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SOME INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper a general integral identity for twice differentiable convex functions is derived. We establish some new Hermite Hadamard's type inequalities for functions whose absolute values of derivatives are convex.

1. Introduction

Let $f : I \subseteq R \to R$ be a convex function defined on the interval I of real numbers; i.e

$$f(tx + (1-t)) \leq tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$. Lots of results associated with convex functions can found in the literature. The following classical double inequality;

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality for convex functions. Both inequalities hold in reversed direction if f is concave. Since its discovery in 1983, Hermite-Hadamard's inequality [9] has been considered the most useful inequality in mathematical analysis. A number of the papers has been written on this inequality, providing new proofs, noteworthy extensions, generalizations and numerous applications. See ([1]-[15]) and references therein.

In [6] Dragomir proved the following lemma for twice differentiable functions.

Lemma 1. *$f : I \subseteq R \to R$ be a twice differentiable mapping on I° with f'' integrable on $[a, b] \subset I^\circ$. Then we have the identity:*

$$\frac{1}{2}\int_a^b (x-a)(b-x) f''(x) dx = \frac{(b-a)}{2}(f(a) + f(b)) - \int_a^b f(x) dx \tag{2}$$

In [13] Sarıkaya and Aktan obtained the following results which are contained Hermite Hadamard type inequality using the above lemma.

Theorem 1. *Let $I \subseteq R$ be an open interval, a,b $\in I$ with $a < b$ and $f : I \to R$ twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. If $|f''|$ is a convex on $[a, b]$ then the following inequality holds;*

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$$\left|(\lambda-1)\, f(\frac{a+b}{2})-\lambda\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\begin{cases}\frac{(b-a)^2}{12}\Big[\left(\lambda^4+(1+\lambda)+(1-\lambda)^3+\frac{5\lambda-3}{4}\right)\left|f''(a)\right| \\ \quad+\left(\lambda^4+(2-\lambda)+\lambda^3+\frac{1-3\lambda}{4}\right)\left|f''(b)\right|\Big], & for \quad 0\leq\lambda\leq\frac{1}{2} \\ \frac{(b-a)^2(3\lambda-1)}{48}\Big[\left|f''(a)\right|+\left|f''(b)\right|\Big] & for \quad \frac{1}{2}\leq\lambda\leq 1.\end{cases}$$

Proposition 1. (*Trapezoid inequality*) *Under the assumptions Theorem 1 with* $\lambda=1$ *in Theorem 1, we have;*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right] \tag{3}$$

Theorem 2. *Let* $I\subseteq R$ *be an open interval,* $a,b\in I$ *with* $a<b$ *and* $f:I\to R$ *twice differentiable mapping such that* f'' *is integrable and* $0\leq\lambda\leq 1$. *If* $|f''|^q$ *is a convex on* $[a,b]$, $q\geq 1$ *then the following inequality hold:*

$$\left|(\lambda-1)\, f\left(\frac{a+b}{2}\right)-\lambda\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\begin{cases}\frac{(b-a)^2}{2}\left(\frac{\lambda^3}{3}+\frac{1-3\lambda}{24}\right)^{1-\frac{1}{q}} \\ \times\begin{cases}\Big([\frac{\lambda^4}{6}+\frac{3-8\lambda}{3.2^6}]\left|f''(a)\right|^q \\ +[\frac{(2-\lambda)\lambda^3}{6}+\frac{5-16\lambda}{3.2^6}]\left|f''(b)\right|^q\Big)^{\frac{1}{q}} \\ +\Big([\frac{1+\lambda}{6}(1-\lambda)^3+\frac{48\lambda-27}{3.2^6}]\left|f''(a)\right|^q \\ +[\frac{\lambda^4}{6}+\frac{3-8\lambda}{3.2^6}]\left|f''(b)\right|^q\Big)^{\frac{1}{q}},\end{cases} & for \quad 0\leq\lambda\leq\frac{1}{2} \\ \frac{(b-a)^2}{2}(\frac{3\lambda-1}{24})^{1-\frac{1}{q}} \\ \times\begin{cases}\left(\frac{8\lambda-3}{3.2^6}\left|f''(a)\right|^q+\frac{16\lambda-5}{3.2^6}\left|f''(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{16\lambda-5}{3.2^6}\left|f''(a)\right|^q+\frac{8\lambda-3}{3.2^6}\left|f''(b)\right|^q\right)^{\frac{1}{q}},\end{cases} & for \quad \frac{1}{2}\leq\lambda\leq 1\end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proposition 2. *Under the assumptions Theorem 2 with* $\lambda=1$ *in Theorem 2 then we get the "trapezoid inequality"*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{4}$$
$$\leq\frac{(b-a)^2}{24}\left\{\left[\frac{5}{16}\left|f''(a)\right|^q+\frac{11}{16}\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\left[\frac{11}{16}\left|f''(a)\right|^q+\frac{5}{16}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right.$$

In this paper, we established some new results related to the right-hand side of Hermite - Hadamard type inequality for functions whose absolute values of second derivatives is convex on the interval $[a,b]$.

2. Main results

In order to prove our main theorems, we need the following Lemma.

Lemma 2. *$f : I \subseteq R \to R$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$, where $a,b \in I$ with $a<b$ and $\alpha, \lambda \in [0,1]$. Then we have the identity;*

$$\begin{aligned}
&\lambda\left[\alpha f(a)+(1-\alpha) f(b)\right]+(1-\lambda)\left[f(\alpha a+(1-\alpha)(b))\right] \\
&-(b-a)\left[\left(\lambda(1-\alpha)-\frac{1}{2}\right) f'(b)+(1-\alpha)(1-\lambda) f'(\alpha a+(1-\alpha) b)\right] \\
&-\frac{1}{b-a}\int_a^b f(x)\,dx \qquad (5)\\
=\;& \frac{(b-a)^2}{2}\left\{\int_0^{1-\alpha} t(2\alpha\lambda-t) f''(tb+(1-t)a)\,dt\right. \\
&\left.+\int_{1-\alpha}^1 t\left[2(1-\lambda(1-\alpha))-t\right] f''(tb+(1-t)a)\,dt\right\}
\end{aligned}$$

Proof. We note that;

$$\begin{aligned}
I &= \int_0^{1-\alpha} t(2\alpha\lambda-t) f''(tb+(1-t)a)\,dt \\
&\quad+\int_{1-\alpha}^1 t\left[2(1-\lambda(1-\alpha))-t\right] f''(tb+(1-t)a)\,dt
\end{aligned}$$

Integrating by parts, we get;

$$\begin{aligned}
I &= \left.\frac{t(2\alpha\lambda-t) f'(tb+(1-t)a)}{b-a}\right|_0^{1-\alpha}-\frac{2}{b-a}\int_0^{1-\alpha}(\alpha\lambda-t) f'(tb+(1-t)a)\,dt \\
&\quad+\left.\frac{t\left[2(1-\lambda(1-\alpha))-t\right] f'(tb+(1-t)a)}{b-a}\right|_{1-\alpha}^1 \\
&\quad-\frac{2}{b-a}\int_{1-\alpha}^1\left[(1-\lambda(1-\alpha))-t\right] f'(tb+(1-t)a)\,dt \\
&= \frac{(1-2\lambda(1-\alpha)) f'(b)-2(1-\alpha)(1-\lambda) f'(\alpha a+(1-\alpha)(b))}{b-a} \\
&\quad-\frac{2}{b-a}\left[\int_0^{1-\alpha}(\alpha\lambda-t) f'(tb+(1-t)a)\,dt\right. \\
&\quad\left.+\int_{1-\alpha}^1\left[(1-\lambda(1-\alpha))-t\right] f'(tb+(1-t)a)\,dt\right].
\end{aligned}$$

Again in a same way, using integrating by parts formula for the last above integrals, we have

$$\begin{aligned}
&\left[\int_0^{1-\alpha}(\alpha\lambda-t) f'(tb+(1-t)a)\,dt+\int_{1-\alpha}^1\left[(1-\lambda(1-\alpha))-t\right] f'(tb+(1-t)a)\,dt\right] \\
&=\left[-\frac{\lambda\left[\alpha f(a)+(1-\alpha) f(b)\right]}{b-a}-\frac{(1-\lambda)\left[f(\alpha a+(1-\alpha)(b))\right]}{b-a}+\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\right].
\end{aligned}$$

Thus

$$I=\frac{(1-2\lambda(1-\alpha))f'(b)-(1-\alpha)2(1-\lambda)f'(\alpha a+(1-\alpha)(b))}{b-a}-$$
$$\frac{2}{b-a}\left[-\frac{\lambda[\alpha f(a)+(1-\alpha)f(b)]}{b-a}-\frac{(1-\lambda)[f(\alpha a+(1-\alpha)b)]}{b-a}+\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\right],$$

multiplying I with $\frac{(b-a)^2}{2}$ we get following result, i.e

$$\frac{(b-a)^2}{2}I=\lambda[\alpha f(a)+(1-\alpha)f(b)]+(1-\lambda)[f(\alpha a+(1-\alpha)(b))]-$$
$$(b-a)\left[\left(\lambda(1-\alpha)-\frac{1}{2}\right)f'(b)+(1-\alpha)(1-\lambda)f'(\alpha a+(1-\alpha)b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx$$

which completes the proof. □

Remark 1. *Taking* $\alpha=\frac{1}{2}$, $\lambda=1$ *and changing variables with* $x=(tb+(1-t)a)$ *in Lemma 2 , inequality 5 becomes inequality 2.*

Theorem 3. $f:I\subseteq R\to R$ *be a twice differentiable mapping on* I° *such that* $f''\in L[a,b]$ *,where* $a,b\in I$ *with* $a<b$ *and* $\alpha,\lambda\in[0,1]$. *If* $|f''|^q$ *is convex on* $[a,b]$ *,* $q\geq 1$ *then the following inequality holds;*

$$\begin{aligned}
&\Bigg|\lambda[\alpha f(a)+(1-\alpha)f(b)]+(1-\lambda)[f(\alpha a+(1-\alpha)(b))]-(b-a)\\
&\times\left[\left(\lambda(1-\alpha)-\frac{1}{2}\right)f'(b)+(1-\alpha)(1-\lambda)f'(\alpha a+(1-\alpha)b)\right]\\
&-\frac{1}{b-a}\int_a^b f(x)\,dx\Bigg|\\
\leq\ &\frac{(b-a)^2}{2}\\
\times&\begin{cases}
\begin{cases} A_1^{1-\frac{1}{q}}\left(B_1|f''(b)|^q+B_3|f''(a)|^q\right)^{\frac{1}{q}} \\ +A_4^{1-\frac{1}{q}}\left(C_2|f''(b)|^q+C_4|f''(a)|^q\right)^{\frac{1}{q}}\end{cases}, & \begin{aligned}2\alpha\lambda&\leq 1-\alpha\\ &\leq 2(1-\lambda(1-\alpha))\end{aligned}\\
\begin{cases} A_1^{1-\frac{1}{q}}\left(B_1|f''(b)|^q+B_3|f''(a)|^q\right)^{\frac{1}{q}} \\ +A_3^{1-\frac{1}{q}}\left(C_1|f''(b)|^q+C_3|f''(a)|^q\right)^{\frac{1}{q}}\end{cases}, & \begin{aligned}2\alpha\lambda&\leq 2(1-\lambda(1-\alpha))\\ &\leq 1-\alpha\end{aligned}\\
\begin{cases} A_2^{1-\frac{1}{q}}\left(B_2|f''(b)|^q+B_4|f''(a)|^q\right)^{\frac{1}{q}} \\ +A_4^{1-\frac{1}{q}}\left(C_2|f''(b)|^q+C_4|f''(a)|^q\right)^{\frac{1}{q}}\end{cases}, & \begin{aligned}1-\alpha&\leq 2\alpha\lambda\\ &\leq 2(1-\lambda(1-\alpha))\end{aligned}
\end{cases}
\end{aligned}\tag{6}$$

where

$$\begin{aligned}
A_1&=\frac{8}{3}(\alpha\lambda)^3-(1-\alpha)^2(\alpha\lambda-\frac{(1-\alpha)}{3}),\\
A_2&=(1-\alpha)^2(\alpha\lambda-\frac{(1-\alpha)}{3}),\\
A_3&=\left[\frac{1}{3}-(1-\lambda(1-\alpha))-(1-\alpha)^2(\frac{(1-\alpha)}{3}-(1-\lambda(1-\alpha)))\right],
\end{aligned}$$

$$
\begin{aligned}
A_4 &= (1-\alpha)^2\left[\frac{(1-\alpha)}{3}-(1-\lambda(1-\alpha))\right]+\left[\frac{1}{3}-(1-\lambda(1-\alpha))\right] \\
&\quad +\frac{8}{3}\left[1-\lambda(1-\alpha)\right]^3, \\
B_1 &= \frac{8}{3}(\alpha\lambda)^4-(1-\alpha)^3\left[\frac{2}{3}(\alpha\lambda)-\frac{(1-\alpha)}{4}\right], \\
B_2 &= (1-\alpha)^3\left[\frac{2}{3}(\alpha\lambda)-\frac{(1-\alpha)}{4}\right], \\
B_3 &= \frac{8}{3}(\alpha\lambda)^3(1-\lambda\alpha)-(1-\alpha)^2(\alpha\lambda)+(1-\alpha)^3\left[\frac{2}{3}(\alpha\lambda)+\frac{1}{3}-\frac{(1-\alpha)}{4}\right], \\
B_4 &= (1-\alpha)^2(\alpha\lambda)-(1-\alpha)^3\left[\frac{2}{3}(\alpha\lambda)+\frac{1}{3}-\frac{(1-\alpha)}{4}\right],
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \left[\frac{1}{4}-\frac{2}{3}(1-\lambda(1-\alpha))\right]-(1-\alpha)^3\left[\frac{(1-\alpha)}{4}-\frac{2(1-\lambda(1-\alpha))}{3}\right], \\
C_2 &= \left[\frac{1}{4}-\frac{2}{3}(1-\lambda(1-\alpha))\right]+(1-\alpha)^3\left[\frac{(1-\alpha)}{4}-\frac{2(1-\lambda(1-\alpha))}{3}\right], \\
&\quad +\frac{8(1-\lambda(1-\alpha))^4}{3}, \\
C_3 &= \left[\frac{1}{12}-\frac{(1-\lambda(1-\alpha))}{3}\right] \\
&\quad -(1-\alpha)^2\left[(1-\alpha)\left(\frac{1}{3}-\frac{(1-\alpha)}{4}\right)-(1-\lambda(1-\alpha))\left(1-\frac{2(1-\alpha)}{3}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
C_4 &= (1-\alpha)^3\left(\frac{1}{3}-\frac{(1-\alpha)}{4}\right)-(1-\alpha)^2(1-\lambda(1-\alpha))\left(1-\frac{2(1-\alpha)}{3}\right), \\
&\quad +\frac{8}{3}(1-\lambda(1-\alpha))^3\lambda(1-\alpha)+\left[\frac{1}{12}-\frac{(1-\lambda(1-\alpha))}{3}\right].
\end{aligned}
$$

Proof. Using Lemma 2 and well known power mean inequality, we have;

$$
\begin{aligned}
& \lambda\left[\alpha f(a)+(1-\alpha)f(b)\right]+(1-\lambda)\left[f(\alpha a+(1-\alpha)(b))\right] \qquad (7) \\
& -\ (b-a)\left[f'(b)\left(\lambda(1-\alpha)-\frac{1}{2}\right)+(1-\alpha)(1-\lambda)f'(\alpha a+(1-\alpha)b)\right] \\
& -\frac{1}{b-a}\int_a^b f(x)\,dx \\
\leq\ & \frac{(b-a)^2}{2} \\
& \times\left\{\begin{array}{l}
\left(\int_0^{1-\alpha} t\left|(2\alpha\lambda-t)\right|dt\right)^{1-\frac{1}{q}} \\
\times\left(\int_0^{1-\alpha} t\left|(2\alpha\lambda-t)\right|.\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
+\left(\int_{1-\alpha}^1 t\left|2(1-\lambda(1-\alpha)-t)\right|dt\right)^{1-\frac{1}{q}} \\
\left(\int_{1-\alpha}^1 t\left|2(1-\lambda(1-\alpha))-t\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}
\end{array}\right.
\end{aligned}
$$

Since $|f''|^q$ *is* convex on $[a,b]$;

$$|f''(tb+(1-t)a)|^q \le t\,|f''(b)|^q + (1-t)\,|f''(a)|^q$$

holds for $t \in [0,1]$. Hence by simple computation we get;

$$\int_0^{1-\alpha} t\,|(2\alpha\lambda - t)|\,dt = \left\{ \begin{array}{ll} A_1, & 2\alpha\lambda \le 1-\alpha \\ A_2, & 2\alpha\lambda \ge 1-\alpha \end{array} \right. . \tag{8}$$

So we note that,

$$\int_{1-\alpha}^{1} t\,|2(1-\lambda(1-\alpha)) - t|\,dt = \left\{ \begin{array}{ll} A_3, & 2(1-\lambda(1-\alpha)) \le 1-\alpha \\ A_4, & 2(1-\lambda(1-\alpha)) \ge 1-\alpha \end{array} \right\}, \tag{9}$$

$$\begin{aligned} & \int_0^{1-\alpha} t\,|(2\alpha\lambda - t)|\,.\,|f''(tb+(1-t)a)|^q\,dt \qquad (10) \\ \le & \int_0^{1-\alpha} t\,|(2\alpha\lambda - t)|\,.\,\left[t\,|f''(b)|^q + (1-t)\,|f''(a)|^q\right]dt \\ = & \left\{ \begin{array}{ll} B_1\,|f''(b)|^q + B_3\,|f''(a)|^q, & 2\alpha\lambda \le 1-\alpha \\ B_2\,|f''(b)|^q + B_4\,|f''(a)|^q, & 2\alpha\lambda \ge 1-\alpha \end{array} \right. \end{aligned}$$

and

$$\begin{aligned} & \int_{1-\alpha}^{1} t\,|2(1-\lambda(1-\alpha) - t)|\,|f''(tb+(1-t)a)|^q\,dt \qquad (11) \\ \le & \int_{1-\alpha}^{1} t\,|2(1-\lambda(1-\alpha) - t)|\,.\,\left[t\,|f''(b)|^q + (1-t)\,|f''(a)|^q\right]dt \\ = & \left\{ \begin{array}{ll} C_1\,|f''(b)|^q + C_3\,|f''(a)|^q, & 2(1-\lambda(1-\alpha)) \le 1-\alpha \\ C_2\,|f''(b)|^q + C_4\,|f''(a)|^q, & 2(1-\lambda(1-\alpha)) \ge 1-\alpha \end{array} \right. . \end{aligned}$$

Combining (8)-(11) in (7) we obtain the inequality (6). This complete the proof. □

Remark 2. *If we choose* $\alpha = \frac{1}{2}$ *and* $\lambda = 1$ *in Theorem 3, inequality* (6) *becomes inequality* (4) *and also if we choose* $q = 1$ *we get the Trapezoid inequality which is the same as proposition 1.*

Theorem 4. *$f : I \subseteq R \to R$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$ and $\alpha,\lambda \in [0,1]$. If $|f''|^q$ is convex on $[a,b]$, $q > 1$ then the following inequality holds:*

$$\begin{aligned} & \Bigg| \lambda\left[\alpha f(a) + (1-\alpha) f(b)\right] + (1-\lambda)\left[f(\alpha a + (1-\alpha)(b))\right] \qquad (12) \\ & - (b-a)\left[\left(\lambda(1-\alpha) - \frac{1}{2}\right) f'(b) + (1-\alpha)(1-\lambda) f'(\alpha a + (1-\alpha) b)\right] \\ & - \frac{1}{b-a}\int_a^b f(x)\,dx \Bigg| \\ \le & \frac{(b-a)^2}{2}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \\ & \times \left\{ \begin{array}{ll} (1-\alpha)^{\frac{1}{q}} E_1^{\frac{1}{p}} D_1^{\frac{1}{q}} + \alpha^{\frac{1}{q}} F_2^{\frac{1}{p}} D_2^{\frac{1}{q}}, & 2\alpha\lambda \le 1-\alpha \le 2(1-\lambda(1-\alpha)) \\ (1-\alpha)^{\frac{1}{q}} E_1^{\frac{1}{p}} D_1^{\frac{1}{q}} + \alpha^{\frac{1}{q}} F_1^{\frac{1}{p}} D_2^{\frac{1}{q}}, & 2\alpha\lambda \le 2(1-\lambda(1-\alpha)) \le 1-\alpha \\ (1-\alpha)^{\frac{1}{q}} E_2^{\frac{1}{p}} D_1^{\frac{1}{q}} + \alpha^{\frac{1}{q}} F_2^{\frac{1}{p}} D_2^{\frac{1}{q}}, & 1-\alpha \le 2\alpha\lambda \le 2(1-\lambda(1-\alpha)) \end{array} \right. \end{aligned}$$

where

$$
\begin{aligned}
E_1 &= \left[(4p+1)(\alpha\lambda)^{2p+1}\right] + (1-\alpha-\alpha\lambda)^{2p+1} \\
E_2 &= \left[(2p+1)(1-\alpha)(\alpha\lambda)^{2p}\right] \\
F_1 &= \left[(\lambda(1-\alpha))^{2p+1} + (\lambda(1-\alpha)-)^{2p+1}\right] \\
F_2 &= \left\{[(4p+1)-(1-\alpha)[(2p+1)(2\lambda+1)-\lambda]](1-\lambda(1-\alpha))^{2p} + (\lambda(1-\alpha))^{2p+1}\right\} \\
D_1 &= \frac{|f''(\alpha a+(1-\alpha)b)|^q + |f''(a)|^q}{2} \\
D_2 &= \frac{|f''(\alpha a+(1-\alpha)b)|^q + |f''(b)|^q}{2}
\end{aligned}
$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 and by Hölder's inequality we have

$$
\begin{aligned}
&\Big|\lambda[\alpha f(a)+(1-\alpha)f(b)] + (1-\lambda)[f(\alpha a+(1-\alpha)(b))] \qquad (13)\\
&-(b-a)\left[\left(\lambda(1-\alpha)-\frac{1}{2}\right)f'(b) + (1-\alpha)(1-\lambda)f'(\alpha a+(1-\alpha)b)\right] \\
&-\frac{1}{b-a}\int_a^b f(x)\,dx\Big| \\
\leq\ & \frac{(b-a)^2}{2}\left[\int_0^{1-\alpha} |t(2\alpha\lambda-t)|\,|f''(tb+(1-t)a)\,dt|\right. \\
&\left.+\int_{1-\alpha}^{1} |t[2(1-\lambda(1-\alpha))-t]|\,|f''(tb+(1-t)a)\,dt|\right] \\
\leq\ & \frac{(b-a)^2}{2}\left[\left(\int_0^{1-\alpha}|t(2\alpha\lambda-t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{1-\alpha}^{1}|t[2(1-\lambda(1-\alpha))-t]|^p\,dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^{1}|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Since $|f''|^q$ is convex on $[a,b]$ for $\alpha,\lambda\in(0,1)$ by the inequality (1)

$$
\begin{aligned}
&\int_0^{1-\alpha}|f''(tb+(1-t)a)|^q\,dt \qquad (14)\\
&= (1-\alpha)\left(\frac{1}{(1-\alpha)(b-a)}\int_a^{\alpha a+(1-\alpha)b}|f''(x)|^q\,dx\right) \\
&\leq (1-\alpha)\frac{|f''(\alpha a+(1-\alpha)b)|^q+|f''(a)|^q}{2}
\end{aligned}
$$

and

$$\int_{1-\alpha}^{1}\left|f''(tb+(1-t)a)\right|^{q}dt \tag{15}$$
$$\leq \alpha\left[\frac{1}{\alpha(b-a)}\int_{\alpha a+(1-\alpha)b}^{b}\left|f''(x)\right|^{q}dx\right]$$
$$\leq \alpha\frac{\left|f''(\alpha a+(1-\alpha)b)\right|^{q}+\left|f''(b)\right|^{q}}{2}.$$

Note that, $xy\leq\left(\frac{x+y}{2}\right)^{2}$for all $x,y>0$. By simple computation

$$\int_{0}^{1-\alpha}\left|t(2\alpha\lambda-t)\right|^{p}dt \tag{16}$$
$$\leq \frac{1}{2p+1}\begin{cases}\left\{(4p+1)(\alpha\lambda)^{2p+1}+(1-\alpha-\alpha\lambda)^{2p+1}\right., & 2\alpha\lambda\leq1-\alpha\\ \left\{(2p+1)(1-\alpha)(\alpha\lambda)^{2p}\right., & 2\alpha\lambda\geq1-\alpha\end{cases},$$

and

$$\int_{1-\alpha}^{1}\left|t\left[2(1-\lambda(1-\alpha))-t\right]\right|^{p}dt=\frac{1}{2p+1} \tag{17}$$
$$\times\begin{cases}\left\{(\lambda(1-\alpha))^{2p+1}-[\lambda(1-\alpha)-\alpha]^{2p+1}\right., & 2(1-\lambda(1-\alpha))\leq1-\alpha\\ \left\{\begin{array}{l}\left[(4p+1)-(1-\alpha)\left[(2p+1)(2\lambda+1)-\lambda\right]\right]\\ \times(1-\lambda(1-\alpha))^{2p}\\ +(\lambda(1-\alpha))^{2p+1}\end{array}\right., & 2(1-\lambda(1-\alpha))\geq1-\alpha\end{cases}.$$

Thus using $(14)-(17)$ in (13) we obtain the inequality(12). This complete the proof. □

Theorem 5. *$f:I\subseteq R\to R$ be a twice differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$ and $\alpha,\lambda\in[0,1]$. If $|f''|^{q}$ is convex on $[a,b]$, $q>1$, then the following inequality holds;*

$$\Big|\lambda\left[\alpha f(a)+(1-\alpha)f(b)\right]+(1-\lambda)f(\alpha a+(1-\alpha)(b)) \tag{18}$$
$$-(b-a)\left[\left(\lambda(1-\alpha)-\frac{1}{2}\right)f'(b)+(1-\alpha)(1-\lambda)f'(\alpha a+(1-\alpha)b)\right]$$
$$-\frac{1}{b-a}\int_{a}^{b}f(x)dx\Big|$$
$$\leq \frac{(b-a)^{2}}{2}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}$$
$$\times\begin{cases}E_{1}^{\frac{1}{p}}(D_{1}^{\star})^{\frac{1}{q}}+F_{2}^{\frac{1}{p}}(D_{2}^{\star})^{\frac{1}{q}}, & 2\alpha\lambda\leq1-\alpha\leq2(1-\lambda(1-\alpha)),\\ E_{1}^{\frac{1}{p}}D_{1}^{\frac{1}{q}}+F_{1}^{\frac{1}{p}}D_{2}^{\frac{1}{q}}, & 2\alpha\lambda\leq2(1-\lambda(1-\alpha))\leq1-\alpha,\\ E_{2}^{\frac{1}{p}}D_{1}^{\frac{1}{q}}+F_{2}^{\frac{1}{p}}D_{2}^{\frac{1}{q}}, & 1-\alpha\leq2\alpha\lambda\leq2(1-\lambda(1-\alpha)),\end{cases}$$

where

$$D_{1}^{\star}=\frac{(1-\alpha)^{2}\left|f''(b)\right|^{q}+\left(1-\alpha^{2}\right)\left|f''(a)\right|^{q}}{2},$$
$$D_{2}^{\star}=\frac{\alpha(2-\alpha)\left|f''(b)\right|^{q}+\alpha^{2}\left|f''(a)\right|^{q}}{2}.$$

E_1, E_2, F_1, F_2 are the same as Theorem 4 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 and Hölder's inequality, we have:

$$\begin{aligned}
&\Bigg|\lambda\left[\alpha f(a)+(1-\alpha) f(b)\right]+(1-\lambda)\left[f(\alpha a+(1-\alpha)(b))\right] \qquad (19)\\
&-(b-a)\left[\left(\lambda(1-\alpha)-\frac{1}{2}\right) f'(b)+(1-\alpha)(1-\lambda) f'(\alpha a+(1-\alpha) b)\right]\\
&-\frac{1}{b-a}\int_a^b f(x)\,dx\Bigg|\\
\leq\ &\frac{(b-a)^2}{2}\left[\left(\int_0^{1-\alpha}|t(2\alpha\lambda-t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{1-\alpha}^1|t[2(1-\lambda(1-\alpha))-t]|^p\,dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^1|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right]\\
\leq\ &\frac{(b-a)^2}{2}\left[\left(\int_0^{1-\alpha}|t(2\alpha\lambda-t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}t|f''(b)|^q+(1-t)|f''(a)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{1-\alpha}^1|t[2(1-\lambda(1-\alpha))-t]|^p\,dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^1 t|f''(b)|^q+(1-t)|f''(a)|^q\,dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|f''|^q$ *is* convex on $[a,b]$;

$$|f''(tb+(1-t)a)|^q\leq t|f''(b)|^q+(1-t)|f''(a)|^q \tag{20}$$

holds for $t\in[0,1]$

Integrating the inequality above (20) on $[0,1]$;

$$\begin{aligned}
\int_0^{1-\alpha}|f''(tb+(1-t)a)|^q\,dt &\leq \int_0^{1-\alpha}t|f''(b)|^q\,dt+\int_0^{1-\alpha}(1-t)|f''(a)|^q\,dt\\
&= \frac{(1-\alpha)^2|f''(b)|^q+\left(1-\alpha^2\right)|f''(a)|^q}{2}; \qquad (21)\\
\int_{1-\alpha}^1|f''(tb+(1-t)a)|^q\,dt &\leq \int_{1-\alpha}^1 t|f''(b)|^q\,dt+\int_{1-\alpha}^1(1-t)|f''(a)|^q\,dt\\
&= \frac{\alpha(2-\alpha)|f''(b)|^q+\alpha^2|f''(a)|^q}{2}.
\end{aligned}$$

The integrals

$$\int_0^{1-\alpha}|t(2\alpha\lambda-t)|^p \, and \int_{1-\alpha}^1|t[2(1-\lambda(1-\alpha))-t]|^p\,dt$$

calculated before as (16), (17) in Theorem 4. Thus using $(20), (21)$ and $(16), (17)$ in the inequality (19) we obtain the inequality (18). This completes the proof. □

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