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FRACTIONAL CALCULUS APPROACH FOR WAVE PROPAGATION IN NONLOCAL ONE DIMENSIONAL ELASTIC SOLIDS (FRACTIONAL CALCULUS)

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Abstract. In this paper an attempt has been made to study the wave propagation through a one dimensional elastic solid in the frame work of Eringen's nonlocal elasticity using the concept of fractional calculus. The fractional differential equation corresponding to the axial harmonic wave propagation through a one dimensional elastic bar of infinite length with uniform cross section is obtained. The dynamic equation for the harmonic wave propagation in terms of Caputo fractional derivatives of various orders α has been derived. The dispersive equation for an axial wave is obtained in terms of fractional order α and a parameter β_2, which accounts the effect of nonlocal long range interactions. The effect of wave number on this ratio is studied numerically for different values of the fractional order and the nonlocal parameters.

1. Introduction

Nonlocal continuum theory developed by Eringen [1] and is found as major field of research due to its application in micro structural behavior of nano materials. The wave propagation through nonlocal materials is able to detect the dispersive nature of traveling disturbances, which is a special property and it cannot be described by a local stress-strain model. Due to the dispersive nature in wave propagation and the study of Born-Karmann model of lattice dynamics at the border of the Brillouin zone as the wave length becomes close to the lattice distance and there exist a large deviation in the speed of the traveling waves [2]. Recently, Zingales [3] presented a detailed study on wave propagation in nonlocal elastic solids which is mechanically based on Kroner-Eringen integral model of nonlocal elasticity in unbounded domain. Another work due to Aydogdu [4] is appeared recently in the scientific literature for a nonlocal one dimensional solids in the study of low dimensional system and nano structures.

Lazopoulos [5] introduced the concept of fractional calculus into the continuum mechanics which provides a natural extension for describing nonlocal constitutive relation of a one dimensional bar under axial extension. It is assumed that the strain energy density not only depends on the local strain but also on the order of the derivative α of the strain where $0 < \alpha < 1$. Cottone et al. [6] discussed the wave propagation in unbounded and bounded domain that includes dispersion of elastic waves using fractional derivatives of order α where $0 < \alpha < 1$. Cornetti et al. [8] proved that if the attenuation function of strain is expressed as power law then the fractional calculus can be used effectively to a Eringen nonlocal elastic model. Sapora [7] presented the wave propagation in one dimensional elastic continuum by means of fractional derivative of order α where $1 < \alpha < 2$. Challamel et al. [9] presented the fractional generalization of Eringen's nonlocal elasticity with free non integer derivatives in the stress-strain fractional order differential equation

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which contains a single length scale and fractional order derivatives as parameters. Recently, Vikash Pandey [10] investigated the spatial dispersion of elastic waves in a one dimensional elastic bar with nonlocal interactions using a different attenuation function containing fractional derivative as a parameter.

In this paper the wave propagation in one dimensional elastic solids with the help of generalized fractional derivative is discussed. The fractional differential equation is considered as Eringen's fractional nonlocal elastic model. The equation contains the percentage of nonlocal interactions, a characteristic length responsible for nonlocal effects and the order of Caputo fractional derivative. Using complex fourier transform the dispersive equation for harmonic wave propagation has been derived in terms of the these three parameters. A numerical interpretation is presented to show the effect of different fractional orders between 1 and 3 on the dispersion curve and the effect of the other parameter, which influences the non locality.

2. Fractional order derivatives

From the Cauchy's definition of a multiple order integral, the function $f(x)$ is given by

$$I^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-\tau)^{n-1} f(\tau) d(\tau) \tag{1}$$

where x is a positive real number, n is a positive integer and it is assumed that $f(x) = 0$ for $x < 0$.

Using the ideas presented by Lazopoulous [5], n has been replaced by a positive real number α then α-fold primitive of $f(x)$ has been obtained as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} f(\tau) d(\tau) \tag{2}$$

where $\Gamma(\alpha)$ is the Euler gamma function and (α) is a non-integer.

From the Riemann - Liouville definition [6], the fractional order derivative of a function $f(x)$ of order α where $n-1 < \alpha < n$ is a positive real number less than n is given by

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x \frac{f(\tau)}{(x-\tau)^{\alpha+1-n}} d(\tau) \tag{3}$$

If $\alpha = m$ then $D^\alpha f(x) = \frac{d^m}{dx^m} f(x)$, where m is a positive integer.

Clearly, we have

$$D^\alpha x^r = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)} x^{r-\alpha} \tag{4}$$

Hence for $\alpha > 0$, the fractional derivative of order α of a constant c is not zero as

$$D^\alpha c = \frac{c x^{-\alpha}}{\Gamma(1-\alpha)} \tag{5}$$

where $n-1 < \alpha < n$. Also, we have $D^\alpha x^{\alpha-1} = 0$.

As the Riemann-Liouville derivative of a constant function is not zero, there is another option [3] for defining fractional derivative known as the Caputo fractional order derivative.

$$_cD_a^\alpha f(x) = \frac{1}{\Gamma(x-\alpha)}\int_a^x \frac{f^n(\tau)}{(x-\tau)^{\alpha+1-n}} d(\tau) \tag{6}$$

where $x > a$ and $n-1 < \alpha < n$.

Thus we have ${}_cD^\alpha(constant) = 0$ and Caputo fractional order derivative [11] is a generalization of the derivation of integral order, it is more useful in practical applications of fractional order derivatives. However, the definition of Riemann-Liouville and Caputo are related by the identity

$$ {}_cD_a^\alpha f(x) = D_{a+}^\alpha \left(f(x) - f(a)\right) \tag{7}$$

.

3. Eringen's Fractional nonlocal model

Consider an elastic bar of uniform cross section S and an infinite in length with an external self equibrated force $N(x) = \sigma(x)s$. The axial equation of motion of the bar is

$$\frac{dN}{dx} = \rho s \frac{\partial^2 u}{\partial t^2}. \tag{8}$$

Thus, we have the equation of motion as

$$\frac{d\sigma}{dx} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{9}$$

where $\sigma(x)$ is the stress at a point, x is the longitudinal co-ordinate where $-\infty < x < \infty$, ρ represents the mass density at any point of the material of the bar and $u(x,t)$ is the axial displacement of any point.

The kinematic relation between the uniaxial strain $\epsilon(x)$ and the axial displacement is

$$\epsilon = \frac{\partial u}{\partial x}. \tag{10}$$

According to the nonlocal elastic stress field theory [1], the stress at any point of the elastic continuum depends not only on the strain at that point but also on the strain at all other points of the body. This theory is based on atomic theory of lattice dynamics with experimental observations of phonon dispersion. In case of an elastic bar of infinite in length, the one dimensional stress strain relation [7] can be taken as

$$\sigma(x) = \beta_1 E\epsilon(x) + \beta_2 E l^\alpha \int_{-\infty}^{\infty} \epsilon(\tau) g(x,\tau) d\tau \tag{11}$$

where E is the young's modulus, l is a dimensionless parameter which is responsive for long range nonlocal effects and α is a positive real number related to the fractional dimensions of the inner micro-structure which has a selected property. Also, here $g(x,\tau)$ is the attenuation function that accounts for the contribution of the strain at a point τ due to nonlocal stress at x. Usally, $g(x-\tau)$ can be considered as a monotonically decreasing function with the distance $\mid x-\tau \mid$. The parameter β_1 and β_2 are considered as weights where $\overline{E} = \beta_1 E$ is the reduced elastic modulus that accounts for long range effects and β_2 is a coefficient pertaining for the percentage of the nonlocal interactions. Always, we have $0 \le \beta_1, \beta_2 \le 1$ and $\beta_1 + \beta_2 = 1$.

It is observed by Paola and Zingales [12] that if the attenuation function of strain is expressed as a power law then the concepts of fractional calculus can be used to construct Eringen's nonlocal elastic model. Thus the corresponding attenuation function can be taken as

$$g(\xi) = \frac{1}{\Gamma(n-\alpha)} \mid \xi \mid^{n-\alpha-1} \tag{12}$$

where $n - 1 < \alpha < n$.

The improved differential based model is obtained by changing the integer order of the derivative in the stress-strain relationship which includes a symmetrized Caputo fractional derivative [11]. It is observed by Challamel [9] that the fractional ordered model is a better match for the Born-Karmann model of lattice dynamics. Thus, the fractional order differential equation [13] corresponding to Eringen's fractional model of nonlocal elasticity is

$$\sigma(x) = \beta_1 E\epsilon(x) + \beta_2 E l_c^{\alpha} D^{\alpha}\sigma(x), \tag{13}$$

where $n-1 < \alpha < n$ is the fractional order of the stress-strain relation. Here ${}_cD^{\alpha}$ is the Caputo fractional derivative of order α defined as

$${}_cD^{\alpha}\sigma(x) = \frac{1}{2}\left({}_cD^{\alpha}_{-\infty} +_c D^{\alpha}_{\infty}\right)\sigma(x), \tag{14}$$

where

$${}_cD^{\alpha}_{-\infty}\sigma(x) = \frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{x}\frac{\sigma^n(\tau)}{(x-\tau)^{\alpha-n+1}}d(\tau), \tag{15}$$

and

$${}_cD^{\alpha}_{\infty}\sigma(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x}^{\infty}\frac{\sigma^n(\tau)}{(\tau-x)^{\alpha-n+1}}d(\tau). \tag{16}$$

The equations (15 and 16) constitutive Eringen's fractional nonlocal model. From these equations it can observed that the fractional nonlocal model reduces to Eringen's nonocal model for $\alpha = 2$ and while for $\alpha = 1$ it reduce to constitutive equation for classical elasticity with $\beta_2 = 0$. Thus we consider the Eringen's fractional nonlocal model with fractional order derivative α where $n-1 < \alpha < n$ and $n \geq 2$.

4. Solution and dispersive equation

For the described harmonic wave propagation, the displacement of the propagation [9] assumed as

$$u(x,t) = u_0 e^{i(kx-\omega t)} \tag{17}$$

where ω is the angular frequency and k is the wave number. Thus the equations (8 and 9) can be reduced as

$$\frac{d\sigma}{dx} = -\rho\omega^2 u \qquad \epsilon = iku \tag{18}$$

From the definition of exponential fourier transform of a function [14] we have

$$Ff(x) = \overline{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} \tag{19}$$

here k is the wave number.

From the above definition, we have the following properties

i $F\int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau = \overline{f}(k)\overline{g}(k)$

ii $F\frac{d^m}{dx^m}f(x) = (ik)^n\overline{f}(k)$

iii $(iii)F\left[\frac{1}{|x|^{\beta}}\right] = 2\Gamma(1-\beta)sin\frac{\beta\pi}{2}\frac{1}{|k|^{1-\beta}}$

iv $(iv)F\left[\frac{sgn(x)}{|x|^{\beta}}\right] = -2i\Gamma(1-\beta)cos\frac{\beta\pi}{2}\frac{sgn(k)}{(|k|)^{1-\beta}}$

By taking the attenuation function $g(\xi)$, the strain of nonlocal elasticity (18) we can express the fractional order differential equation 11 as

$$\sigma(x) = \beta_1 E\epsilon(x) + \frac{\beta_2 E l^\alpha}{2\Gamma(x-\alpha)} \int_{-\infty}^{\infty} \frac{\sigma^n(\xi) sgn(x-\xi)}{\mid x-\xi \mid^{\alpha-n+1}} d\xi \tag{20}$$

$$\sigma(x) = \beta_1 E\epsilon(x) + \frac{\beta_2 E l^\alpha}{2\Gamma(x-\alpha)} \frac{sgn(x)}{\mid x \mid^{\alpha-n+1}} * \sigma^n(x) \tag{21}$$

Taking fourier transform as defined above to the equation (18), we have

$$(ik)\overline{\sigma} = -\rho\iota\omega^2\overline{u} \qquad and \qquad \overline{\epsilon} = (ik)\overline{u} \tag{22}$$

From these equations, we have a relation

$$\frac{\overline{\sigma}(x)}{\overline{\epsilon}(k)} = \frac{\rho\omega^2}{k^2} \tag{23}$$

.

Again applying Fourier transform to be fractional differential order gives in (22), we have

$$\overline{\sigma}(k) = E\beta_1\overline{\epsilon}(k) + \frac{E\beta_2 l^\alpha}{2\Gamma(x-\alpha)} F\left[\frac{sgn(x)}{\mid x \mid^{\alpha-n+1}} * \sigma^n(x)\right] \tag{24}$$

Using Fourier theorem and Fourier properties (22), we obtain

$$\overline{\sigma}(k) = E\beta_1\overline{\epsilon}(k) + E\beta_2 l^\alpha cos(\frac{\alpha\pi}{2})\mid k \mid^\alpha \quad \overline{\sigma}(k) \tag{25}$$

The above equation is defined separately using for nonlocal model as

$$\frac{\overline{\sigma}}{\overline{\epsilon}} = \frac{E\beta_1}{1 - E\beta_2 l^\alpha cos(\frac{\alpha\pi}{2})\mid k \mid^\alpha} \tag{26}$$

where $n-1 < \alpha < n$ and $n \geq 2$

$$\frac{\omega}{C_0} = k\sqrt{\frac{\beta_1}{1 - cos(\frac{\alpha\pi}{2})\,[l \mid K \mid]^\alpha}} \tag{27}$$

where $C_0 = \sqrt{E/\rho}$ is the speed of the nonlocal wave. The above equation (27) is the dispersive equation for the nonlocal elasticity with small order of $\alpha\epsilon[n-1, n]$. Thus, in the dispersive equation all the frequencies depends on fractional order (α).

5. Numerical Results and Discussion

For the described model of wave propagation in fractional nonlocal one dimensional bar, the parameter for computation has been considered as $E = 72KN/mm^2$, $\beta_1 = 0.76$, $l = 1.1$.

From the observations of the wave propagation in the bar shows that the different values of the fractional derivative order $1 < \alpha < 2$ and $2 < \alpha < 3$ and the parameter β_2 which influences the non locality. It is observed from the dispersive equation that as α increases from $1 < \alpha < 2$ the angular frequency increases of wave number K. while α varies from $2 < \alpha < 3$, the behavior is different as α approaches to 3. The same behavior can be observed for different values of the parameter $\beta_2 = 0.2$ and 0.4.

It is also possible to determined from the effect of dispersive equation for axial wave propagation for another strength of nonlocality α. The dynamic properties of the nonlocal model consumed will be same for the value of fractional derivative α more than 3.

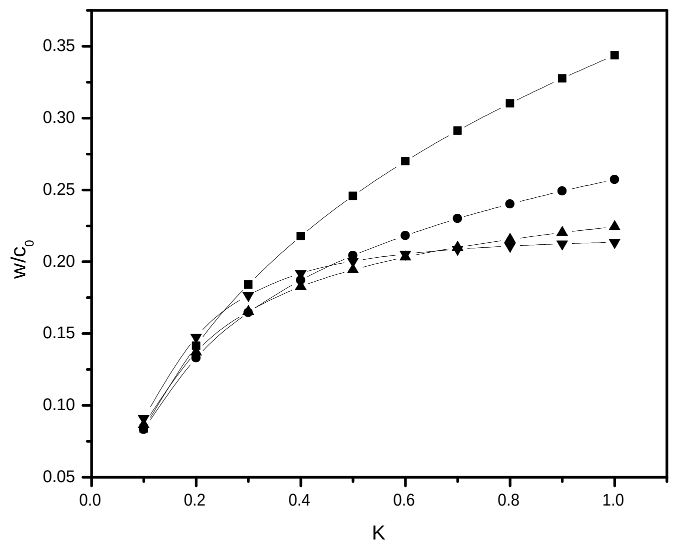

FIGURE 1. Nonlocal Fractional derivative effect for different values of $1 < \alpha < 2$

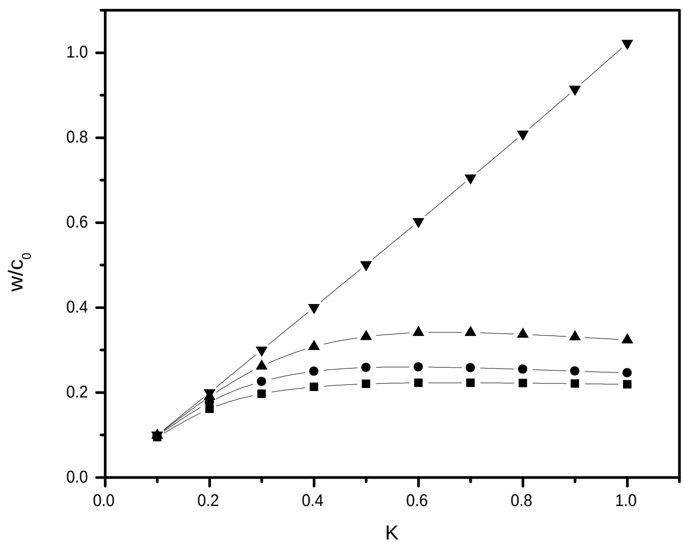

FIGURE 2. Nonlocal Fractional derivative effect for different values of $2 < \alpha < 3$

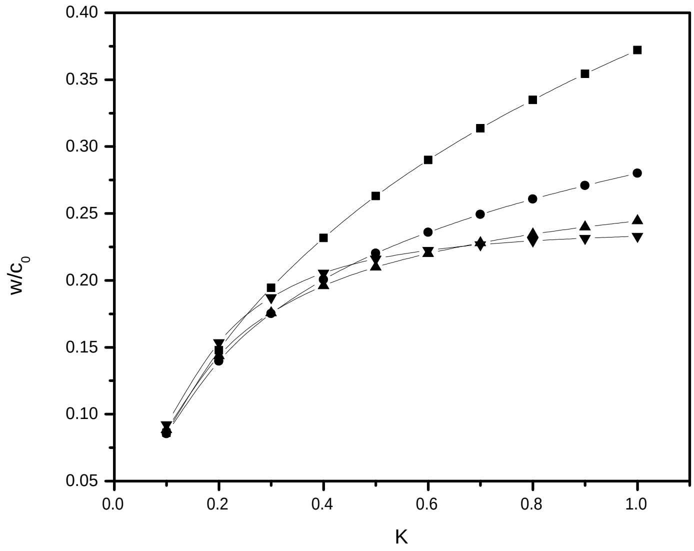

FIGURE 3. Nonlocal Fractional derivative of $\beta = 0.2$ effect for different values of $1 < \alpha < 2$

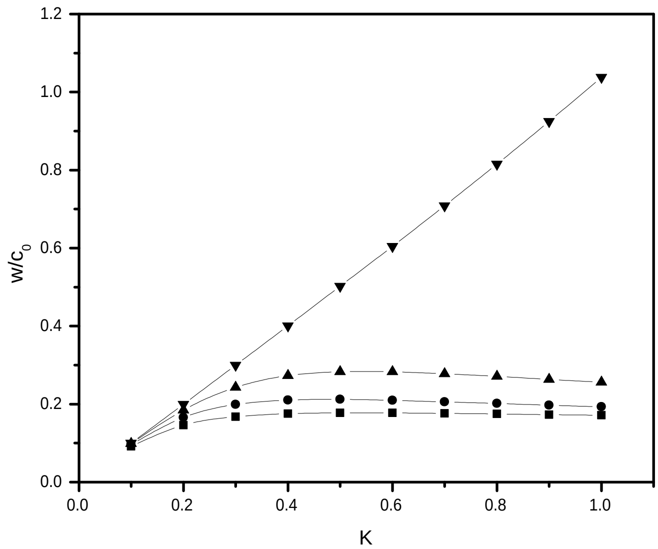

FIGURE 4. Nonlocal Fractional derivative of $\beta = 0.4$ effect for different values of $2 < \alpha < 3$

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