TJMM
9 (2017), No. 1, 35-42

ON CO-ORDINATED OSTROWSKI AND HADAMARD'S TYPE INEQUALITIES FOR CONVEX FUNCTIONS II

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Abstract. In this article, we establish various inequalities for some differentiable mappings that are linked with the illustrious Hermite-Hadamard and Ostrowski integral inequality for convex functions of several-variables on the co-ordinates.The generalized integral inequalities contribute some better estimates than some already presented.

1. Introduction

In 1938, A. Ostrowski proved a sharp estimate of a differentiable function, whose first derivative is bounded by its integral mean as follows:

Theorem 1. [12] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with bounded derivative, that is, $\|f'\|_\infty < \infty$, then*

$$\left| f(x) - \frac{1}{b-a}\int_0^1 f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f\|_\infty, \tag{1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible one.

Inequality (1) has a lot of applications in different fields of mathematics such as probability and numerical analysis etc. Due to that reason (1) make the cause of attention among mathematicians and researchers. Inequality (1) was improved, generalized and extended in different directions by using different techniques [1, 3, 5, 7]. Dragomir [6] defined convex function on the co-ordinates as follows:

Definition 1. *Let us consider the bi-dimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b, c < d$. A function $f : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex whose defined for all $y \in [c, d]$ and $x \in [a, b]$.*

Recall that the function $f : \Delta \to \mathbb{R}$ is convex on Δ if

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w),$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

Clearly, every convex (concave) function $f : \Delta \to \mathbb{R}$ is convex (concave) on the co-ordinates but converse may not be true [6]. By using this definition many researchers formulated Ostrowski type inequalities and in particular gave some sharp estimates for the left and right Hadamard inequality and some related results [2, 4, 6, 8, 9, 13].

The main aim of this paper is to establish some new Ostrowski type inequalities for co-ordinated convex functions and as an applications we have derived left Hadamard type

2010 *Mathematics Subject Classification.* Primary 26B25, 26D10, 26D15; Secondary 26A48, 26A51, 52A41.

Key words and phrases. co-ordinated convex function; Ostrowski inequality; Hermite- Hadamard inequality.

inequalities. Among those some are sharper than the exiting one's and some generalized the exiting one's.

This paper is organized in the following way. After this Introduction in Section 2, we formulated our main results and some related applications.

2. Main Results

Lemma 1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable function on $\Delta := [a, b] \times [c, d]$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then*

$$f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)dudv - A = \sum_{\substack{\mu \in \{a,b\} \\ \nu \in \{c,d\}}} I_{\mu,\nu},$$

where,

$$A = \frac{1}{b-a} \int_a^b f(u,y)du - \frac{1}{d-c} \int_c^d f(x,v)dv,$$

$$I_{\mu,\nu} = \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(s-1) \frac{\partial^2 f}{\partial t \partial s}(t\mu + (1-t)x, s\nu + (1-s)y)dtds$$

with $I_{\mu,\nu}$ is positive when $(\mu,\nu) = (a,c)$ and (b,d) otherwise negative.

Proof. Consider

$$\begin{aligned}
I_{a,c} &= \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(s-1) \frac{\partial^2 f}{\partial t \partial s}(ta + (1-t)x, sc + (1-s)y)dtds \\
&= \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 (s-1) \left(\int_0^1 (t-1) \frac{\partial^2 f}{\partial t \partial s}(ta + (1-t)x, sc + (1-s)y)dt \right) ds \\
&= \frac{(x-a)(y-c)^2}{(b-a)(d-c)} \left[\int_0^1 (1-s) \frac{\partial f}{\partial s}(x, sc + (1-s)y)ds - \right. \\
&\qquad \left. \int_0^1 \int_0^1 (1-s) \frac{\partial f}{\partial s}(ta + (1-t)x, sc + (1-s)y)dsdt \right] \\
&= \frac{(x-a)(y-c)}{(b-a)(d-c)} \left[f(x,y) - \int_0^1 f(x, sc + (1-s)y)ds - \int_0^1 f(ta + (1-t)x, y)dt \right. \\
&\qquad \left. + \int_0^1 \int_0^1 f(ta + (1-t)x, sc + (1-s)y)dsdt \right].
\end{aligned}$$

Similarly

$$\begin{aligned}
I_{a,d} &= -\frac{(x-a)^2(y-d)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(s-1) \frac{\partial^2 f}{\partial t \partial s}(ta + (1-t)x, sd + (1-s)y)dtds \\
&= \frac{(x-a)(y-d)}{(b-a)(d-c)} \left[f(x,y) - \int_0^1 f(x, sd + (1-s)y)ds - \int_0^1 f(ta + (1-t)x, y)dt \right. \\
&\qquad \left. + \int_0^1 \int_0^1 f(ta + (1-t)x, sd + (1-s)y)dsdt \right],
\end{aligned}$$

$$I_{b,c} = -\frac{(x-b)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(s-1) \frac{\partial^2 f}{\partial t \partial s}(tb + (1-t)x, sc + (1-s)y) dtds$$
$$= \frac{(x-b)(y-c)}{(b-a)(d-c)} \left[f(x,y) - \int_0^1 f(x, sc + (1-s)y) ds - \int_0^1 f(tb + (1-t)x, y) dt \right.$$
$$\left. + \int_0^1 \int_0^1 f(tb + (1-t)x, sc + (1-s)y) dsdt \right],$$

and

$$I_{b,d} = \frac{(x-b)^2(y-d)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(s-1) \frac{\partial^2 f}{\partial t \partial s}(tb + (1-t)x, sd + (1-s)y) dtds$$
$$= \frac{(x-b)(y-d)}{(b-a)(d-c)} \left[f(x,y) - \int_0^1 f(x, sd + (1-s)y) ds - \int_0^1 f(tb + (1-t)x, y) dt \right.$$
$$\left. + \int_0^1 \int_0^1 f(tb + (1-t)x, sd + (1-s)y) dsdt \right],$$

which completes the proof. □

Theorem 2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable function on $\Delta := [a,b] \times [c,d]$ with $a < b$ and $c < d$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$ is a convex function on the co-ordinates on Δ, then*

$$\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) dudv - A \right| \leq \sum_{\substack{\mu \in \{a,b\} \\ \nu \in \{c,d\}}} \frac{(x-\mu)^2(y-\nu)^2}{36(b-a)(d-c)}$$
$$\times \left[\left| \frac{\partial^2 f}{\partial t \partial s}(\mu,\nu) \right| + 2 \left| \frac{\partial^2 f}{\partial t \partial s}(\mu,y) \right| + 2 \left| \frac{\partial^2 f}{\partial t \partial s}(x,\nu) \right| + 4 \left| \frac{\partial^2 f}{\partial t \partial s}(x,y) \right| \right] \tag{2}$$

Proof. By lemma 1, we have

$$\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) dudv - A \right| \leq \sum_{\substack{\mu \in \{a,b\} \\ \nu \in \{c,d\}}} |I_{\mu,\nu}|$$

$$|I_{\mu,\nu}| \leq \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(t\mu + (1-t)x, s\nu + (1-s)y) \right| dtds$$

Since $f : \Delta \to \mathbb{R}$ is coordinated convex on Δ

$$|I_{\mu,\nu}| \leq \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (1-s)(1-t) \left\{ t \left| \frac{\partial^2 f}{\partial t \partial s}(\mu, s\nu + (1-s)y) \right| + \right.$$
$$\left. (1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(x, s\nu + (1-s)y) \right| \right\} dtds$$
$$\leq \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (1-s)(1-t) \left\{ st \left| \frac{\partial^2 f}{\partial t \partial s}(\mu,\nu) \right| + (1-s)t \times \right.$$
$$\left. \left| \frac{\partial^2 f}{\partial t \partial s}(\mu,y) \right| + (1-t)s \left| \frac{\partial^2 f}{\partial t \partial s}(x,\nu) \right| + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(x,y) \right| \right\} dtds,$$

which completes the proof. □

Corollary 1. *By setting* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in* (2) *we have*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}, \quad \frac{c+d}{2}\right) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)dudv - A \right| \\
&\leq \frac{(b-a)(d-a)}{36 \times 16} \sum_{\substack{\mu \in \{a,b\} \\ \nu \in \{c,d\}}} \left\{ \left| \frac{\partial^2 f}{\partial t \partial s}(\mu,\nu) \right| + 2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\mu, \frac{c+d}{2}\right) \right| + \right. \\
&\qquad \left. 2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \nu\right) \right| + 4 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \right\} \\
&\leq \frac{(b-a)(d-a)}{64} \sum_{\substack{\mu \in \{a,b\} \\ \nu \in \{c,d\}}} \left| \frac{\partial^2 f}{\partial t \partial s}(\mu,\nu) \right| \qquad (3)
\end{aligned}$$

Remark 1. *It may be noted that* (3) *gives better approximation to* [9, Theorem 2].

Theorem 3. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable function on* $\Delta := [a,b] \times [c,d]$ *with* $a < b$ *and* $c < d$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q > 1$ *with* $p = \frac{q}{q-1}$, *is a convex function on the co-ordinates on* Δ, *then*

$$\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)dudv - A \right| \leq \sum_{\substack{\mu \in \{a,b\} \\ \nu \in \{c,d\}}} \frac{2^{-2/q}(x-\mu)^2(y-\nu)^2}{(p+1)^{2/p}(b-a)(d-c)}$$

$$\times \left(\left| \frac{\partial^2 f}{\partial t \partial s}(\mu,\nu) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(\mu,y) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(x,\nu) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(x,y) \right|^q \right)^{1/q} \qquad (4)$$

Proof. By lemma 1, we have

$$\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)dudv - A \right| \leq \sum_{\substack{\mu \in \{a,b\} \\ \nu \in \{c,d\}}} |I_{\mu,\nu}|$$

$$|I_{\mu,\nu}| \leq \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(t\mu + (1-t)x, s\nu + (1-s)y) \right| dtds$$

By Höder's inequality

$$\begin{aligned}
|I_{\mu,\nu}| \leq \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)} &\left(\int_0^1 \int_0^1 (1-t)^p(1-s)^p dsdt \right)^{1/p} \times \\
&\left(\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(t\mu + (1-t)x, s\nu + (1-s)y) \right|^q dtds \right)^{1/q}
\end{aligned}$$

By coordinated convexity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ on Δ, we have

$$\begin{aligned}
|I_{\mu,\nu}| \leq &\frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)(p+1)^{2/p}} \left(\int_0^1 \int_0^1 \left\{ st \left| \frac{\partial^2 f}{\partial t \partial s}(\mu,\nu) \right|^q + (1-s)t \left| \frac{\partial^2 f}{\partial t \partial s}(\mu,y) \right|^q \right. \right. \\
&\left. \left. + \ (1-t)s \left| \frac{\partial^2 f}{\partial t \partial s}(x,\nu) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(x,y) \right|^q \right\} dtds \right)^{1/q},
\end{aligned}$$

which completes the proof. □

Remark 2. *By setting* $\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right| \le M$ *for* $(t,s)\in[a,b]\times[c,d]$*, Theorem 3 reduces to* [10, Theorem 4].

The following theorem gives tighter estimate than that of previous theorem 3.

Theorem 4. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable function on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$ *and* $c<d$*. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$*,* $q\ge 1$*, is a convex function on the co-ordinates on* Δ*, then*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dudv - A\right| \le \sum_{\substack{\mu\in\{a,b\}\\ \nu\in\{c,d\}}} \frac{3^{-2/q}(x-\mu)^2(y-\nu)^2}{4(b-a)(d-c)}$$

$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}(\mu,\nu)\right|^q + 2\left|\frac{\partial^2 f}{\partial t\partial s}(\mu,y)\right|^q + 2\left|\frac{\partial^2 f}{\partial t\partial s}(x,\nu)\right|^q + 4\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q\right)^{1/q} \tag{5}$$

Proof. By lemma 1, we have

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dudv - A\right| \le \sum_{\substack{\mu\in\{a,b\}\\ \nu\in\{c,d\}}} |I_{\mu,\nu}|$$

$$|I_{\mu,\nu}| \le \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)}\int_0^1\int_0^1 (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(t\mu+(1-t)x, s\nu+(1-s)y)\right| dtds$$

By Höder's inequality:

$$|I_{\mu,\nu}| \le \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (1-t)(1-s)dsdt\right)^{1/p}\times$$
$$\left(\int_0^1\int_0^1 (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(t\mu+(1-t)x, s\nu+(1-s)y)\right|^q dtds\right)^{1/q}$$

By coordinated convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ on Δ, we have

$$|I_{\mu,\nu}| \le \frac{(x-\mu)^2(y-\nu)^2}{2^{2/p}(b-a)(d-c)}\left(\int_0^1\int_0^1 (1-t)(1-s)\left\{st\left|\frac{\partial^2 f}{\partial t\partial s}(\mu,\nu)\right|^q + (1-s)t\right.\right.$$
$$\left.\left.\left|\frac{\partial^2 f}{\partial t\partial s}(\mu,y)\right|^q + (1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(x,\nu)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|^q\right\} dtds\right)^{1/q},$$

which complete the proof. □

Remark 3. *By setting* $\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right| \le M$ *for* $(t,s)\in[a,b]\times[c,d]$*, Theorem 4 reduces to* [10, Theorem 3].

Theorem 5. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable function on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$ *and* $c<d$*. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$*,* $q>1$ *with* $1/p+1/q=1$ *is a concave function on the* *co-ordinates on* Δ*, then*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dudv-A\right|\leq \sum_{\substack{\mu\in\{a,b\}\\ \nu\in\{c,d\}}}\frac{(x-\mu)^2(y-\nu)^2}{(p+1)^{2/p}(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{\mu+x}{2},\frac{\nu+y}{2}\right)\right| \tag{6}$$

Proof. By lemma 1, we have

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dudv-A\right|\leq \sum_{\substack{\mu\in\{a,b\}\\ \nu\in\{c,d\}}}|I_{\mu,\nu}|$$

$$|I_{\mu,\nu}|\leq\frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)}\int_0^1\int_0^1(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(t\mu+(1-t)x,s\nu+(1-s)y)\right|dtds$$

By Höder's inequality:

$$|I_{\mu,\nu}|\leq\frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1(1-t)^p(1-s)^p dsdt\right)^{1/p}\times \left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(t\mu+(1-t)x,s\nu+(1-s)y)\right|^q dtds\right)^{1/q}$$

By coordinated concavity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ on Δ, we have

$$\begin{aligned}\int_0^1\int_0^1&\left|\frac{\partial^2 f}{\partial t\partial s}(t\mu+(1-t)x,s\nu+(1-s)y)\right|^q dtds\\ &\leq\frac{1}{2}\left[\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\mu+(1-t)x,\frac{\nu+y}{2}\right)\right|^q dt\right.\\ &\qquad\left.+\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{\mu+x}{2},s\nu+(1-s)y\right)\right|^q ds\right]\\ &\leq\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{\mu+x}{2},\frac{\nu+y}{2}\right)\right|^q,\end{aligned}$$

which completes the proof. □

Theorem 6. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$ *and* $c<d$*. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$*, is a concave function on the co-ordinates on* Δ*, then*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dudv-A\right|\leq \sum_{\substack{\mu\in\{a,b\}\\ \nu\in\{c,d\}}}\frac{(x-\mu)^2(y-\nu)^2}{4(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{\mu+2x}{3},\frac{\nu+2y}{3}\right)\right| \tag{7}$$

Proof. By the concavity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ on the co-ordinates on Δ and power-mean inequality, the following inequality holds:

$$\left|\frac{\partial^2 f}{\partial t \partial s}(t\mu+(1-t)x,v)\right|^q \geq t\left|\frac{\partial^2 f}{\partial t \partial s}(\mu,v)\right|^q+(1-t)\left|\frac{\partial^2 f}{\partial t \partial s}(x,v)\right|^q$$
$$\geq \left(t\left|\frac{\partial^2 f}{\partial t \partial s}(\mu,v)\right|+(1-t)\left|\frac{\partial^2 f}{\partial t \partial s}(x,v)\right|\right)^q$$

for all $\mu, x \in [a,b]$, $t \in [0,1]$ and fixed $v \in [c,d]$

$$\left|\frac{\partial^2 f}{\partial t \partial s}(t\mu+(1-t)x,v)\right| \geq t\left|\frac{\partial^2 f}{\partial t \partial s}(\mu,v)\right|+(1-t)\left|\frac{\partial^2 f}{\partial t \partial s}(x,v)\right|.$$

Similarly

$$\left|\frac{\partial^2 f}{\partial t \partial s}(u,s\nu+(1-s)y)\right| \geq s\left|\frac{\partial^2 f}{\partial t \partial s}(u,\nu)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(u,y)\right|,$$

for all $\nu, y \in [c,d]$, $s \in [0,1]$ and fixed $u \in [a,b]$, showing that $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is concave on co-ordinates on Δ. Now by lemma 1

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dudv-A\right| \leq \sum_{\substack{\mu\in\{a,b\}\\ \nu\in\{c,d\}}} |I_{\mu,\nu}|.$$

$$|I_{\mu,\nu}| \leq \frac{(x-\mu)^2(y-\nu)^2}{(b-a)(d-c)}\int_0^1\int_0^1(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(t\mu+(1-t)x,s\nu+(1-s)y)\right|dtds.$$

By coordinated concavity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ on Δ, and Jensen's integral inequality, we have

$$\int_0^1\int_0^1(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(t\mu+(1-t)x,s\nu+(1-s)y)\right|dtds$$
$$=\int_0^1(1-t)\left[\int_0^1(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(t\mu+(1-t)x,s\nu+(1-s)y)\right|ds\right]ds$$
$$\leq \int_0^1(1-t)\left[\left(\int_0^1(1-s)ds\right)\left|\frac{\partial^2 f}{\partial t \partial s}\left(t\mu+(1-t)x,\frac{\int_0^1(1-s)(s\nu+(1-s)y)ds}{\int_0^1(1-s)ds}\right)\right|\right]dt$$
$$=\frac{1}{2}\int_0^1(1-t)\left|\frac{\partial^2 f}{\partial t \partial s}\left(t\mu+(1-t)x,\frac{\nu+2y}{3}\right)\right|dt$$
$$\leq \frac{1}{2}\left(\int_0^1(1-t)dt\right)\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{\int_0^1(1-t)(t\mu+(1-t)x)dt}{\int_0^1(1-t)dt},\frac{\nu+2y}{3}\right)\right|$$
$$=\frac{1}{4}\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{\mu+2x}{3},\frac{\nu+2y}{3}\right)\right|dt,$$

which completes the proof. □

Remark 4. *By setting* $x=\frac{a+b}{2}, y=\frac{c+d}{2}$ *, Theorem 6 reduces to* [9, Theorem 5].

Authors' contributions: All authors contributed equally in this article. They read and approved the final manuscript.

Acknowledgments: This research paper is made possible through the help and support from HEC, Pakistan and Univ. of the Engg. & Tech. Taxila, Pakistan. We gratefully

acknowledge the time and expertise devoted to reviewing papers by the advisory editors, the members of the editorial board, and the referees.
Finally, We would like to express our tribute to Prof. Dr. Muhammad Iqbal Bhatti (Chairman, Dept. of Math., UET, Taxila), Our doctoral Supervisor, who died young in his work. As a talented and outstanding young professor, you sacrificed your life to the country when your career was thriving. We will never forget how much you cared about our study and daily life. We will miss you forever!

References

[1] Alomari, M. and Darus, M., *Some Ostrowski type inequalities for convex functions with applications,* RGMIA **13** (1) 2010. Article No. 3.
[2] Alomari, M. and Darus, M., *The Hadamard's inequality for s-convex functions of 2-variables,* Int. Jour. Math. Anal. **2** (13), (2008), 629–638.
[3] Alomari, M., Darus, M., Dragomir, S.S., Cerone, P., *Ostroski type inequalities for functions whose derivatives are s-convex in second sense*, Appl. Math. Lett. **23** (2010), 1071–1076.
[4] Bakula M.K., and Pečarić, J., *On the Jensen's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese Journal of Mathematics **5** (2006), 1271–1292.
[5] Cerone, P., Dragomir, S.S., Roumeliots, J., *An inequality of Ostrowski-Grüss type for twice differentiable mappings and applications in numerical integration*, Kyungpook Math. J. **39** (1999), 331–341.
[6] Dragomir, S.S., *On Hadamard inequality for convex functions on co-ordinates in rectangle from the plane*, Taiwanese Journal of Mathematics, **5** (2001), 775–788.
[7] Dragomir, S.S., Cerone, P., Roumelities, J., *A new generalization of Ostrowski integral inequality for mapping whose derivatives are bounded and applications in numerical integration and for special means*, Appl. Math. Lett. **13** (2000), 19–25.
[8] Latif, M.A. and Alomari, M., *Hadamard-type inequalities for product two convex functions on the co-ordinates*, Int. Math. Forum **4** (47), (2009), 2327–2338.
[9] Latif, M.A. and Dragomir, S.S., *On some new inequalities for differentiable co-ordinated convex functions*, Jour. Ineq. Appl. 2012:28, doi:10.1186/1029-242X-2012-28.
[10] Latif, M.A., Hussain, S. and Dragomir, S.S., *New Ostrowski type inequalities fo co-ordinated convex functions*, ON-LINE:`http://rgmia.org/papers/v14/v14a49.pdf`.
[11] Mitrinović, D.S., Pečarić, J.E. and Fink, A.M., *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dortrecht, 1991.
[12] Ostrowski, A., *Über die Absolutabweichung einer differentiebaren funktin von ihren intergralmittelwert*, Comment. Math. Helv. **10** (1938), 226–227.
[13] Özdemir, M.E., Kavurmaci, H., Akdemir, A.O. and Avci, M., *Inequalities for convex and s-convex functions on* $\Delta = [a, b] \times [c, d]$, Jour. Ineq. Appl., 2012.

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