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ON NONLINEAR CAPUTO FRACTIONAL q-DIFFERENCE BOUNDARY VALUE PROBLEMS WITH MULTI-POINT CONDITIONS

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Abstract. In this paper, we investigate a class of nonlinear boundary value problems of Caputo type fractional q-difference equations supplemented with nonlocal multi-point conditions. By making use of some well known tools of the fixed point theory, the existence and uniqueness of solutions for the proposed problems are established. As applications, two examples are presented to demonstrate the effectiveness of the obtained results.

1. Introduction

Fractional differential equations have been paid attentions due to the fact that fractional differential equations can describe better natural phenomena than ordinary differential equations. In the past serval decades, a lot of books and papers on fractional differential equations have been published, for example, [4, 10, 15, 17, 18]. However, fractional differential equations do not give discrete physical phenomena. In [2, 7], Al-Salam and Agarwal proposed the fractional q-difference calculus. In this way, fractional q-difference equations can depict discrete problems. What is more, the fractional q-difference calculus plays an important role in quantum calculus. Recently, boundary value problems for nonlinear fractional q-difference equations have been addressed extensively by many scholars. For the development of fractional q-difference equations, see [1, 8, 11, 12, 16, 19, 20, 22, 23, 24, 25, 26, 27] and the references therein. For instance, the authors [3, 5] obtained some existence results for sequential q-fractional integrodifferential equations with q-antiperiodic boundary conditions and nonlocal four-point boundary conditions by applying some standard fixed point theorems, respectively. By using the Guo-Krasnoselskii fixed point theorem, Graef and Kong [13, 14] considered the existence of positive solutions for boundary value problems with fractional q-derivatives in terms of different ranges of λ, respectively. By means of the Banach contraction mapping principle and Scheafer fixed point theorem, Yang [21] gave existence and uniqueness of solutions for nonlinear fractional q-difference equations with nonlocal Riemann-Liouville q-integral boundary conditions. In [6], existence results for nonlinear fractional q-difference equation with nonlocal boundary conditions were shown by applying some well-known tools of fixed-point theory such as Banach's contraction principle, Krasnoselskii's fixed-point theorem, and the Leray-Schauder nonlinear alternative.

In this work, motivated by the above mentioned papers, we investigate the existence and uniqueness of solutions for the multi-point boundary value problems of nonlinear fractional q-difference equations of the form

$$^{c}D_q^{\alpha}u(t) = f(t, u(t)), \quad t \in [0,1],\ 1 < \alpha \le 2, \tag{1}$$

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$$u(0)=\delta u(\sigma),\quad a^{c}D_q^{\beta}u(\zeta_1)+b^{c}D_q^{\beta}u(\zeta_2)=\sum_{i=1}^{m-2}\gamma_i u(\xi_i),\quad 0<\beta\leq 1, \tag{2}$$

where $0\leq\sigma\leq\zeta_1<\xi_1<\cdots<\xi_{m-2}<\zeta_2\leq 1$ $(i=1,2,\ldots,m-2)$, D_q^{α} represents the Caputo type fractional q-derivative of order α. The nonlinear function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function and $\delta,a,b,\gamma_i\in\mathbb{R}$. Here we point out that multi-point boundary conditions in (2) include the values of the unknown function and its fractional derivative. Due to the fact that if $q\to 1^-$, the q-derivative and q-integral are the usual derivative and integral, we should point out that if $q\to 1^-$, the above boundary value problem can reduce to the problem considered by Agarwal *et al.* [4]. In this paper, we will establish the existence and uniqueness of solutions for a class of nonlinear multi-point boundary value problems (1)-(2) by using the Banach contraction principle, Krasnoselskii's fixed point theorem and Schauder fixed point theorem. Finally, we give two examples to show the effectiveness of the obtained results.

2. Preliminaries

For the convenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory. These details can be found in the recent literature; see [9] and references therein.

Definition 1. [9] Let $\alpha\geq 0$, $0<q<1$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x)=f(x)$ and

$$I_q^{\alpha}f(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)}f(t)d_qt,\quad \alpha>0, x\in[0,1],$$

where $\Gamma_q(\alpha)=(1-q)^{(\alpha-1)}(1-q)^{1-\alpha}$, $0<q<1$, and satisfies the relation: $\Gamma_q(\alpha+1)=[\alpha]_q\Gamma_q(\alpha)$, with

$$[\alpha]_q=\frac{q^{\alpha}-1}{q-1},\quad (1-q)^{(0)}=1,\quad (1-q)^{(n)}=\prod_{k=0}^{n-1}(1-q^{k+1}),\quad n\in\mathbb{N}.$$

More generally, if $\alpha\in\mathbb{R}$, then $(1-q)^{(\alpha)}=\prod_{n=0}^{\infty}\big((1-q^{n+1})/(1-q^{1+\alpha+n})\big)$.

For $0<q<1$,the q-derivative of a real valued function f is here defined by

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\ x\neq 0,\quad \text{and } D_qf(0)=\lim_{x\to 0}D_qf(x),$$

and q-derivatives of higher order by

$$D_q^0 f(x)=f(x)\quad\text{and}\quad D_q^n f(x)=D_qD_q^{n-1}f(x),\quad n\in\mathbb{N}.$$

Definition 2. [9] The fractional q-derivative of the Caputo type of order $\alpha\geq 0$ is defined by

$$({}^cD_q^{\alpha}f)(x)=(I_q^{m-\alpha}D_q^m f)(x),\quad \alpha>0,$$

where m is the smallest integer greater than or equal to α.

Lemma 1. [9] *Let $\alpha,\beta\geq 0$ and f be a function defined on $[0,1]$. Then the next formulas hold:* (1) $I_q^{\beta}I_q^{\alpha}f(x)=I_q^{\alpha+\beta}f(x)$, (2) $D_q^{\alpha}I_q^{\alpha}f(x)=f(x)$.

Lemma 2. [9] *Let $\alpha>0$ and $\alpha\in\mathbb{R}^+\setminus\mathbb{N}$. Then the following equality holds:*

$$(I_q^{\alpha c}D_q^{\alpha}f)(x)=f(x)-\sum_{k=0}^{m-1}\frac{x^k}{\Gamma_q(k+1)}(D_q^k f)(0),$$

where m is the smallest integer greater than or equal to α.

Lemma 3. [9] *For $\alpha \in \mathbb{R}^+$, $\lambda \in (-1, \infty)$, the following is valid:*

$$I_q^\alpha((x-a)^{(\lambda)}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}(x-a)^{(\alpha+\lambda)}, \quad 0 < a < x < b.$$

In particular, for $\lambda = 0$, $a = 0$, using q-integration by parts, we have

$$\begin{aligned}(I_q^\alpha 1)(x) &= \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} d_q t = \frac{1}{\Gamma_q(\alpha)} \int_0^x \frac{D_q((x-t)^{(\alpha)})}{-[\alpha]_q} d_q t \\ &= -\frac{1}{\Gamma_q(\alpha+1)} \int_0^x D_q((x-t)^{(\alpha)}) d_q t = \frac{x^{(\alpha)}}{\Gamma_q(\alpha+1)}.\end{aligned}$$

Lemma 4. *Let $h \in C[0,1]$. Then for $0 < \beta \leq 1 < \alpha \leq 2$, the unique solution of the linear problem*

$$^cD_q^\alpha u(t) = h(t), \quad t \in [0,1],\ 1 < \alpha \leq 2, \tag{3}$$

supplemented with boundary conditions (2) *is given by*

$$\begin{aligned}u(t) =& I_q^\alpha h(t) + \frac{\delta}{1-\delta} I_q^\alpha h(\sigma) + \frac{\delta\sigma + (1-\delta)t}{A(1-\delta)} \\ &\times \left(\frac{\delta \sum_{i=1}^{m-2} \gamma_i}{1-\delta} I_q^\alpha h(\sigma) + \sum_{i=1}^{m-2} \gamma_i I_q^\alpha h(\xi_i) - a I_q^{\alpha-\beta} h(\zeta_1) - b I_q^{\alpha-\beta} h(\zeta_2) \right),\end{aligned} \tag{4}$$

where

$$A = \frac{a\zeta_1^{(1-\beta)} + b\zeta_2^{(1-\beta)}}{\Gamma_q(2-\beta)} - \sum_{i=1}^{m-2} \gamma_i \xi_i - \frac{\delta\sigma}{1-\delta} \sum_{i=1}^{m-2} \gamma_i \neq 0.$$

Proof. Let $u(t)$ be a solution of (3). In view of Lemmas 1 and 2, (3) is equivalent to the integral equation

$$u(t) = I_q^\alpha h(t) + c_0 + c_1 t, \quad \text{for } c_0, c_1 \in \mathbb{R}. \tag{5}$$

From (5), we can get

$$D_q^\beta u(t) = I_q^{\alpha-\beta} h(t) + \frac{t^{(1-\beta)}}{\Gamma_q(2-\beta)} c_1. \tag{6}$$

Using the boundary conditions $u(0) = \delta u(\sigma)$ and $a^cD_q^\beta u(\zeta_1) + b^cD_q^\beta u(\zeta_2) = \sum_{i=1}^{m-2} \gamma_i u(\xi_i)$ in (5) and (6), we obtain

$$\begin{aligned}c_0 =& \frac{\delta}{1-\delta} \Bigg(I_q^\alpha h(\sigma) + \frac{\sigma}{A} \Bigg(\frac{\delta \sum_{i=1}^{m-2} \gamma_i}{1-\delta} I_q^\alpha h(\sigma) \\ &+ \sum_{i=1}^{m-2} \gamma_i I_q^\alpha h(\xi_i) - a I_q^{\alpha-\beta} h(\zeta_1) - b I_q^{\alpha-\beta} h(\zeta_2) \Bigg) \Bigg), \\ c_1 =& \frac{1}{A} \left(\frac{\delta \sum_{i=1}^{m-2} \gamma_i}{1-\delta} I_q^\alpha h(\sigma) + \sum_{i=1}^{m-2} \gamma_i I_q^\alpha h(\xi_i) - a I_q^{\alpha-\beta} h(\zeta_1) - b I_q^{\alpha-\beta} h(\zeta_2) \right).\end{aligned}$$

Substituting the values of c_0 and c_1 in (5), we can obtain (4). This completes the proof. □

3. Main results

Let $\mathscr{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm: $\|u\| = \sup\{|u(t)|, t \in [0,1]\}$. In the following we define the operator $\mathscr{T} : \mathscr{C} \to \mathscr{C}$:

$$\begin{aligned}(\mathscr{T}u)(t) =& \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s))d_qs + \int_0^\sigma \frac{\delta(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-\delta)} f(s,u(s))d_qs \\ &+ \frac{\delta\sigma+(1-\delta)t}{A(1-\delta)}\Bigg(\frac{\delta\sum_{i=1}^{m-2}\gamma_i}{1-\delta}\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s))d_qs \\ &+ \sum_{i=1}^{m-2}\gamma_i\int_0^{\xi_i}\frac{(\xi_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s))d_qs - a\int_0^{\zeta_1}\frac{(\zeta_1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)} \\ &\times f(s,u(s))d_qs - b\int_0^{\zeta_2}\frac{(\zeta_2-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)} f(s,u(s))d_qs\Bigg). \end{aligned} \tag{7}$$

From Lemma 4, we can see that the problem (1)–(2) has solutions if and only if the operator $\mathscr{T}$ has fixed points.

For the sake of convenience, in the sequel we always set

$$\begin{aligned}\mathfrak{M} =& \frac{1}{\Gamma_q(\alpha+1)} + \frac{|\delta|\sigma^{(\alpha)}}{\Gamma_q(\alpha+1)|1-\delta|} + \frac{|\delta|\sigma+|1-\delta|}{|A||1-\delta|} \\ &\times \left(\frac{\delta\sigma^{(\alpha)}}{\Gamma_q(\alpha+1)|1-\delta|}\sum_{i=1}^{m-2}|\gamma_i| + \sum_{i=1}^{m-2}\frac{|\gamma_i|\xi_i^{(\alpha)}}{\Gamma_q(\alpha+1)} + \frac{|a|\zeta_1^{(\alpha-\beta)}+|b|\zeta_2^{(\alpha-\beta)}}{\Gamma_q(\alpha-\beta+1)}\right). \end{aligned} \tag{8}$$

Now we present the first result of this paper. By using the Banach contraction principle, the existence and uniqueness of solutions for problem (1)–(2) are given.

Theorem 1. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition:*

(H_1) $|f(t,u) - f(t,v)| \le \mathfrak{L}|u-v|$, $\mathfrak{L} > 0$, $\forall t \in [0,1]$, $u,v\in\mathbb{R}$.

Then the problem (1)–(2) *has a unique solution if $\mathfrak{L}\mathfrak{M} < 1$, where $\mathfrak{M}$ is given by* (8).

Proof. Now we firstly prove that the operator $\mathscr{T}$ defined by (7) satisfies the relation $\mathscr{T}\mathscr{B}_r \subset \mathscr{B}_r$, where $\mathscr{B}_r = \{u \in \mathscr{C} : \|u\| \le r\}$, $r \ge \mathfrak{M}\varphi/(1-\mathfrak{L}\mathfrak{M})$, $\varphi = \sup_{t\in[0,1]}|f(t,0)|$. For $x \in \mathscr{B}_r$, $t\in[0,1]$, from the assumption (H_1), we obtain

$$|f(t,u(t))| \le |f(t,u(t)) - f(t,0)| + |f(t,0)| \le \mathfrak{L}\|u\| + \varphi \le \mathfrak{L}r + \varphi. \tag{9}$$

In view of (8) and (9), we have

$$\begin{aligned}\|\mathscr{T}u\| \le& \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs + \int_0^\sigma \frac{|\delta|(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)|1-\delta|}|f(s,u(s))|d_qs \\ &+ \left|\frac{\delta\sigma+(1-\delta)t}{A(1-\delta)}\right|\Bigg(\frac{|\delta|\sum_{i=1}^{m-2}|\gamma_i|}{|1-\delta|}\int_0^\sigma\frac{(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\ &+ \sum_{i=1}^{m-2}|\gamma_i|\int_0^{\xi_i}\frac{(\xi_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs + |a|\int_0^{\zeta_1}\frac{(\zeta_1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)} \\ &\times |f(s,u(s))|d_qs + |b|\int_0^{\zeta_2}\frac{(\zeta_2-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|f(s,u(s))|d_qs\Bigg)\Bigg\} \\ \le&(\mathfrak{L}r+\varphi)\sup_{t\in[0,1]}\Bigg\{\frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)} + \frac{|\delta|\sigma^{(\alpha)}}{\Gamma_q(\alpha+1)|1-\delta|} + \left|\frac{\delta\sigma+(1-\delta)t}{A(1-\delta)}\right|\end{aligned}$$

$$\times\left(\frac{\delta\sigma^{(\alpha)}}{\Gamma_q(\alpha+1)|1-\delta|}\sum_{i=1}^{m-2}|\gamma_i|+\sum_{i=1}^{m-2}\frac{|\gamma_i|\xi_i^{(\alpha)}}{\Gamma_q(\alpha+1)}+\frac{|a|\zeta_1^{(\alpha-\beta)}+|b|\zeta_2^{(\alpha-\beta)}}{\Gamma_q(\alpha-\beta+1)}\right)\Bigg\}$$
$$\leq(\mathfrak{L}r+\varphi)\mathfrak{M}\leq r.$$

This implies that $\mathscr{T}\mathscr{B}_r\subset\mathscr{B}_r$. Nextly, making use of the condition (H_1) and (8), we get

$$\|\mathscr{T}u-\mathscr{T}v\|\leq\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs+\int_0^\sigma\frac{|\delta|(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)|1-\delta|}d_qs+\left|\frac{\delta\sigma+(1-\delta)t}{A(1-\delta)}\right|$$
$$\times\left(\frac{|\delta|\sum_{i=1}^{m-2}|\gamma_i|}{|1-\delta|}\int_0^\sigma\frac{(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs+\sum_{i=1}^{m-2}|\gamma_i|\int_0^{\xi_i}\frac{(\xi_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs\right.$$
$$\left.+|a|\int_0^{\zeta_1}\frac{(\zeta_1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}d_qs+|b|\int_0^{\zeta_2}\frac{(\zeta_2-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}d_qs\right)\Bigg\}\mathfrak{L}\|u-v\|$$
$$\leq\mathfrak{L}\mathfrak{M}\|u-v\|,\quad\forall u,v\in\mathscr{C},$$

which shows that the operator $\mathscr{T}$ is a contraction according to the given assumption $\mathfrak{L}\mathfrak{M}<1$. Thus, by using the Banach contraction principle, there exists a unique fixed point for the operator $\mathscr{T}$ which corresponds to the unique solution for the problem (1)–(2). This completes the proof. □

Theorem 2. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (H_1). Furthermore, we assumed that $|f(t,u)|\leq\mu(t)$, $\forall(t,u)\in[0,1]\times\mathbb{R}$, and $\mu\in C([0,1],\mathbb{R}^+)$. Then the problem* (1)*–*(2) *has at least one solution on* $[0,1]$ *if $\mathfrak{L}(\mathfrak{M}-1/\Gamma_q(\alpha+1))<1$, where $\mathfrak{M}$ is given by* (8).

Proof. Let us consider the set $\mathfrak{B}_\nu=\{u\in\mathscr{C}:\|u\|\leq\nu\}$ with $\nu\geq\mathfrak{M}\|\mu\|$, where $\|\mu\|=\sup_{t\in[0,1]}|\mu(t)|$. In order to satisfy the hypothesis of Krasnoselskii's fixed point theorem, we define two operators $\mathscr{T}_1$ and $\mathscr{T}_2$ on $\mathfrak{B}_\nu$ as

$$(\mathscr{T}_1u)(t)=\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs,$$
$$(\mathscr{T}_2u)(t)=\int_0^\sigma\frac{\delta(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-\delta)}f(s,u(s))d_qs+\frac{\delta\sigma+(1-\delta)t}{A(1-\delta)}\left(\frac{\delta\sum_{i=1}^{m-2}\gamma_i}{1-\delta}\int_0^\sigma\frac{(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\right.$$
$$\times f(s,u(s))d_qs+\sum_{i=1}^{m-2}\gamma_i\int_0^{\xi_i}\frac{(\xi_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs$$
$$\left.-a\int_0^{\zeta_1}\frac{(\zeta_1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}f(s,u(s))d_qs-b\int_0^{\zeta_2}\frac{(\zeta_2-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}f(s,u(s))d_qs\right).$$

For $u,v\in\mathfrak{B}_\nu$, it is easy to prove that $\|\mathscr{T}_1u+\mathscr{T}_2v\|\leq\|\mu\|\mathfrak{M}\leq\nu$, which means that $\mathscr{T}_1u+\mathscr{T}_2v\in\mathfrak{B}_\nu$.

Using the assumption (H_1), for $u,v\in\mathscr{C}$, $t\in[0,1]$, we get

$$\|\mathscr{T}_2u-\mathscr{T}_2v\|=\sup_{t\in[0,1]}\Bigg\{\int_0^\sigma\frac{|\delta|(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)|1-\delta|}d_qs+\left|\frac{\delta\sigma+(1-\delta)t}{A(1-\delta)}\right|$$
$$\left(\frac{|\delta|\sum_{i=1}^{m-2}|\gamma_i|}{|1-\delta|}\int_0^\sigma\frac{(\sigma-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs+\sum_{i=1}^{m-2}|\gamma_i|\int_0^{\xi_i}\frac{(\xi_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs\right.$$
$$\left.+|a|\int_0^{\zeta_1}\frac{(\zeta_1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}d_qs+|b|\int_0^{\zeta_2}\frac{(\zeta_2-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}d_qs\right)\Bigg\}\mathfrak{L}\|u-v\|$$
$$\leq\mathfrak{L}(\mathfrak{M}-1/\Gamma_q(\alpha+1))\|u-v\|,$$

which implies that $\mathscr{T}_2$ is a contraction in view of the condition $\mathfrak{L}(\mathfrak{M} - 1/\Gamma_q(\alpha+1)) < 1$.

On the other hand, continuity of f implies that the operator $\mathscr{T}_1$ is continuous. Also, $\mathscr{T}_1$ is uniformly bounded on $\mathfrak{B}_\nu$ as

$$\|\mathscr{T}_1 u\| \leq \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s))| d_q s \right\} \leq \frac{\|\mu\|}{\Gamma_q(\alpha+1)}.$$

Furthermore, letting $\sup_{(t,u)\in[0,1]\times\mathfrak{B}_\nu} |f(t,u)| = \mathfrak{F} < \infty$ and $0 < t_1 < t_2 < 1$, we get

$$\|(\mathscr{T}_1 u)(t_2) - (\mathscr{T}_1 u)(t_1)\| \leq \frac{\mathfrak{F}}{\Gamma_q(\alpha+1)} |t_2^{(\alpha)} - t_1^{(\alpha)}|,$$

which tends to zero independent of u as $(t_2 - t_1) \to 0$. This implies that $\mathscr{T}_1$ is relatively compact on $\mathfrak{B}_\nu$. Hence by the Arzelá–Ascoli theorem, $\mathscr{T}_1$ is compact on $\mathfrak{B}_\nu$. Thus the hypothesis of Krasonselskii's fixed theorem is satisfied and consequently the problem (1)–(2) has at least one solution on $[0,1]$. This completes the proof. □

Theorem 3. *Assume that exists a positive constant $\mathscr{L}_1$ such that $|f(t,u)| \leq \mathscr{L}_1$ for all $t \in [0,1]$, $u \in \mathbb{R}$. Then there exists at least one solution for the problem* (1)–(2) *on* $[0,1]$.

Proof. We firstly show that the operator $\mathscr{T}$ is completely continuous. Clearly continuity of $\mathscr{T}$ follows from the continuity of f and it is easy to establish by the given assumption that $|(\mathscr{T}u)(t)| \leq \mathscr{L}_1\mathfrak{M} = \mathscr{L}_2$. Let $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}\|(\mathscr{T}u)(t_2) - (\mathscr{T}u)(t_1)\| \leq &\mathscr{L}_1 \Bigg\{ \frac{|t_2^{(\alpha)} - t_1^{(\alpha)}|}{\Gamma_q(\alpha+1)} + \frac{|t_2 - t_1|}{|A|} \Bigg(\frac{\delta\sigma^{(\alpha)}}{\Gamma_q(\alpha+1)|1-\delta|} \sum_{i=1}^{m-2} |\gamma_i| \\ &+ \sum_{i=1}^{m-2} \frac{|\gamma_i|\xi_i^{(\alpha)}}{\Gamma_q(\alpha+1)} + \frac{|a|\zeta_1^{(\alpha-\beta)} + |b|\zeta_2^{(\alpha-\beta)}}{\Gamma_q(\alpha-\beta+1)} \Bigg) \Bigg\},\end{aligned}$$

Clearly, the right-hand side tends to zero independently of u as $(t_2 - t_1) \to 0$. Thus, by the Arzelá–Ascoli theorem, the operator $\mathscr{T}$ is completely continuous.

Nextly, we consider the set $\mathscr{V} = \{u \in \mathscr{C} : u = \varepsilon\mathscr{T}u, 0 < \varepsilon < 1\}$. To show that $\mathscr{V}$ is bounded, let $t \in [0,1]$. Then it is easy to see that $|u(t)| = \varepsilon|\mathscr{T}u| \leq \mathscr{L}_1\mathfrak{M} = \mathscr{L}_2$. Hence, for any $u \in \mathscr{V}$, $t \in [0,1]$, we have $\|u\| \leq \mathscr{L}_2$. So $\mathscr{V}$ is bounded. Thus, the Schauder fixed point theorem applies and the problem (1)–(2) has at least one solution on $[0,1]$. This completes the proof. □

4. Two examples

In this section, we give two examples to show the effectiveness of the obtained results.

Example 1. Consider the boundary value problem

$$\begin{aligned}&D_q^{\frac{3}{2}} u(t) = \frac{\sin u}{\sqrt{t^2+225}} + \sqrt{t+7}, \quad t \in [0,1], \\ &u(0) = \frac{1}{2}u(\frac{1}{6}), \quad \frac{2}{3}{}^cD_q^{\frac{1}{2}} u(\frac{1}{3}) + \frac{1}{2}{}^cD_q^{\frac{1}{2}} u(\frac{4}{5}) = \frac{1}{3}u(\frac{1}{2}) + \frac{1}{4}u(\frac{2}{3}) + \frac{1}{3}u(\frac{3}{4}).\end{aligned} \tag{10}$$

Here $q = 1/2$, $f(t,u(t)) = \frac{\sin u}{\sqrt{t^2+225}} + \sqrt{t+7}$ is continuous function. It is easy to see that $A \approx 0.2712$, $\mathfrak{M} = 8.5474$ and $\mathfrak{L} = 1/15$. We can see that the conditions of Theorem 1 are satisfied. From Theorem 1, the problem (10) has a unique solution.

Example 2. Consider the boundary value problem

$$D_q^{\frac{3}{2}}u(t) = \frac{\sin u}{\sqrt{t^2+4}} + \cos t + 1, \quad t \in [0,1],$$
$$u(0) = \frac{1}{2}u(\frac{1}{6}), \quad \frac{2}{3}{}^cD_q^{\frac{1}{2}}u(\frac{1}{3}) + \frac{1}{2}{}^cD_q^{\frac{1}{2}}u(\frac{4}{5}) = \frac{1}{3}u(\frac{1}{2}) + \frac{1}{4}u(\frac{2}{3}) + \frac{1}{3}u(\frac{3}{4}). \tag{11}$$

Here $q = 1/2$, $f(t,u(t)) = \frac{\sin u}{\sqrt{t^2+4}} + \cos t + 1$ is continuous function. On the other hand, we have $|f(t,u(t))| = |\frac{\sin u}{\sqrt{t^2+4}} + \cos t + 1| \leq 1/2 + 1 + 1 = 5/2 = \mathscr{L}_1$. Therefore, by Theorem 3, the problem (11) has at least one solution.

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