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OSCILLATION THEOREMS FOR SECOND ORDER QUASI-LINEAR DIFFERENCE EQUATIONS WITH LINEAR NEUTRAL TERMS

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Abstract. This paper studies the oscillatory behavior of solutions of second order quasi-linear difference equation with several neutral terms of the from

$$\Delta\left(a_n\left(\Delta\left(\sum_{i=1}^{m}p_{in}x_{\tau_i(n)}\right)\right)^{\alpha}\right)+q_nf(x_{\sigma(n)})=0,\ n\geq n_0,$$

where α is a ratio of odd positive integers. Some new sufficient conditions are presented which include several existing results. Examples are provided to illustrate the main results.

1. Introduction

Neutral difference equations have many applications in various problems of population dynamics, economics, biology, etc. Therefore, analysis of qualitative behavior of solutions of such equations is important for applications. In particular, there has been much interest in studying the oscillatory and nonoscillatory behavior of solutions of various types of second order neutral difference equations, see for example [1, 2, 3, 6, 4, 5, 7, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], and the references cited therein.

In this paper, we are concerned with the oscillation of second order neutral difference equation of the form

$$\Delta(a_n(\Delta z_n)^{\alpha})+q_nf(x_{\sigma(n)})=0,\ n\geq n_0, \tag{1}$$

where n_0 is a nonnegative integer and $z_n=x_n+\sum_{i=1}^{m}p_{in}x_{\tau_i(n)},\ m\geq 1$ is an integer, subject to the following conditions:

(H_1) α is a ratio of odd positive integers;
(H_2) f is a real valued continuous and nondecreasing function with $uf(u)>0$, for $u\neq 0$ and $\frac{f(u)}{u^{\beta}}\geq M>0$, where β is a ratio of odd positive integers;
(H_3) $\{a_n\}$ and $\{q_n\}$ are positive real sequences;
(H_4) $\{p_{in}\}$ is a nonnegative real sequences such that $0\leq p_{in}\leq c_i<\infty$ for $i=1,2,...,m$;
(H_5) $\{\tau_i(n)\}$ and $\{\sigma(n)\}$ are sequences of integers such that $\lim_{n\to\infty}\tau_i(n)=\lim_{n\to\infty}\sigma(n)=\infty$ and $\tau_i\circ\sigma=\sigma\circ\tau_i$ for $i=1,2,...,m$;
(H_6) $\lim_{n\to\infty}R_n<\infty$, where $R_n=\sum_{s=n_0}^{n-1}\frac{1}{a_s^{\frac{1}{\alpha}}}$.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined and satisfying the equation (1) for all $n\geq n_0$. A nontrivial solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

In the following, we present some background details that motivate our study. In [18, 19], the authors studied the neutral difference equation

$$\Delta(a_n(\Delta(x_n+p_nx_{\tau(n)}))^{\alpha})+q_nx^{\beta}_{\sigma(n)}=0 \tag{2}$$

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and established some oscillation criteria in the case

$$0 \leq p_n \leq p < \infty \text{ and } \tau \circ \sigma = \sigma \circ \tau.$$

In [2, 7, 9, 10, 15], the authors considered the oscillation of equation (1) in the case where $m = 1$, $\alpha = \beta$, and

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty. \tag{3}$$

In [13], the author studied the oscillatory behavior of equation (1) in the case $m = 1$, $\alpha = 1$ and condition (H_6). Recently in [16], the authors discussed the oscillation of equation (1) under the condition (3). We stress that the results obtained in [13, 15, 18, 22] cannot be applied to equation (1) in the case where (H_6) holds and $m \neq 1$. Our aim in this paper is to obtain some new criteria for the oscillation of equation (1) via the comparison method suggested by Thandapani et al.[16], under assumption (H_6). In Section 2, we present some sufficient conditions for the oscillation of all solutions equation (1), and in Section 3, we provide some examples to illustrate the main results. The results obtained in this paper generalize and complement to some of the results established in [2, 7, 9, 10, 12, 15, 16, 18, 19, 22].

2. Oscillation Theorems

In what follows, we use the following notations without further mention:

$$\begin{aligned} Q_n &= \min\{q_n, q_{\tau_i(n)}, ..., q_{\tau_m(n)}\} \\ Q_{n,\,N} &= Q_n(R_{\eta(n)} - R_N)^{\beta}, \text{ and } A_n = \sum_{s=n}^{\infty} \frac{1}{a_s^{\frac{1}{\alpha}}}\ , \end{aligned}$$

where $n \geq N \geq n_0$ is sufficiently large. Note that from the assumptions, it is enough to state and prove the lemmas and theorems for the case $\{x_n\}$ is eventually positive solutions since the opposite case can be proved similarly. We begin with the following lemma.

Lemma 1. *Let $\{x_n\}$ be an eventually positive solution of equation (1). Then one of the following two cases holds for all sufficiently large n :*

(I) $z_n > 0, a_n(\Delta z_n)^{\alpha} > 0,\ \Delta(a_n(\Delta z_n)^{\alpha}) < 0;$
(II) $z_n > 0, a_n(\Delta z_n)^{\alpha} < 0,\ \Delta(a_n(\Delta z_n)^{\alpha}) < 0.$

Proof. The proof of the lemma is similar to that of in [18], and hence the details are omitted. □

Lemma 2. *Assume that $y_i \geq 0$ for $i = 1, 2, ..., m$. Then*

$$(a) \qquad \sum_{i=1}^{m} y_i^r \geq \frac{1}{m^{r-1}} \left(\sum_{i=1}^{m} y_i\right)^r \quad \text{for } r \geq 1;$$

$$(b) \qquad \sum_{i=1}^{m} y_i^r \geq \left(\sum_{i=1}^{m} y_i\right)^r \quad \text{for } 0 < r < 1.$$

Proof. The proof is similar to that of in [16], and the details are omitted. □

Lemma 3. *Let $\gamma > 1$ and $\{e_n\}$ be a positive real sequence. If*

$$\sum_{n=n_0}^{\infty} e_n = \infty,$$

then the inequality $\Delta x_n - e_n x_{n+g}^{\gamma} \geq 0$, *where* g *is a positive integer, has no positive solution.*

Proof. Let $\{x_n\}$ be a positive solution of the given inequality, then we have $x_n > 0$ and nondecreasing for all $n \geq N \geq n_0$. Now from the given inequality we have

$$\frac{\Delta x_n}{x_{n+g}^{\gamma}} \geq e_n$$

or

$$\int_{x_n}^{x_{n+1}} \frac{ds}{s^{\gamma}} \geq \frac{\Delta x_n}{x_{n+g}^{\gamma}} \geq e_n.$$

Summing the last inequality from N to ∞, we have

$$\sum_{n=N}^{\infty} e_n \leq \frac{1}{x_N^{\gamma}(\gamma - 1)} < \infty$$

a contradiction. This completes the proof. □

Theorem 1. *Let* $\beta \geq 1$. *Assume that there exists two sequences* $\{\eta(n)\}$ *and* $\{\xi(n)\}$ *of integers such that* $\eta(n) \leq \sigma(n) \leq \xi(n)$ *and* $\lim_{n\to\infty} \eta(n) = \infty$. *If the first order difference inequalities*

$$\Delta\left(y_n + \sum_{i=1}^{m} c_i^{\beta} y_{\tau_i(n)}\right) + \frac{MQ_{n,N}}{(m+1)^{\beta-1}} y_{\eta(n)}^{\frac{\beta}{\alpha}} \leq 0, \tag{4}$$

and

$$\Delta\left(u_n + \sum_{i=1}^{m} c_i^{\beta} u_{\tau_i(n)}\right) - \frac{MQ_n A_{\sigma(n)}^{\beta}}{(m+1)^{\beta-1}} u_{\xi(n)}^{\frac{\beta}{\alpha}} \geq 0 \tag{5}$$

have no positive solutions, then every solution of equation (1) *is oscillatory.*

Proof. Assume to the contrary that there exists a nonoscillatory solution $\{x_n\}$ of equation (1). Without loss of generality, we may assume that $x_n > 0$, $x_{\tau_i(n)} > 0$ and $x_{\sigma(n)} > 0$ for all $n \geq N \geq n_0$, where N is chosen so that one of the two cases of Lemma 1 holds for all $n \geq N$. We shall show that in each case we are led to a contradiction.

Case (I): In view of (H_2), we have from equation (1) that

$$\Delta(a_n(\Delta z_n)^{\alpha}) + Mq_n x_{\sigma(n)}^{\beta} \leq 0, \ n \geq N, \tag{6}$$

and

$$\sum_{i=1}^{m} c_i^{\beta} \Delta(a_{\tau_i(n)}(\Delta z_{\tau_i(n)})^{\alpha}) + M \sum_{i=1}^{m} c_i^{\beta} q_{\tau_i(n)} x_{\tau_i(\sigma(n))}^{\beta} \leq 0, \ n \geq N. \tag{7}$$

Combining (6) and (7), we are led to

$$\Delta(a_n(\Delta z_n)^{\alpha}) + \sum_{i=1}^{m} c_i^{\beta} \Delta(a_{\tau_i(n)}(\Delta z_{\tau_i(n)})^{\alpha}) + MQ_n\left(x_{\sigma(n)}^{\beta} + \sum_{i=1}^{m} c_i^{\beta} x_{\sigma(\tau_i(n))}^{\beta}\right) \leq 0. \tag{8}$$

Using Lemma 2 (a) in (8), we obtain

$$\Delta(a_n(\Delta z_n)^{\alpha}) + \sum_{i=1}^{m} c_i^{\beta} \Delta(a_{\tau_i(n)}(\Delta z_{\tau_i(n)})^{\alpha}) + \frac{MQ_n}{(m+1)^{\beta-1}} z_{\sigma(n)}^{\beta} \leq 0, \ n \geq N, \tag{9}$$

where we have used $\tau_i(\sigma(n)) = \sigma(\tau_i(n))$, for $i = 1, 2, ..., m$. It follows from Lemma 1(I) that $y_n = a_n(\Delta z_n)^{\alpha}$ positive and decreasing, and so

$$z_n = z_N + \sum_{s=N}^{n-1} \frac{a_s^{\frac{1}{\alpha}} \Delta z_s}{a_s^{\frac{1}{\alpha}}} \geq y_n^{\frac{1}{\alpha}}(R_n - R_N). \tag{10}$$

From (9) and (10), and the fact $\eta(n) \leq \sigma(n)$, we obtain (4). Thus, $\{y_n\}$ is a positive solution of the inequality (4), which is a contradiction.

Case(II): From Lemma 1(II), we have $\Delta(a_n(-\Delta z_n)^\alpha) > 0$, and hence

$$\Delta z_s \leq \frac{a_n^{\frac{1}{\alpha}} \Delta z_n}{a_s^{\frac{1}{\alpha}}}, \; n \geq s \geq N,$$

which, upon summation, tends to

$$z_l \leq z_n + a_n^{\frac{1}{\alpha}} z_n^{\frac{1}{\alpha}} \sum_{s=n}^{l} \frac{1}{a_s^{\frac{1}{\alpha}}}.$$

Since $z_n > 0$, and taking the limit as $l \to \infty$, we obtain

$$z_n \geq -A_n a_n^{\frac{1}{\alpha}} \Delta z_n. \tag{11}$$

Since $\Delta z_n < 0$, inequality (9) reduces to

$$\Delta(a_n(-\Delta z_n)^\alpha) + \sum_{i=1}^{m} c_i^\beta \Delta(a_{\tau_i(n)}(-\Delta z_{\tau_i(n)})^\alpha) - \frac{MQ_n}{(m+1)^{\beta-1}} z_{\sigma(n)}^\beta \geq 0, \; n \geq N.$$

Note that $z_n > 0$ and $\Delta z_n < 0$, then z_n^β is decreasing. It follows from $\sigma(n) \leq \xi(n)$ that $z_{\sigma(n)}^\beta \geq z_{\xi(n)}^\beta$ and

$$\Delta(a_n(-\Delta z_n)^\alpha) + \sum_{i=1}^{m} c_i^\beta \Delta(a_{\tau_i(n)}(-\Delta z_{\tau_i(n)})^\alpha) - \frac{MQ_n}{(m+1)^{\beta-1}} z_{\xi(n)}^\beta \geq 0. \tag{12}$$

Combining (11) and (12), we have

$$\Delta(a_n(-\Delta z_n)^\alpha) + \sum_{i=1}^{m} c_i^\beta \Delta(a_{\tau_i(n)}(-\Delta z_{\tau_i(n)})^\alpha) - \frac{MQ_n}{(m+1)^{\beta-1}} A_{\xi(n)}^\beta (-a_{\xi(n)}(\Delta z_{\xi(n)})^\alpha)^{\frac{\beta}{\alpha}} \geq 0.$$

Set $u_n = a_n(-\Delta z_n)^\alpha$ in the last inequality, we see that $\{u_n\}$ is a positive solution of the inequality (5). which is again a contradiction. This completes the proof. □

Using additional assumptions on the coefficients of equation (1), one can deduce from Theorem 1, a number of oscillation criteria applicable to different types of neutral difference equations. In the following, we use the notations $\tau_*(n) = \max\{\tau_i(n) : i = 1, 2, ..., m\}$ and $\tau(n) = \min\{\tau_i(n) : i = 1, 2, ..., m\}$, and the notation τ_*^{-1} and τ^{-1} stand for the inverses of the functions τ_* and τ respectively.

Theorem 2. *Let $\beta \geq 1$ and $\tau_i(n) \geq n$ for $i = 1, 2, ..., m$. Assume that there exist two sequences $\{\eta(n)\}$ and $\{\xi(n)\}$ of integers such that $\eta(n) \leq \sigma(n) \leq \xi(n)$ and $\lim_{n\to\infty} \eta(n) = \infty$. If the first order difference inequalities*

$$\Delta w_n + \frac{MQ_{n,N}}{(m+1)^{\beta-1}\left(1 + \sum_{i=1}^{m} c_i^\beta\right)^{\frac{\beta}{\alpha}}} w_{\xi(n)}^{\frac{\beta}{\alpha}} \leq 0, \; n \geq N, \tag{13}$$

and

$$\Delta v_n - \frac{MQ_n A_{\xi(n)}^\beta}{(m+1)^{\beta-1}\left(1 + \sum_{i=1}^{m} c_i^\beta\right)^{\frac{\beta}{\alpha}}} v_{\tau_*^{-1}(\xi(n))}^{\frac{\beta}{\alpha}} \geq 0, \; n \geq N \tag{14}$$

have no positive solutions, then every solution of equation (1) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1). As in the proof of Theorem 1, there exists two possible cases for $\{z_n\}$ as stated in Lemma 1.
Case(I): Proceeding as in the proof of case (I) of Theorem 1, we see that $y_n = a_n(\Delta z_n)^\alpha$ is a positive decreasing solution of the inequality (4). Now define

$$w_n = y_n + \sum_{i=1}^{m} c_i^\beta y_{\tau_i(n)}.$$

Then $w_n > 0$, and in view of $\tau_i(n) \geq n$, we obtain

$$w_n \leq y_n \left(1 + \sum_{i=1}^{m} c_i^\beta\right).$$

Substituting the last inequality into inequality (4), we see that $\{w_n\}$ is a positive solution of the inequality (13), a contradiction.
Case (II): Proceeding as in the proof of case (II) of Theorem 1, we see that $u_n = a_n(-\Delta z_n)^\alpha$ is a positive increasing solution of the inequality (5), we now define

$$v_n = u_n + \sum_{i=1}^{m} c_i^\beta u_{\tau_i(n)}. \tag{15}$$

Then $v_n > 0$ and inview of $\tau_i(n) \geq n$, we obtain

$$v_n \leq u_{\tau_*(n)} \left(1 + \sum_{i=1}^{m} c_i^\beta\right).$$

Substituting the last inequality into (5), we see that $\{v_n\}$ is a positive solution of the inequality (14). This contradiction completes the proof. □

Next by adding additional assumptions on $\alpha, \beta, \{\tau_i(n)\}, \{\sigma(n)\}, \{\eta(n)\}$ and $\{\xi(n)\}$ one can deduce explicit oscillation criteria for the equation (1) from Theorem 2.

Corollary 1. *Assume that* $\alpha = \beta \geq 1$, $\tau_*(n) = n + k$, $\eta(n) = n - d$, $\xi(n) = n + r$ *with* $n - d \leq \sigma(n) \leq n + r$ *and* $r \geq k + 2$, *where* k, d, r *are positive integers. If*

$$\lim_{n\to\infty} \inf \sum_{s=n-d}^{n-1} Q_{s,N} > \frac{(m+1)^{\beta-1}\left(1 + \sum_{i=1}^{m} c_i^\beta\right)}{M} \left(\frac{d}{d+1}\right)^{d+1}, \tag{16}$$

and

$$\lim_{n\to\infty} \inf \sum_{s=n+1}^{n+r-k-1} Q_s A_{s+r}^\beta > (m+1)^{\beta-1}\left(1 + \sum_{i=1}^{m} c_i^\beta\right)\left(\frac{r-k-1}{r-k}\right)^{r-k} \tag{17}$$

hold, then every solution of equation (1) *is oscillatory.*

Proof. By Lemma 6.1.6 of [2], the assumption (16) ensures that the difference inequality (13) has no positive solution. Further by Lemma 6.1.7 of [2], the condition (17) guaranteed that the difference inequality (14) has no positive solution. Now the result follows from Theorem 2. □

Remark 1. *Let* $\alpha = \beta = m = 1$. *Then Corollary 1 includes some results in [13] as special case.*

Corollary 2. *Let* $\beta \geq 1$ *with* $\alpha < \beta$, $\tau_*(n) = n + k$, $\eta(n) = n - d$, $\xi(n) = n + r$ *with* $n - d \leq \sigma(n) \leq n + r$ *where* k, d *and* r *are positive integers. If there exists a* $\lambda > \frac{1}{d}\log\frac{\beta}{\alpha}$ *such that*

$$\lim_{n\to\infty} \inf \left[Q_{n,N} \exp(-e^{\lambda n})\right] > 0, \tag{18}$$

and

$$\lim_{n\to\infty}\sum_{s=N}^{n-1} Q_s A_{s+r}^{\beta} = \infty \tag{19}$$

then every solution of equation (1) *is oscillatory.*

Proof. From the hypotheses, the inequalities (13) and (14) in Theorem 2 take the form

$$\Delta w_n + \frac{MQ_{n,N}}{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)^{\frac{\beta}{\alpha}}} w_{n-d}^{\frac{\beta}{\alpha}} \le 0 \tag{20}$$

and

$$\Delta v_n - \frac{MQ_n A_{n+r}^{\beta}}{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)^{\frac{\beta}{\alpha}}} v_{n+r-k}^{\frac{\beta}{\alpha}} \ge 0, \tag{21}$$

respectively. By Theorem 2 of [11], the condition (18) ensures that the inequality (20) has no positive solution. By Lemma 3, we see that condition (19) implies that the inequality (21) has no positive solution. The result now follows from Theorem 2. □

Remark 2. *Let* $\alpha = 1$, $\beta > 1$, *and* $m = 1$. *Then Corollary 2 includes some results in* [15] *as special case.*

Theorem 3. *Assume that* $\beta \ge 1$ *and let* $\tau_*(n) \le n$ *for all* $n \ge n_0$. *Suppose that there exists two sequences* $\{\eta(n)\}$ *and* $\{\xi(n)\}$ *of integers such that* $\eta(n) \le \sigma(n) \le \xi(n)$ *and* $\lim_{n\to\infty}\eta(n) = \infty$. *If the first order difference inequalities*

$$\Delta w_n + \frac{MQ_{n,N}}{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)^{\frac{\beta}{\alpha}}} w_{\tau^{-1}(\eta(n))}^{\frac{\beta}{\alpha}} \le 0, \tag{22}$$

and

$$\Delta v_n - \frac{MQ_n A_{\xi(n)}^{\beta}}{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)^{\frac{\beta}{\alpha}}} v_{\xi(n)}^{\frac{\beta}{\alpha}} \ge 0 \tag{23}$$

have no positive solutions, then every solution of equation (1) *is oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1). As in the proof of Theorem 1, there exists two positive cases for $\{z_n\}$ as stated in Lemma 1.
Case (I): Assume first that case (I) of Lemma 1 holds. Following the same argument as in Theorem 3 of [16], we conclude that the inequality (22) has a positive solution, which is a contradiction.
Case (II): Proceeding as in the case (II) of Theorem 1, we see that $u_n = a_n(-\Delta z_n)^{\alpha}$ is positive, increasing and satisfies the inequality (5). By virtue of $\tau_*(n) \le n$, the inequality

$$v_n \le u_n\left(1+\sum_{i=1}^{m} c_i^{\beta}\right) \tag{24}$$

holds for the sequence defined by (15). Using the inequality (24) in (5), we conclude that $\{v_n\}$ is a positive solution of (23). This contradiction completes the proof. □

Corollary 3. *Let $\alpha = \beta \geq 1$ and $\tau_*(n) = n - k$ where k is a non-negative integer. Also assume that $\eta(n) = n - d$ and $\xi(n) = n + r$ such that $n - d \leq \sigma(n) \leq n + r$ where d and r are positive integers with $d > k$ and $r \geq 2$. If*

$$\lim_{n\to\infty} \inf \sum_{s=n+k-d}^{n-1} Q_{s,N} > \frac{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)}{M} \left(\frac{d-k}{d-k+1}\right)^{d-k+1} \tag{25}$$

and

$$\lim_{n\to\infty} \inf \sum_{s=n+1}^{n+r-1} Q_s A_{s+r}^{\beta} > \frac{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)}{M} \left(\frac{r-1}{r}\right)^{r}, \tag{26}$$

then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Corollary 1, and hence the details are omitted. □

Corollary 4. *Let $\beta \geq 1$ with $\alpha < \beta$, and $\tau_*(n) = n - k$ where k are non-negative integer. Also assume that $\eta(n) = n - d$ and $\xi(n) = n + r$ such that $n - d \leq \sigma(n) \leq n + r$ where d and r are positive integers with $d > k$. If there exists a $\lambda > \frac{1}{d-k}\log\frac{\beta}{\alpha}$ such that* (18) *and* (19) *hold, then every solution of equation* (1) *is oscillatory.*

Proof. The proof is similar to that of corollary 2 and hence the details are omitted. □

Next we turn our attention to the case $0 < \beta < 1$.

Theorem 4. *Let $0 < \beta < 1$. Assume that there exists two sequences $\{\eta(n)\}$ and $\{\xi(n)\}$ of integers such that $\eta(n) \leq \sigma(n) \leq \xi(n)$ and $\lim_{n\to\infty} \eta(n) = \infty$. If the first order difference inequalities*

$$\Delta\left(y_n + \sum_{i=1}^{m} c_i^{\beta} y_{\tau_i(n)}\right) + MQ_{n,N} y_{\eta(n)}^{\frac{\beta}{\alpha}} \leq 0,$$

and

$$\Delta\left(u_n + \sum_{i=1}^{m} c_i^{\beta} u_{\tau_i(n)}\right) - MQ_n A_{\xi(n)}^{\beta} u_{\xi(n)}^{\frac{\beta}{\alpha}} \geq 0$$

have no positive solutions, then every solution of equation (1) *is oscillatory.*

Proof. The proof is exactly the same as that of Theorem 1 except by using Lemma 2(b) instead of Lemma 2(a). Hence the details are omitted. □

Theorem 5. *Let $0 < \beta < 1$ and $\tau_i(n) \geq n$ for $i = 1, 2, ..., m$. Assume that there exist two sequences $\{\eta(n)\}$ and $\{\xi(n)\}$ of integers such that $\eta(n) \leq \sigma(n) \leq \xi(n)$ and $\lim_{n\to\infty} \eta(n) = \infty$. If the first order difference inequalities*

$$\Delta w_n + \frac{MQ_{n,N}}{\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)^{\frac{\beta}{\alpha}}} w_{\xi(n)}^{\frac{\beta}{\alpha}} \leq 0,$$

and

$$\Delta v_n - \frac{MQ_n A_{\xi(n)}^{\beta}}{\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)^{\frac{\beta}{\alpha}}} v_{\tau_*^{-1}(\xi(n))}^{\frac{\beta}{\alpha}} \geq 0$$

have no positive solutions, then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2 and hence the details are omitted. □

Theorem 6. *Assume that* $0 < \beta < 1$ *and let* $\tau_*(n) \le n$ *for all* $n \ge n_0$. *Suppose that there exists two sequences* $\{\eta(n)\}$ *and* $\{\xi(n)\}$ *of integers such that* $\eta(n) \le \sigma(n) \le \xi(n)$ *and* $\lim_{n\to\infty} \eta(n) = \infty$. *If the first order difference inequalities*

$$\Delta w_n + \frac{MQ_{n,N}}{\left(1+\sum_{i=1}^m c_i^\beta\right)^{\frac{\beta}{\alpha}}} w^{\frac{\beta}{\alpha}}_{\tau^{-1}(\eta(n))} \le 0,$$

and

$$\Delta v_n - \frac{MQ_n A^\beta_{\xi(n)}}{\left(1+\sum_{i=1}^m c_i^\beta\right)^{\frac{\beta}{\alpha}}} v^{\frac{\beta}{\alpha}}_{\xi(n)} \ge 0$$

have no positive solutions, then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 3 except using Lemma 2(b) instead of Lemma 2(a) and so the details are omitted. □

Corollary 5. *Assume that* $0 < \alpha = \beta < 1$, $\tau_*(n) = n + k$, $\eta(n) = n - d$, $\xi(n) = n + r$ *with* $n - d \le \sigma(n) \le n + r$ *and* $r \ge k + 2$, *where* k, d *and* r *are positive integers. If*

$$\liminf_{n\to\infty} \sum_{s=n-d}^{n-1} Q_{n,\,N} > \frac{\left(1+\sum_{i=1}^m c_i^\beta\right)}{m}\left(\frac{d}{d+1}\right)^{d+1} \tag{27}$$

and

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+r-k-1} Q_s A^\beta_{s+r} > \left(1+\sum_{i=1}^m c_i^\beta\right)\left(\frac{r-k-1}{r-k}\right)^{r-k} \tag{28}$$

then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Corollary 1 and hence the details are omitted. □

Corollary 6. *Assume that* $0 < \alpha = \beta < 1$, $\tau_*(n) = n - k$, $\eta(n) = n - d$, $\xi(n) = n + r$ *with* $n - d \le \sigma(n) \le n + r$, *where* d *and* r *are positive integers with* $d > k$ *and* $r \ge 2$. *If*

$$\liminf_{n\to\infty} \sum_{s=n+k-d}^{n-1} Q_{s,N} > \frac{\left(1+\sum_{i=1}^m c_i^\beta\right)}{m}\left(\frac{d-k}{d-k+1}\right)^{d-k+1} \tag{29}$$

and

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+r-1} Q_s A^\beta_{s+r} > \left(1+\sum_{i=1}^m c_i^\beta\right)\left(\frac{r-1}{r}\right)^{r} \tag{30}$$

then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Corollary 3 and hence the details are omitted. □

3. Examples

In this section, we provide some examples to illustrate the main results.

Example 1. *Consider the second order neutral type difference equation of the from*

$$\Delta\left(2^{3n}\left(\Delta\left(x_n + 4\,x_{n+1} + 2\,x_{n+2} + 3x_{n+3}\right)\right)^3\right) + (2^{4n})x^3_{n-1} = 0,\ n \ge 1. \tag{31}$$

Here $A_n = \frac{1}{2^{n-1}}$, $R_n = \frac{2^n-2}{2^n}$, $Q_n = 2^{4n}$, *and* $\tau_*(n) = n + 3$. *It is easy to see that all conditions of Corollary 1 are satisfied and hence every solution of equation* (31) *is oscillatory.*

Example 2. *Consider a second order difference equations of the form*

$$\Delta\left(2^{\frac{n}{3}}\left(\Delta\left(x_n + 4\,x_{n-2} + x_{n-3}\right)\right)^{\frac{1}{3}}\right) + (2 + 2^{\frac{4}{3}})2^{\frac{n}{3}} x_{n-1}^{\frac{1}{3}} = 0,\ n \geq 1. \tag{32}$$

Here $\tau_*(n) = n - 2$, $Q_n = (2 + 2^{\frac{4}{3}})2^{\frac{n}{3}}$, $R_n = (\frac{2^n - 2}{2^n})$, $A_n = \frac{1}{2^{n-1}}$. *It is easy to see that all conditions of Corollary 6 are satisfied and hence every solution of equation* (32) *is oscillatory. In fact one such solution of equation* (32) *is* $\{x_n\} = \{(-1)^n\}$.

Remark 3. *The results presented in* [2, 5, 13, 15, 18, 22] *cannot be applied to examples 1 and 2 since* $m \neq 1$. *Therefore our results extend and compliment to some of the results reported in the literature.*

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