j-1} = 1 + \sum_{j=n+1}^\infty p_j z^{j-1}, \end{aligned}$$

for $z \in U$, $p \in \mathcal{H}[1, n]$.

Differentiating, we obtain $(RI_{m,\lambda,l}^\alpha f(z))' = p(z) + zp'(z)$, for $z \in U$ and (15) becomes

$$p(z) + zp'(z) \prec h(z), \quad \text{for } z \in U.$$

Using Lemma 1, we have

$$p(z) \prec q(z), \quad \text{for } z \in U, \quad \text{i.e.} \quad \frac{RI_{m,\lambda,l}^\alpha f(z)}{z} \prec q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1} dt, \quad \text{for } z \in U,$$

and q is the best dominant. □

Theorem 6. Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$.

If $\alpha, \lambda, l \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination

$$\left(\frac{zRI_{m+1,\lambda,l}^\alpha f(z)}{RI_{m,\lambda,l}^\alpha f(z)} \right)' \prec h(z), \quad \text{for } z \in U \tag{16}$$

holds, then

$$\frac{RI_{m+1,\lambda,l}^\alpha f(z)}{RI_{m,\lambda,l}^\alpha f(z)} \prec g(z), \quad \text{for } z \in U,$$

and this result is sharp.

Proof. For $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^\infty a_j z^j$ we have

$$RI_{m,\lambda,l}^\alpha f(z) = z + \sum_{j=n+1}^\infty \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C_{m+j-1}^m \right\} a_j z^j, \quad \text{for } z \in U.$$

Consider

$$p(z) = \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{RI_{m,\lambda,l}^\alpha f(z)} = \frac{z + \sum_{j=n+1}^\infty \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^{m+1} + (1-\alpha) C_{m+j}^{m+1} \right\} a_j z^j}{z + \sum_{j=n+1}^\infty \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C_{m+j-1}^m \right\} a_j z^j}.$$

We have $p'(z) = \frac{(RI_{m+1,\lambda,l}^\alpha f(z))'}{RI_{m,\lambda,l}^\alpha f(z)} - p(z) \cdot \frac{(RI_{m,\lambda,l}^\alpha f(z))'}{RI_{m,\lambda,l}^\alpha f(z)}$ and we obtain $p(z) + z \cdot p'(z) = \left(\frac{zRI_{m+1,\lambda,l}^\alpha f(z)}{RI_{m,\lambda,l}^\alpha f(z)} \right)'$.

Relation (16) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z), \quad \text{for } z \in U.$$

By using Lemma 2, we have

$$p(z) \prec g(z), \quad \text{for } z \in U, \quad \text{i.e.} \quad \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{RI_{m,\lambda,l}^\alpha f(z)} \prec g(z), \quad \text{for } z \in U.$$

□

Theorem 7. Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$.

If $\alpha, \lambda, l \geq 0, n, m \in \mathbb{N}, f \in \mathcal{A}_n$ and the differential subordination

$$\begin{aligned} & \frac{(m+1)(m+2)}{z} RI_{m+2,\lambda,l}^\alpha f(z) - \frac{(m+1)(2m+1)}{z} RI_{m+1,\lambda,l}^\alpha f(z) + \frac{m^2}{z} RI_{m,\lambda,l}^\alpha f(z) + \\ & \frac{\alpha}{z} \left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2) \right] I(m+2, \lambda, l) f(z) - \\ & \frac{\alpha}{z} \left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2} - (m+1)(2m+1) \right] I(m+1, \lambda, l) f(z) + \\ & \frac{\alpha}{z} \left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2 \right] I(m, \lambda, l) f(z) \prec h(z), \quad \text{for } z \in U \end{aligned} \tag{17}$$

holds, then

$$[RI_{m,\lambda,l}^\alpha f(z)]' \prec g(z), \quad \text{for } z \in U.$$

This result is sharp.

Proof. Let

$$\begin{aligned} p(z) &= (RI_{m,\lambda,l}^\alpha f(z))' = (1-\alpha)(R^m f(z))' + \alpha(I(m, \lambda, l) f(z))' = \\ & 1 + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C_{m+j-1}^m \right\} j a_j z^{j-1} = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots \end{aligned} \tag{18}$$

We deduce that $p \in \mathcal{H}[1, n]$.

By using the properties of operators $RI_{m,\lambda,l}^\alpha, R^m$ and $I(m, \lambda, l)$, after a short calculation, we obtain $p(z) + zp'(z) = \frac{(m+1)(m+2)}{z} RI_{m+2,\lambda,l}^\alpha f(z) - \frac{(m+1)(2m+1)}{z} RI_{m+1,\lambda,l}^\alpha f(z) + \frac{m^2}{z} RI_{m,\lambda,l}^\alpha f(z) + \frac{\alpha}{z} \left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2) \right] I(m+2, \lambda, l) f(z) - \frac{\alpha}{z} \left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2} - (m+1)(2m+1) \right] I(m+1, \lambda, l) f(z) + \frac{\alpha}{z} \left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2 \right] I(m, \lambda, l) f(z)$.

Using the notation in (18), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 2, we have

$$p(z) \prec g(z), \quad \text{for } z \in U, \quad \text{i.e.} \quad (RI_{m,\lambda,l}^\alpha f(z))' \prec g(z), \quad \text{for } z \in U,$$

and this result is sharp.

□

Theorem 8. Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U , where $0 \leq \beta < 1$.

If $\alpha, \lambda, l \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\begin{aligned} & \frac{(m+1)(m+2)}{z} RI_{m+2, \lambda, l}^\alpha f(z) - \frac{(m+1)(2m+1)}{z} RI_{m+1, \lambda, l}^\alpha f(z) + \frac{m^2}{z} RI_{m, \lambda, l}^\alpha f(z) + \\ & \frac{\alpha}{z} \left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2) \right] I(m+2, \lambda, l) f(z) - \\ & \frac{\alpha}{z} \left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2} - (m+1)(2m+1) \right] I(m+1, \lambda, l) f(z) + \\ & \frac{\alpha}{z} \left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2 \right] I(m, \lambda, l) f(z) \prec h(z), \quad \text{for } z \in U, \end{aligned} \quad (19)$$

then

$$(RI_{m, \lambda, l}^\alpha f(z))' \prec q(z), \quad \text{for } z \in U,$$

where q is given by $q(z) = 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt$, for $z \in U$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 7 and considering $p(z) = (RI_{m, \lambda, l}^\alpha f(z))'$, the differential subordination (19) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \quad \text{for } z \in U.$$

By using Lemma 1 for $\gamma = 1$ and $n = 1$, we have $p(z) \prec q(z)$, i.e.,

$$\begin{aligned} (RI_{m, \lambda, l}^\alpha f(z))' \prec q(z) &= \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{1+(2\beta-1)t}{1+t} dt \\ &= 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt, \quad \text{for } z \in U. \end{aligned}$$

□

Theorem 9. Let h be an holomorphic function which satisfies the inequality $-\frac{1}{2} < \operatorname{Re} \left[1 + \frac{zh''(z)}{h'(z)} \right]$, for $z \in U$, and $h(0) = 1$.

If $\alpha, \lambda, l \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\begin{aligned} & \frac{(m+1)(m+2)}{z} RI_{m+2, \lambda, l}^\alpha f(z) - \frac{(m+1)(2m+1)}{z} RI_{m+1, \lambda, l}^\alpha f(z) + \frac{m^2}{z} RI_{m, \lambda, l}^\alpha f(z) + \\ & \frac{\alpha}{z} \left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2) \right] I(m+2, \lambda, l) f(z) - \\ & \frac{\alpha}{z} \left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2} - (m+1)(2m+1) \right] I(m+1, \lambda, l) f(z) + \\ & \frac{\alpha}{z} \left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2 \right] I(m, \lambda, l) f(z) \prec h(z), \quad \text{for } z \in U, \end{aligned} \quad (20)$$

then

$$(RI_{m, \lambda, l}^\alpha f(z))' \prec q(z), \quad \text{for } z \in U,$$

where q is given by $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best dominant.

Proof. Using the properties of operator $RI_{m,\lambda,l}^\alpha$ and considering $p(z) = \left(RI_{m,\lambda,l}^\alpha f(z) \right)'$, we obtain

$$\begin{aligned} p(z) + zp'(z) &= \frac{(m+1)(m+2)}{z} RI_{m+2,\lambda,l}^\alpha f(z) - \frac{(m+1)(2m+1)}{z} RI_{m+1,\lambda,l}^\alpha f(z) + \\ &\frac{m^2}{z} RI_{m,\lambda,l}^\alpha f(z) + \frac{\alpha}{z} \left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2) \right] I(m+2, \lambda, l) f(z) - \\ &\frac{\alpha}{z} \left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2} - (m+1)(2m+1) \right] I(m+1, \lambda, l) f(z) + \\ &\frac{\alpha}{z} \left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2 \right] I(m, \lambda, l) f(z), \quad \text{for } z \in U. \end{aligned}$$

Then (20) becomes

$$p(z) + zp'(z) \prec h(z), \quad \text{for } z \in U.$$

Since $p \in \mathcal{H}[1, n]$, using Lemma 1, we deduce

$$p(z) \prec q(z), \quad \text{for } z \in U, \text{ i.e. } \left(RI_{m,\lambda,l}^\alpha f(z) \right)' \prec q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt, \quad \text{for } z \in U,$$

and q is the best dominant. \square

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