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IDENTITIES FOR HYPERGEOMETRIC INTEGRALS AND EULER SUMS

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Abstract. We provide an explicit analytical representation for a number of logarithmic and hypergeometric function integrals in terms of the polygamma function and other special functions. The integrals in question will be associated with harmonic numbers of positive terms. A few examples of integrals will be given an identity in terms of some special functions including the Riemann zeta function.

1. Introduction and Preliminaries

In this paper we will develop explicit analytical representations, identities, new families of integral representations, of the form:

$$\int_0^1 x^{p-1} \ln(1-x) \; {}_2F_1 \left[\begin{array}{c} 1,2 \\ 2+k \end{array} \middle| x^p \right] dx \tag{1}$$

for (k,p) the set of positive integers and where ${}_2F_1 \left[\begin{array}{c} \cdot,\cdot \\ \cdot \end{array} \middle| z \right]$ is the classical Gauss hypergeometric function. Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers and let $\mathbb{N} := \{1,2,3,\cdots\}$ be the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Riemann zeta function is defined, for $s \in \mathbb{C}$ with $\Re(s) > 1$ by $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$. For $p \in \mathbb{N} := \{1,2,3,...\}$ we define the generalized harmonic number of order m as $H_p^{(m)} = \zeta_p(m) = \sum_{j=1}^{p} \frac{1}{j^m}$. Let

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1), \; H_0 := 0 \tag{2}$$

be the nth harmonic number, where γ denotes the Euler-Mascheroni constant and $\psi(z)$ is the Psi (or Digamma) function defined by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(t)\, dt.$$

In the case of non-integer values of n such as (for example) a value $\rho \in \mathbb{R}$, the generalized harmonic numbers $H_\rho^{(m+1)}$ may be defined, in terms of the polygamma functions

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}}\{\log \Gamma(z)\} \qquad (n \in \mathbb{N}_0),$$

by

$$H_\rho^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\, \psi^{(m)}(\rho+1) \tag{3}$$
$$(\rho \in \mathbb{R} \setminus \{-1,-2,-3,\cdots\}; \; m \in \mathbb{N}),$$

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where $\zeta(z)$ is the Riemann zeta function. Whenever we encounter harmonic numbers of the form $H_\rho^{(m)}$ at admissible real values of ρ, they may be evaluated by means of this known relation (3). In the exceptional case of (3) when $m=0$, we may define $H_\rho^{(1)}$ by

$$H_\rho^{(1)} = H_\rho = \gamma + \psi(\rho+1) \qquad (\rho \in \mathbb{R}\setminus\{-1,-2,-3,\cdots\}).$$

We assume that

$$H_0^{(m)} = 0 \qquad (m \in \mathbb{N}).$$

In the case of non integer values of the argument $z=\frac{r}{q}$, we may write the generalized harmonic numbers, $H_z^{(\alpha+1)}$, in terms of polygamma functions

$$H_{\frac{r}{q}}^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{q}+1\right), \ \frac{r}{q} \neq \{-1,-2,-3,...\},$$

where $\zeta(z)$ is the zeta function. When we encounter harmonic numbers at possible rational values of the argument, of the form $H_{\frac{r}{q}}^{(\alpha)}$ they maybe evaluated by an available relation in terms of the polygamma function $\psi^{(\alpha)}(z)$ or, for rational arguments $z=\frac{r}{q}$, and we also define

$$H_{\frac{r}{q}}^{(1)} = \gamma + \psi\left(\frac{r}{q}+1\right), \text{ and } H_0^{(\alpha)} = 0.$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [6], or Choi and Cvijovic [3] in terms of the polylogarithmic or other special functions. Polygamma functions at negative rational values of the argument can also be explicitly evaluated, for example

$$H_{-\frac{1}{6}} = -\frac{\sqrt{3}\pi}{2} - \frac{3}{2}\ln 3 - 2\ln 2, \ H_{-\frac{1}{4}}^{(2)} = 8G - 5\zeta(2), \ H_{-\frac{3}{4}}^{(3)} = -\pi^3 - 27\zeta(3).$$

Some specific values are listed in the books [12], [13] and [14]. Some results for sums of harmonic numbers may be seen in the works of [4], [15] and references therein.

The following lemma is proved in [7].

Lemma 1. *Let k be a positive integer. Then:*

$$\begin{aligned} M(k) &= \sum_{n\geq 1} \frac{H_n}{n\binom{n+k}{k}} \\ &= -\frac{1}{1+k}\int_0^1 \ln(1-x)\ {}_2F_1\left[\begin{matrix}1,2\\2+k\end{matrix}\middle| x\right] dx && (4) \\ &= \zeta(2) + \frac{1}{2}\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\left(H_{j-1}^2 + H_{j-1}^{(2)}\right) && (5) \end{aligned}$$

The following Lemma is given in [10]

Lemma 2. *Let $p\in\mathbb{N}$ and $r=1,2,3,...,p-1$. Then:*

$$S(k,p,r) = \frac{1}{1+k}\int_0^1 \frac{\ln(1-x)}{x^{\frac{r}{p}}}\left(\begin{matrix}\frac{r}{p}\ {}_2F_1\left[\begin{matrix}1,1\\2+k\end{matrix}\middle| x\right] \\ -\ {}_2F_1\left[\begin{matrix}1,2\\2+k\end{matrix}\middle| x\right]\end{matrix}\right) dx \qquad (6)$$

$$= \sum_{n\geq 1} \frac{H_{n-\frac{r}{p}}}{n\left(\begin{array}{c} n+k \\ k \end{array}\right)}$$

$$= H_{-\frac{r}{p}}\left(\frac{r-pk}{rk}+H_{\frac{r}{p}}\right) \tag{7}$$
$$+\sum_{j=1}^{k}(-1)^{j+1}\left(\begin{array}{c} k \\ j \end{array}\right)\left(H_{j-1+\frac{r}{p}}\left(H_{j-1}-H_{-\frac{r}{p}}\right)-\sum_{\mu=1}^{j-1}\frac{1}{\mu}H_{\mu-1+\frac{r}{p}}\right).$$

In the case when $r=0$ (7) reduces to (5).

The next few theorems relate the main results of this investigation, namely the closed form representation of integral (1).

2. Main results

The following main Theorem is proved

Theorem 1. *Let $k,p\geq 1$ be real positive integer, then*

$$V(k,p) = \sum_{n=1}^{\infty}\frac{H_{pn}}{n\left(\begin{array}{c} n+k \\ k \end{array}\right)}$$

$$= -\frac{p}{1+k}\int_0^1 x^{p-1}\ln(1-x)\ {}_2F_1\left[\left.\begin{array}{c} 1,2 \\ 2+k \end{array}\right| x^p\right]dx \tag{8}$$

$$= \frac{1}{k}\ln p+\frac{1}{p}M(k)+\frac{1}{p}\sum_{r=1}^{p-1}S(k,p,r). \tag{9}$$

where $M(k)$ is given by (5) and $S(k,p,r)$ is given by (7).

Proof. For $p\in\mathbb{N}$ and from the properties of the polygamma function with multiple argument

$$\psi^{(n)}(pz)=\delta_{n,0}+\frac{1}{p^{n+1}}\sum_{r=0}^{p-1}\psi^{(n)}(z+\frac{r}{p}),$$

where $\delta_{n,0}$ is the Kronecker delta, we are able to rewrite, in terms of harmonic numbers, and using the properties of the digamma function, as

$$H_{pn}=\ln p+\frac{1}{p}H_n+\frac{1}{p}\sum_{r=1}^{p-1}H_{n-\frac{r}{p}}.$$

Here $H_{n-\frac{r}{p}}$ may be thought of as shifted harmonic numbers, other results on summing shifted harmonic numbers are published in [9],[10] and [11]. Now summing over the integers n

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{pn}}{n\binom{n+k}{k}} &= \sum_{n=1}^{\infty}\frac{1}{n\binom{n+k}{k}}\left(\ln p+\frac{1}{p}H_n+\frac{1}{p}\sum_{r=1}^{p-1}H_{n-\frac{r}{p}}\right)\\ &= \frac{1}{k}\ln p+\frac{1}{p}\sum_{n=1}^{\infty}\frac{H_n}{n\binom{n+k}{k}}+\frac{1}{p}\sum_{r=1}^{p-1}\sum_{n=1}^{\infty}\frac{H_{n-\frac{r}{p}}}{n\binom{n+k}{k}}\\ &= \frac{1}{k}\ln p+\frac{1}{p}M(k)+\frac{1}{p}\sum_{r=1}^{p-1}S(k,p,r)\end{aligned}$$

which is the result (9). For the integral representation (8) consider, for $|t|\le 1$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{t^n}{n\binom{n+k}{k}\binom{pn+j}{j}} &= \sum_{n=1}^{\infty}\frac{p\ t^n\ \Gamma(j+1)\Gamma(pn)}{\binom{n+k}{k}\Gamma(pn+j+1)}\\ &= p\sum_{n=1}^{\infty}\frac{t^n\ B(j+1,pn)}{\binom{n+k}{k}}. \qquad (10)\end{aligned}$$

$$\Gamma(z) = \int_0^{\infty} e^{-t}\ t^{z-1}\ dt,\ \Re(z)>0$$

is the Euler gamma function and

$$B(a,b)=B(b,a)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}(1-t)^{b-1}\ dt,$$

where $\Re(a)>0, \Re(b)>0$ is the beta function. From (10)

$$\begin{aligned}p\sum_{n=1}^{\infty}\frac{t^n\ B(j+1,pn)}{\binom{n+k}{k}} &= p\int_0^1\frac{(1-x)^j}{x}\sum_{n=1}^{\infty}\frac{(tx^p)^n}{\binom{n+k}{k}}dx\\ &= \frac{pt}{1+k}\int_0^1(1-x)^j x^{p-1}\ {}_2F_1\left[\left.\begin{matrix}1,2\\2+k\end{matrix}\right|tx^p\right]dx.\end{aligned}$$

Putting $t=1$, differentiating with respect to j then putting $j\to 0$, that is

$$\begin{aligned}&\lim_{j\to 0}\left(\frac{d}{dj}\sum_{n=1}^{\infty}\frac{t^n}{n\binom{n+k}{k}\binom{pn+j}{j}}\right)\\ &= \lim_{j\to 0}\left(\frac{d}{dj}\left(\frac{pt}{1+k}\int_0^1(1-x)^j x^{p-1}\ {}_2F_1\left[\left.\begin{matrix}1,2\\2+k\end{matrix}\right|tx^p\right]dx\right)\right),\end{aligned}$$

and

$$V(k,p) = -\frac{p}{1+k}\int_0^1 x^{p-1}\ln(1-x)\ {}_2F_1\left[\begin{array}{c}1,2\\2+k\end{array}\middle| x^p\right]dx$$

which is the result (8). □

We give two examples to demonstrate the power of the above Theorem

Example 1.

$$V(3,6) = -\frac{3}{2}\int_0^1 x^5\ln(1-x)\ {}_2F_1\left[\begin{array}{c}1,2\\5\end{array}\middle| x^6\right]dx$$

$$= \frac{1}{6}\zeta(2) + \frac{1}{3}\ln 6 + \frac{2646289}{1633632}\ln 3 + \frac{1346713}{765765}\ln 2$$

$$-\frac{1067097\sqrt{3}\pi}{2722720} - \frac{20671373}{24504480}.$$

$$V(2,p) = -\frac{p}{3}\int_0^1 x^{p-1}\ln(1-x)\ {}_2F_1\left[\begin{array}{c}1,2\\4\end{array}\middle| x^p\right]dx$$

$$= \frac{1}{p}\zeta(2) + \frac{1}{2}\ln p + H_p - H_{2p} - \frac{1}{2p}$$

$$+\frac{1}{2p}\sum_{r=1}^{p-1}\frac{(r-p)(r+2p)}{r(r+p)}H_{-\frac{r}{p}}.$$

The case $k = 1$ is interesting in its own right and therefore we have the following result.

Corollary 1. *Under the assumptions of Theorem 1, with $k = 1$, we have,*

$$V(1,p) = \sum_{n=1}^{\infty}\frac{H_{pn}}{n(n+1)}$$

$$= \frac{1}{p}\zeta(2) - \sum_{r=1}^{p-1}\frac{1}{r}H_{-\frac{r}{p}} \tag{11}$$

$$= \frac{1}{p}\zeta(2) + pH_{p-1}^{(2)} - \sum_{r=1}^{p-1}\frac{1}{r}\left(H_{\frac{r}{p}} + \pi\cot\left(\frac{\pi r}{p}\right)\right) \tag{12}$$

$$= p\int_0^1\frac{\ln(1-x)}{x^{p+1}}\ (x^p + \ln(1-x^p))\ dx. \tag{13}$$

Proof. From (8)

$$V(1,p) = \sum_{n=1}^{\infty}\frac{H_{pn}}{n(n+1)} = -\frac{p}{2}\int_0^1 x^{p-1}\ln(1-x)\ {}_2F_1\left[\begin{array}{c}1,2\\3\end{array}\middle| x^p\right]dx$$

$$= p\int_0^1\frac{\ln(1-x)}{x^{p+1}}\ (x^p + \ln(1-x^p))\ dx$$

which is the result (13). From (9)

$$\begin{aligned} V(1,p) &= \ln p + \frac{1}{p}M(1) + \frac{1}{p}\sum_{r=1}^{p-1} S(1,p,r) \\ &= \ln p + \frac{1}{p}\zeta(2) + \frac{1}{p}\sum_{r=1}^{p-1}\left(\frac{r-p}{r}\right)H_{-\frac{r}{p}}. \end{aligned}$$

since

$$\sum_{r=1}^{p-1} H_{-\frac{r}{p}} = -p\ln p,$$

then

$$V(1,p) = \frac{1}{p}\zeta(2) - \sum_{r=1}^{p-1}\frac{1}{r}H_{-\frac{r}{p}}$$

which is the result (11). From the reflection relation of the digamma function

$$\psi(1-z) = \psi(z) + \pi\cot(\pi z)$$

we have

$$H_{-\frac{r}{p}} = H_{\frac{r}{p}-1} + \pi\cot\left(\frac{\pi r}{p}\right)$$

then

$$\begin{aligned} V(1,p) &= \frac{1}{p}\zeta(2) - \sum_{r=1}^{p-1}\frac{1}{r}\left(H_{\frac{r}{p}} - \frac{p}{r} + \pi\cot\left(\frac{\pi r}{p}\right)\right) \\ &= \frac{1}{p}\zeta(2) + pH_{p-1}^{(2)} - \sum_{r=1}^{p-1}\frac{1}{r}\left(H_{\frac{r}{p}} + \pi\cot\left(\frac{\pi r}{p}\right)\right) \end{aligned}$$

hence (12) follows. The identity (12) is noteworthy because it introduces finite cotangent sums, which is a separate field of study in itself. Finite cotangent sums of the form

$$\sum_{r=1}^{p-1}\cot^m\left(\frac{\pi r}{p}\right),$$

and their variations, have been investigated, see [1], [2] and [5]. The author has not seen an investigation of

$$\sum_{r=1}^{p-1} r^q\cot^m\left(\frac{\pi r}{p}\right),$$

$q \in \mathbb{Z}\setminus\{0\}, m \in \mathbb{N}$, in the published literature. From (13) we can also extract the following interesting, and new identity

$$\int_0^1 \frac{\ln(1-x)\ln(1-x^p)}{x^{p+1}}\,dx = \frac{1+p^2}{p^2}\zeta(2) - \frac{1}{p}\sum_{r=1}^{p-1}\frac{1}{r}H_{-\frac{r}{p}} \tag{14}$$

$$= \frac{1+p^2}{p^2}\zeta(2) + H_{p-1}^{(2)} - \frac{1}{p}\sum_{r=1}^{p-1}\frac{1}{r}\left(H_{\frac{r}{p}} + \pi\cot\left(\frac{\pi r}{p}\right)\right) \tag{15}$$

The following new identity is obtained by using the result, from [8]

$$\int_0^1 \frac{\ln(1-x)}{x^{p+1}}\ln\left(\frac{1+x^p}{1-x^p}\right)dx = \frac{2}{p}H_{p-1}\ln 2 + \frac{1}{p}\sum_{r=1}^{p-1}\frac{1}{r}H_{-\frac{r+p}{2p}} - \frac{2}{p}G(p) \tag{16}$$

where

$$G(p)=\sum_{r=0}^{p-1}\ln^2\left(2\sin\left(\frac{(2r+1)\pi}{2p}\right)\right).$$

Now rearranging (16) and using (14) yields the new identity

$$\begin{aligned}\int_0^1\frac{\ln(1-x)\ln(1+x^p)}{x^{p+1}}\,dx &= \frac{2}{p}H_{p-1}\ln 2+\frac{1}{p}\sum_{r=1}^{p-1}\frac{1}{r}\left(H_{-\frac{r+p}{2p}}-H_{-\frac{r}{p}}\right)\\ &\quad -\frac{1}{2}\zeta(2)-\frac{2}{p}G(p).\end{aligned} \tag{17}$$

None of the integrals (14), (16) and (17) and their generalizations can be evaluated with mathematical packages such as *Mathematica*. □

The closed form (9) of the integral (8) is an exact identity which is expressed in finite sums of harmonic numbers and special functions. The following Theorem gives a bound on the intgral (8).

Theorem 2. *Let $k,p\in\mathbb{N}$ then*

$$\frac{H_p}{1+k}<V(k,p)\leq\frac{H_p}{k-1} \tag{18}$$

where

$$V(k,p)=-\frac{p}{1+k}\int_0^1 x^{p-1}\ln(1-x)\;{}_2F_1\left[\begin{matrix}1,2\\2+k\end{matrix}\,\middle|\,x^p\right]dx$$

Proof. The infinite sum $V(k,p)$ is one of positive terms, monotonic increasing and therefore

$$V(k,p)>\frac{H_p}{1+k}.$$

Consider the integral inequality

$$\int_{x_0}^{x_1}|f(x)\,g(x)|\;dx\leq\sup_{x\in[x_0,x_1]}|f(x)|\int_{x_0}^{x_1}|g(x)|\;dx$$

for integrable functions $f(x)$ and $g(x)$ and $0\leq x_0<x_1\in\mathbb{R}$. Now

$$\sup_{x\in[x_0,x_1]}|f(x)|=\sup_{x\in[0,1]}\left|{}_2F_1\left[\begin{matrix}1,2\\2+k\end{matrix}\,\middle|\,x^p\right]\right|=\frac{k+1}{k-1}$$

since for $x\in[0,1]$ and $p\in\mathbb{N}$, ${}_2F_1\left[\begin{matrix}1,2\\2+k\end{matrix}\,\middle|\,x^p\right]$ is monotonic. Also

$$\int_{x_0}^{x_1}|g(x)|\;dx=\int_0^1\left|x^{p-1}\ln(1-x)\right|\;dx=\frac{H_p}{p},$$

therefore

$$\frac{H_p}{1+k}<V(k,p)\leq\frac{p}{k+1}.\frac{H_p}{p}.\frac{k+1}{k-1}$$

and (18) follows. □

A generalization of Theorem 1 has been submitted for publication.

References

[1] Alzer, H. and Chu, W., *Two trigonometric identities*, Irish Math. Soc. Bulletin. **73** (2014), 21–28.

[2] Berndt, B.C. and Boon Pin Yeap, *Explicit evaluations and reciprocity theorems for finite trigonometric sums*, Adv. in Appl. Math. **29** (2002), 358–385.

[3] Choi, J. and Cvijović, D., *Values of the polygamma functions at rational arguments*, J. Phys. A: Math. Theor. **40** (2007), 15019–15028, Corrigendum, ibidem, **43** (2010), 239801 (1 p).

[4] Choi, J. and Srivastava, H.M., *Some summation formulas involving harmonic numbers and generalized harmonic numbers*, Math. Comput. Modelling. **54** (2011), 2220–2234.

[5] Cvijović, D., *Summation formulae for finite cotangent sums*, Appl. Math. Comput. **215** (2009), 1135–1140.

[6] Kölbig, K., *The polygamma function* $\psi(x)$ *for* $x = 1/4$ *and* $x = 3/4$, J. Comput. Appl. Math. **75** (1996), 43–46.

[7] Sofo, A. and Cvijović, D., *Extensions of Euler harmonic sums*, Appl. Anal. Discrete Math. **6** (2012), 317–328.

[8] Sofo, A., *Evaluation of integrals with hypergeometric and logarithmic functions*, Submitted 2016.

[9] Sofo, A., *Harmonic numbers at half integer values*, Integral Transforms Spec. Funct. **27** (2016), no. 6, 430–442.

[10] Sofo, A. and Srivastava, H.M., *A family of shifted harmonic sums*, Ramanujan J. **37** (2015), no. 1, 89–108.

[11] Sofo, A., *New families of alternating harmonic number sums*, Tbilisi Math. J. **8** (2015), no. 2, 195–209.

[12] Sofo, A., *Computational Techniques for the Summation of Series*, Kluwer Academic/Plenum Publishers, New York, 2003.

[13] Srivastava, H.M. and Choi, J., *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, London, 2001.

[14] Srivastava, H.M. and Choi, J., *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[15] Xu, C., Yan, Y. and Shi, Z., *Euler sums and integrals of polylogarithmic functions*, J. Number Theory. **165** (2016), 84–108.

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