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CERTAIN CLASSES OF ANALYTIC FUNCTIONS INVOLVING A FAMILY OF GENERALIZED DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper, we define a new class of analytic function using generalized differential operator. Coefficient inequalities, sufficient condition, distortion theorems, radii of starlikeness, convexity and close - to - convexity results are obtained.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The area of the q-analysis has attracted serious attention of researchers. The great interest is due to its applications in various branches of mathematics and physics, as for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference and $q-$integral equations and in q-transform analysis. The generalized $q-$Taylor formula in the fractional $q-$calculus was introduced by Purohit and Raina [22]. The application of $q-$calculus was initiated by Jackson [13, 14]. He was the first to develop the $q-$integral and $q-$derivative in a systematic way. Later, geometrical interpretation of the $q-$analysis has been recognized through studies on quantum groups. Simply, the quantum calculus is ordinary classical calculus without the notion of limits. It defines $q-$calculus and $h-$calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Mohammed and Darus [18] studied approximation and geometric properties of these $q-$operators in some subclasses of analytic functions in compact disk. Recently, Purohit and Raina [22, 23] have used the fractional $q-$calculus operators in investigating certain classes of functions which are analytic in the open disk. Also Purohit [21] also studied these $q-$operators, defined by using the convolution of normalized analytic functions and $q-$hypergeometric functions. A comprehensive study on the applications of $q-$calculus in the operator theory may be found in [5].

Further the possibility of extension of the q-calculus to postquantum calculus denoted by the $(p,\, q)$-calculus. Actually such an extension of quantum calculus cannot be obtained directly by substitution of q by q/p in q-calculus. When the case $p = 1$ in $(p,\, q)$-calculus, the q-calculus may be obtained (see [1, 12]). Recently, several relations involving $(p,\, q)$-derivative and divided differences were obtained by Serkan Araci et. al.[4]. For detailed development of the so called post-quantum calculus or $(p,\, q)$-calculus, refer to [5] and references therein.

Let p and q be elements of complex numbers and $D = D_{p,q} \subset \mathbb{C}$ such that $x \in D$ implies $px \in D$ and $qx \in D$.

Definition 1. [4] *Let $0 < |q| < |p| \leq 1$. A given function $f : D_{p,q} \to \mathbb{C}$ is called $(p,\, q)$-differentiable under the restriction that, if $0 \in D_{p,q}$, then $f'(0)$ exists.*

Definition 2. [4] *Let $0 < |q| < |p| \leq 1$. A given function $f : D_{p,q} \to \mathbb{C}$ is called $(p,\, q)$-differentiable of order n, if and only if $0 \in D_{p,q}$, then $f^{(n)}(0)$ exists.*

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The $(p,\, q)$-derivative operator of a function f is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x} \quad (x \neq 0) \tag{1}$$

and $(D_{p,q}f)(0) = f'(0)$, provided that the function f is differentiable at 0. We note that $D_{p,q} = D_{q,\, p}$.

Let us denote $\mathcal{U}_r = \{z \in \mathbb{C} \setminus \mid z \mid < r\}$ and let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0, \tag{2}$$

which are analytic in the open disc $\mathcal{U} = \mathcal{U}_1$.

If $f(z)$ is of the form (2), a simple computation yields

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} \frac{p^n - q^n}{p-q} a_n\, z^{n-1}\,, \quad (z \in \mathcal{U}). \tag{3}$$

The Hadamard product of two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k\, b_k\, z^k. \tag{4}$$

In univalent function theory, all geometrically defined subclasses does have beautiful analytic characterization defined in terms of differential inequality. So extending the existing subclasses in q-calculus has numerous applications. To provide a unified approach to the study of various properties of the certain subclasses of $\mathcal{A}$, we define a $(p,\, q)$-analogue of the Sălăgean differential operator.

The $(p,\, q)$-analogue of Sălăgean differential operator (see [26]) $R^m_{p,\, q}f(z) : \mathcal{A} \to \mathcal{A}$ for $m \in N$, is formed as follows.

$$\begin{aligned} R^0_q f\,(z) &= f\,(z) \\ R^1_{p,\, q} f\,(z) &= z(D_{p,\, q}f(z)) \\ &\vdots \\ R^m_{p,\, q} f\,(z) &= R^1_{p,\, q}\left(R^{m-1}_{p,\, q} f\,(z)\right). \end{aligned}$$

From the definition of $R^m_{p,\, q}f\,(z)$, we get

$$R^m_{p,\, q}f\,(z) = z + \sum_{k=2}^{\infty} \left(\frac{p^k - q^k}{p-q}\right)^m a_k z^k, \qquad (z \in \mathcal{U}). \tag{5}$$

It can be seen that if we let $p = 1$ and $q \to 1^-$, then $R^m_{p,\, q}f\,(z)$ reduces to the well-known Sălăgean differential operator [26].

Let $\mathcal{P}$ denote the class of functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ which are analytic and convex in $\mathcal{U}$ and satisfy the condition $Re\,(p(z)) > 0,\, (z \in \mathcal{U})$.

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathcal{U}$. We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Our favorite references of the field are [9, 10, 11] which covers most of the topics in a lucid and economical style.

The concept of starlike functions and convex functions were further extended as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : Re\frac{zf'(z)}{f(z)} > \alpha, z \in \mathcal{U} \right\}, \tag{6}$$

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathcal{U} \right\}. \tag{7}$$

We note that

$$f \in \mathcal{C}(\alpha) \Leftrightarrow zf' \in \mathcal{S}^*(\alpha), \tag{8}$$

where $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are respectively, the classes of starlike of order α and convex of order α in $\mathcal{U}$ (see Robertson [24]).

Similarly, close-to convex functions and quasi-convex functions were further extended as follows:

$$\mathcal{K}(\alpha, \beta) = \left\{ f \in \mathcal{A} : Re\frac{zf'(z)}{g(z)} > \alpha,\ g \in \mathcal{S}^*(\beta),\ z \in \mathcal{U} \right\}, \tag{9}$$

$$\mathcal{C}^*(\alpha, \beta) = \left\{ f \in \mathcal{A} : Re\left(\frac{\left(zf'(z)\right)'}{g'(z)}\right) > \alpha,\ g \in \mathcal{C}(\beta),\ z \in \mathcal{U} \right\}, \tag{10}$$

where $\mathcal{K}(\alpha, \beta)$ and $\mathcal{C}^*(\alpha, \beta)$ are respectively, the classes of close-to-convex of order α type β and quasi-convex of order α and type β in $\mathcal{U}$ (see K. I. Noor and D. K. Thomas [20]).

Using the operator $R^m_{p,\,q}f(z)$, we say that the function $f \in \mathcal{A}$ is in the class $\mathcal{A}^{m,\lambda}_{p,q}(A, B, \alpha, \gamma, \beta)$ if and only if

$$\left| \frac{\frac{zF'_{m,\,\lambda}(z)}{F_{m,\,\lambda}(z)} - 1}{(B - A)\gamma\left[\frac{zF'_{m,\,\lambda}(z)}{F_{m,\,\lambda}(z)} - \alpha\right] - B\left[\frac{zF'_{m,\,\lambda}(z)}{F_{m,\,\lambda}(z)} - 1\right]} \right| < \beta, \qquad (m \in \mathbb{N}_0,\ z \in \mathcal{U}), \tag{11}$$

$$\left(-1 \le B < A \le 1,\ B > 0,\ 0 \le \lambda \le 1,\ 0 \le \alpha, \beta < 1,\ 0 \le \gamma \le \frac{B}{B - A}\right)$$

where, for convenience, $F_{m,\,\lambda}(z) = (1 - \lambda)\, R^m_{p,\,q}f(z) + \lambda R^{m+1}_{p,\,q}f(z)$.

Let $\mathcal{T}$ denote the subclass of $f \in \mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \tag{12}$$

and also we define the class $\mathcal{T}^{m,\,\lambda}_{p,q}(A,\ B,\ \alpha,\ \gamma,\ \beta)$ by

$$\mathcal{T}^{m,\,\lambda}_{p,q}(A,\ B,\ \alpha,\ \gamma,\ \beta) = \mathcal{A}^{m,\,\lambda}_{p,q}(A,\ B,\ \alpha,\ \gamma,\ \beta) \cap \mathcal{T}.$$

Remark 1. *The class* $\mathcal{T}^{0,\,0}_{1,1^-}(-1,\ 1,\ 0,\ \frac{1}{2},\ 1)$ *reduces to the class of functions* $f \in \mathcal{A}$ *that satisfies* $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$*, namely these functions are starlike introduced by Silverman* [27]*. Further, we note that by specializing* m*,* A*,* B *and* λ*, we obtain several subclasses of analytic and univalent functions studied by various authors in the earlier papers (see e.g.* [2, 3, 17, 19, 29]*).*

2. Sufficient condition

Unless otherwise mentioned, we assume in the remainder of this paper that $-1 \leq B < A \leq 1$, $B > 0$, $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 \leq \gamma \leq \frac{B}{B-A}$ and $m \in \mathbb{N}_0$.

Theorem 1. *A function $f(z)$ of the form (2) is in the class $\mathcal{A}_{p,q}^{m,\lambda}(A, B, \alpha, \gamma, \beta)$ if*

$$\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1)-\beta\gamma(B-A)(k+\alpha)] \leq \beta\gamma(B-A)(1-\alpha), \qquad (z\in\mathcal{U}). \tag{13}$$

Proof. Suppose that the inequality (13) holds. Then for $z \in \mathcal{U}$,

$$\begin{aligned}&\left|zF'_{m,\lambda}(z)-F_{m,\lambda}(z)\right| \\ &-\beta\left|(B-A)\gamma\left[zF'_{m,\lambda}(z)-\alpha F_{m,\lambda}(z)\right]-B\left[zF'_{m,\lambda}(z)-F_{m,\lambda}(z)\right]\right| \\ &=\left|\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[(k-1)\left(1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right)\right]a_kz^k\right|- \\ &\beta\left|(B-A)\gamma\left(z(1-\alpha)+\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[(k-\alpha)\left(1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right)\right]\right)a_kz^k\right. \\ &\left.-B\left[\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[(k-1)\left(1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right)\right]a_kz^k\right]\right| \\ &\leq\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1) \\ &-\beta\gamma(B-A)(k+\alpha)]\left|a_k\right|r^k-\beta\gamma(B-A)(1-\alpha)r\end{aligned} \tag{14}$$

Letting $r \longrightarrow 1^-$, then we have

$$\begin{aligned}&\left|zF'_{m,\lambda}(z)-F_{m,\lambda}(z)\right|-\beta\left|(B-A)\gamma\left[zF'_{m,\lambda}(z)-\alpha F_{m,\lambda}(z)\right]-B\left[zF'_{m,\lambda}(z)-F_{m,\lambda}(z)\right]\right| \\ &\leq\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1) \\ &-\beta\gamma(B-A)(k+\alpha)]\left|a_k\right|r^k-\beta\gamma(B-A)(1-\alpha)r\leq 0.\end{aligned} \tag{15}$$

Hence it follows that

$$\left|\frac{\frac{zF'_{m,\lambda}(z)}{F_{m,\lambda}(z)}-1}{(B-A)\gamma\left[\frac{zF'_{m,\lambda}(z)}{F_{m,\lambda}(z)}-\alpha\right]-B\left[\frac{zF'_{m,\lambda}(z)}{F_{m,\lambda}(z)}-1\right]}\right|<\beta, \qquad (z\in\mathcal{U}). \tag{16}$$

Letting

$$w(z)=\frac{\frac{zF'_{m,\lambda}(z)}{F_{m,\lambda}(z)}-1}{\beta(B-A)\gamma\left[\frac{zF'_{m,\lambda}(z)}{F_{m,\lambda}(z)}-\alpha\right]-\beta B\left[\frac{zF'_{m,\lambda}(z)}{F_{m,\lambda}(z)}-1\right]},$$

then $w(0)=0$, $w(z)$ is analytic in $|z|<1$ and $|w(z)|<1$. Hence we have

$$\frac{zF'_{m,\lambda}(z)}{F_{m,\lambda}(z)}=\frac{1+\beta[B+\alpha\gamma(A-B)]w(z)}{1+\beta[B+\gamma(A-B)]w(z)}$$

which shows that $f(z)$ belongs to $\mathcal{A}_{p,q}^{m,\lambda}(A, B, \alpha, \gamma, \beta)$. $\square$

3. Coefficient estimates

Theorem 2. *A function $f(z)$ of the form (12) is in the class $\mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$ if*

$$\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1)-\beta\gamma(B-A)(k+\alpha)] \leq \beta\gamma(B-A)(1-\alpha), \qquad (z\in\mathcal{U})\,. \tag{17}$$

Proof. Suppose that the condition (17) holds. In view of (11), for $z\in\mathcal{U}$

$$\begin{aligned}
&\left|zF'_{m,\,\lambda}(z)-F_{m,\,\lambda}(z)\right|\\
&-\beta\left|(B-A)\,\gamma\left[zF'_{m,\,\lambda}(z)-\alpha F_{m,\,\lambda}(z)\right]-B\left[zF'_{m,\,\lambda}(z)-F_{m,\,\lambda}(z)\right]\right|\\
&=\left|-\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[(k-1)\left(1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right)\right]a_kz^k\right|-\\
&\beta\left|(B-A)\,\gamma\left(z(1-\alpha)-\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[(k-\alpha)\left(1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right)\right]\right)a_kz^k\right.\\
&\left.-B\left[-\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[(k-1)\left(1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right)\right]a_kz^k\right]\right|\\
&\leq\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1)\\
&-\beta\gamma(B-A)(k+\alpha)]\,|a_k|\,r^k-\beta\gamma(B-A)(1-\alpha)r.
\end{aligned}\tag{18}$$

For $|z|=r,\ 0<r<1$, we have

$$\begin{aligned}
&\left|zF'_{m,\,\lambda}(z)-F_{m,\,\lambda}(z)\right|\\
&-\beta\left|(B-A)\,\gamma\left[zF'_{m,\,\lambda}(z)-\alpha F_{m,\,\lambda}(z)\right]-B\left[zF'_{m,\,\lambda}(z)-F_{m,\,\lambda}(z)\right]\right|\\
&\leq\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1)\\
&-\beta\gamma(B-A)(k+\alpha)]\,|a_k|-\beta\gamma(B-A)(1-\alpha)\leq 0.
\end{aligned}\tag{19}$$

Or, equivalently,

$$\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1)-\beta\gamma(B-A)(k+\alpha)] \leq \beta\gamma(B-A)(1-\alpha), \qquad (z\in\mathcal{U})\,.$$

This completes the proof of Theorem 2. □

Corollary 1. *Let the function $f(z)$ be defined by (12) and let it be in the class $\mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$. Then*

$$a_k\leq\frac{\beta\gamma(B-A)(1-\alpha)}{\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1)-\beta\gamma(B-A)(k+\alpha)]} \tag{20}$$

$$(z\in\mathcal{U},\ k\geq 2).$$

The result is sharp for the function

$$f(z)=z-\frac{\beta\gamma(B-A)(1-\alpha)}{\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1)-\beta\gamma(B-A)(k+\alpha)]}z^k. \tag{21}$$

Hereafter, we let

$$\Omega_k^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)$$
$$=\left(\frac{p^k-q^k}{p-q}\right)^m\left[1+\lambda\left(\frac{p^k-q^k}{p-q}-1\right)\right][(k-1)+\beta B(k+1)-\beta\gamma(B-A)(k+\alpha)]\,. \tag{22}$$

4. Distortion Theorems

Theorem 3. *Let the function $f(z)$ defined by (12) be in the class $\mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$. Then for $z\in\mathcal{U}$,*

$$|z|-\frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}|z|^2\le|f(z)|\le|z|+\frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}|z|^2\,. \tag{23}$$

Proof. Since $\Omega_k^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)$ is an increasing function of $k\,(k\ge 2)$. Then we have,

$$\Omega_2^{p,q}(m,\lambda;A,B;\alpha,\gamma,\beta)\sum_{k=2}^{\infty}|a_k|\le\sum_{k=2}^{\infty}\Omega_k^{p,q}(m,\lambda;A,B;\alpha,\gamma,\beta)\,|a_k|\le\beta\gamma(B-A)(1-\alpha),$$

which implies

$$\sum_{k=2}^{\infty}|a_k|\le\frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}.$$

Thus we have

$$|f(z)|\le|z|+|z|^2\sum_{k=2}^{\infty}|a_k|\le|z|+\frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}|z|^2\,.$$

Similarly, we get,

$$|f(z)|\ge|z|-\sum_{k=2}^{\infty}|a_k|\,|z|^2\ge|z|-\frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}|z|^2\,.$$

Finally the result is sharp for the function

$$f(z)=z-\frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}z^2, \tag{24}$$

at $|z|=r$, which gives the required assertion of this theorem. □

Theorem 4. *Let the function $f(z)$ defined by (12) be in the class $\mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$. Then for $z\in\mathcal{U}$,*

$$1-\frac{2\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}|z|\le|f'(z)|\le1+\frac{2\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}|z|\,. \tag{25}$$

Proof. We have

$$\frac{\Omega_2^{p,q}\left(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta\right)}{2}\sum_{k=2}^{\infty}k\left|a_k\right|$$

$$\leq\sum_{k=2}^{\infty}\frac{\Omega_k^{p,q}\left(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta\right)}{k}k\left|a_k\right|\leq\beta\gamma(B-A)(1-\alpha),$$

which implies

$$\sum_{k=2}^{\infty}k\left|a_k\right|\leq\frac{2\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}\left(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta\right)}.$$

Thus we have

$$\left|f'(z)\right|\leq 1+|z|\sum_{k=2}^{\infty}k\left|a_k\right|\leq 1+\frac{2\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}\left(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta\right)}\left|z\right|.$$

Similarly, we get,

$$\left|f'(z)\right|\geq 1-|z|\sum_{k=2}^{\infty}k\left|a_k\right|\geq 1-\frac{2\beta\gamma(B-A)(1-\alpha)}{\Omega_2^{p,q}\left(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta\right)}\left|z\right|.$$

□

Corollary 2. *For the choices of* $m,\,\alpha,\,\beta\,\lambda,\,\gamma$ *and* $A=-1, B=1$, *(23) reduces to*

$$r-\frac{(1-\alpha)}{(2-\alpha)}r^2\leq|f(z)|\leq r+\frac{(1-\alpha)}{(2-\alpha)}r^2,$$

which was proved in [27].

Corollary 3. *For the choices of* $m,\,\alpha,\,\beta\,\lambda,\,\gamma$ *and* $A=-1, B=1$, *(25) reduces to*

$$1-\frac{2(1-\alpha)}{(2-\alpha)}r\leq|f'(z)|\leq 1+\frac{2(1-\alpha)}{(2-\alpha)}r,$$

which was proved in [[27]*, Theorem 6].*

5. Radii of Starlikeness, Convexity and Close-to-Convexity

The real number $r_\rho^*(f)=\sup\{r>0\,|Re\,(k(z))>\rho \text{ for all } z\in\mathcal{U}_r\}$ is called the radius of starlikeness of order ρ of the function f when $k(z)=\frac{zf'(z)}{f(z)}$. Note that $r_\rho^*(f)=r_0^*(f)$ is in fact the largest radius such that the image region $f\left(\mathcal{U}_r^*(f)\right)$ is a starlike domain with respect to the origin. Similar definition is used to define radius of convexity and close to convexity by equivalently replacing $k(z)$ with $1+\frac{zf'(z)}{f(z)}$ and $\frac{f'(z)}{g'(z)}$ respectively. For the study of various radius problems, we refer to [8, 15, 16, 25, 30].

Theorem 5. *Let the function* $f(z)$ *defined by (12) be in the class* $\mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$. *Then* $f(z)$ *is starlike of order* $\rho\,(0\leq\rho<1)$ *in* $|z|<R_1$, *where*

$$R_1=\inf_{k\geq 2}\left\{\frac{\Omega_k^{p,q}\left(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta\right)}{\beta\gamma(B-A)(1-\alpha)}\times\left(\frac{1-\rho}{k-\rho}\right)\right\}^{\frac{1}{k-1}}\qquad(z\in\mathcal{U}).\tag{26}$$

Proof. Given $f\in\mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$ and f is starlike of order ρ, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\rho.\tag{27}$$

For the left hand side of (27), we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{\sum_{k=2}^{\infty}(k-1)\,a_k z^{k-1}}{1-\sum_{k=2}^{\infty}a_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k\,|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k\,|z|^{k-1}}.$$

The last expression is less than $(1-\rho)$ if

$$\frac{\sum_{k=2}^{\infty}(k-1)a_k\,|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k\,|z|^{k-1}} < 1-\rho,$$

which implies

$$\sum_{k=2}^{\infty}\frac{(k-\rho)a_k\,|z|^{k-1}}{(1-\rho)} < 1. \tag{28}$$

In view of Theorem 2, (28) will be true if

$$\frac{(k-\rho)a_k\,|z|^{k-1}}{(1-\rho)} \leq \frac{\Omega_k^{p,q}\,(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}{\beta\gamma(B-A)(1-\alpha)} \tag{29}$$

where $\Omega_k^{p,q}\,(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)$ is defined by (22). From (29), implies

$$|z|^{k-1} \leq \frac{\Omega_k^{p,q}\,(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}{\beta\gamma(B-A)(1-\alpha)}\left(\frac{1-\rho}{k-\rho}\right).$$

The last inequality leads us immediately to the disc $|z| < R_1$, where R_1 is given by (26). □

Theorem 6. *Let the function $f(z)$ defined by (12) be in the class $\mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$. Then $f(z)$ is close-to-convex of order $\rho\,(0 \leq \rho < 1)$ in $|z| < R_2$, where*

$$R_2 = \inf_{k\geq 2}\left\{\frac{\Omega_k^{p,q}\,(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}{\beta\gamma(B-A)(1-\alpha)}\times\left(\frac{1-\rho}{k}\right)\right\}^{\frac{1}{k-1}} \qquad (z\in\mathcal{U}), \tag{30}$$

where $\Omega_k^{p,q}\,(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)$ defined by (22).

Proof. Given $f \in \mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$ and f is close - to - convex of order ρ, we have

$$|f'(z)-1| < 1-\rho. \tag{31}$$

Now proceeding as in Theorem 5, we can prove the Theorem. Therefore we choose to omit the proof. □

We just state the following result.

Theorem 7. *Let the function $f(z)$ defined by (12) be in the class $\mathcal{T}_{p,q}^{m,\,\lambda}(A,\,B,\,\alpha,\,\gamma,\,\beta)$. Then $f(z)$ is convex of order $\rho\,(0 \leq \rho < 1)$ in $|z| < R_3$, where*

$$R_3 = \inf_{k\geq 2}\left\{\frac{\Omega_k^{p,q}\,(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)}{\beta\gamma(B-A)(1-\alpha)}\times\left(\frac{1-\rho}{k(k-\rho)}\right)\right\}^{\frac{1}{k-1}} \qquad (z\in\mathcal{U}), \tag{32}$$

where $\Omega_k^{p,q}\,(m,\,\lambda;\,A,\,B;\,\alpha,\,\gamma,\,\beta)$ defined by (22).

6. Extreme Points

The study of the convex hulls and extreme points of various families of univalent functions was initiated by L. Brickman, T. H. MacGregor, and D. R. Wilken in [6]. The importance of determining the extreme points of a compact family F lies in the fact that the maximum or minimum value of any continuous linear functional defined over the set of analytic functions occurs at one of the extreme points of the closed convex hull of F. There have been numerous papers recently dealing with the extreme points for the closed convex hull of several compact families of univalent functions, but for the classical analysis of the significance of extreme points can be found in [6, 7, 28] . But we employ the technique adopted by Silverman in [28] to find the extreme points for our class.

Theorem 8. *Let* $f_1(z) = z$ *and* $f_k(z) = z - \dfrac{\beta\gamma(B-A)(1-\alpha)}{\Omega_k^{p,q}(m, \lambda; A, B; \alpha, \gamma, \beta)} z^k, (k = 2, 3, \ldots)$. *Then* $f(z) \in \mathcal{T}_{p,q}^{m,\,\lambda}(A,\, B,\, \alpha,\, \gamma,\, \beta)$ *if and only if it can be expressed in the following form:*

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z), \tag{33}$$

where $c_k \geq 0, \quad \sum_{k=1}^{\infty} c_k = 1.$

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z) = z - \sum_{k=2}^{\infty} c_k \frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_k^{p,q}(m,\, \lambda;\, A,\, B;\, \alpha,\, \gamma,\, \beta)} z^k.$$

Then, from Theorem 2, we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \left[\{\Omega_k^{p,q}(m,\, \lambda;\, A,\, B;\, \alpha,\, \gamma,\, \beta)\} \times \frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_k^{p,q}(m,\, \lambda;\, A,\, B;\, \alpha,\, \gamma,\, \beta)} c_k \right] \\ &= \beta\gamma(B-A)(1-\alpha) \sum_{k=2}^{\infty} c_k = \beta\gamma(B-A)(1-\alpha)(1-c_k) = \beta\gamma(B-A)(1-\alpha). \end{aligned} \tag{34}$$

Thus, in view of Theorem 2, we find that $f(z) \in \mathcal{T}_{p,q}^{m,\,\lambda}(A,\, B,\, \alpha,\, \gamma,\, \beta)$.

Conversely, let us suppose that $f(z) \in \mathcal{T}_{p,q}^{m,\,\lambda}(A,\, B,\, \alpha,\, \gamma,\, \beta)$, then, since

$$a_k \leq \frac{\beta\gamma(B-A)(1-\alpha)}{\Omega_k^{p,q}(m,\, \lambda;\, A,\, B;\, \alpha,\, \gamma,\, \beta)}, \quad (k = 2,\, 3,\, \ldots). \tag{35}$$

Set

$$c_k \leq \frac{\Omega_k^{p,q}(m,\, \lambda;\, A,\, B;\, \alpha,\, \gamma,\, \beta)}{\beta\gamma(B-A)(1-\alpha)}, \qquad c_1 = 1 - \sum_{k=2}^{\infty} c_k. \tag{36}$$

Thus clearly, we have $f(z) = \sum_{k=1}^{\infty} c_k f_k(z)$. This completes the proof of Theorem 8. □

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