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ON $H_3(1)$ HANKEL DETERMINANT FOR UNIVALENT FUNCTIONS DEFINED BY USING q-DERIVATIVE OPERATOR

ZELIHA KARAHUSEYIN, ŞAHSENE ALTINKAYA, AND SIBEL YALÇIN

Abstract. In this work, we introduce a new subclass of univalent functions defined by using q-derivative operator. Moreover, for functions belonging to this class, we establish third Hankel determinant denoted by $H_3(1)$.

1. Introduction, Notations and Definitions

Let A represent the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let S be the subclass of A consisting of the form (1) which are also univalent in E.

Let f be given by (1). Then $f \in R$ if it satisfies the inequality

$$\Re\left(f'(z)\right) > 0, \qquad z \in E.$$

The subclass R was studied systematically by MacGregor [8] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

In the field of Geometric Function Theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q-calculus is the important tools that are used to investigate subclasses of analytic functions. For example, the extension of the theory of univalent functions can be described by using the theory of q-calculus. Historically speaking, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [14]). In fact, the theory of univalent functions can be described by using the theory of the q-calculus. Moreover, such q-calculus operators as the fractional q-integral and fractional q-derivative operators, are used to construct several subclasses of analytic functions (see, for example, [1], [6], [11], [15]). In a recent paper Purohit and Raina [13], investigated applications of fractional q-calculus operators to defined certain new classes of functions which are analytic in the open disk. Later, Mohammed and Darus [9] studied approximation and geometric properties of these q-operators in some subclasses of analytic functions in compact disk.

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We shall

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follow the notation and terminology in [3]. We recall the definitions of fractional q-calculus operators of complex valued function $f(z)$.

Definition 1. *Let $q \in (0,1)$ and define*

$$[n]_q = \frac{1-q^n}{1-q},$$

for $n \in \mathbb{N} = \{1, 2, 3 \ldots\}$.

Definition 2. *Let $q \in (0,1)$ and define the q-fractional $[n]_q!$ by*

$$[n]_q! = \begin{cases} \prod\limits_{k=1}^{n} [k]_q\,, & n = 1, 2, \ldots \\ \\ 1, & n = 0 \end{cases}$$

for $n \in \mathbb{N}$.

Definition 3. *For $\alpha \in \mathbb{C}$, the q-shifted factorial is defined as a product of $n \in \mathbb{N}_0 = \{0, 1, \ldots\}$ factors by*

$$(\alpha; q)_0 = 1, \quad (\alpha; q)_n = \prod_{i=0}^{n-1}(1-\alpha q^i), \quad (\alpha; q)_\infty = \prod_{i=0}^{\infty}(1-\alpha q^i).$$

Definition 4. *(see [5]) The q-derivative of a function f is defined on a subset of $\mathbb{C}$ is given by*

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad if \quad z \neq 0, \tag{2}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = \frac{df(z)}{dz}$$

if f is differentiable. From (2), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q\, a_n z^{n-1}. \tag{3}$$

Definition 5. *A function $f \in A$ is said to be in the class $R(q)$, if the following condition holes*

$$\Re\left((D_q f)(z)\right) > 0, \qquad z \in E.$$

We note that

$$\lim_{q \to 1^-} R(q) = \left\{ f \in A : \lim_{q \to 1^-} \Re\left((D_q f)(z)\right) > 0, \quad z \in E \right\} = R.$$

Definition 6. *(see [2]) Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $t \geq 1$, we define the t-th Hankel determinants with Fekete-Szegö parameter λ, that is $H_t^{\lambda}(n)$, as*

$$H_t^{\lambda}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & \lambda a_{n+t-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+t-1} & \cdots & \cdots & a_{n+2(t-1)} \end{vmatrix}.$$

The well known Hankel determinant is the case $\lambda = 1$. In 1976, Noonan and Thomas [10] defined the $H_t(n)$ Hankel determinant. We also defined other similar determinant as:

Definition 7. *(see [2]) Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $t \geq 1$, we define the $B_t^{\lambda}(n)$ determinant as*

$$B_t^{\lambda}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+t-1} \\ a_{n+t} & a_{n+t-1} & \cdots & a_{n+2t-1} \\ a_{n+2t} & a_{n+2t+1} & \cdots & a_{n+3t-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+t(t-1)} & \cdots & \cdots & \lambda a_{n+t^2-1} \end{vmatrix}$$

In this paper, we further extend these definitions to include finitely many Fekete-Szegö parameters λ_j, $j = 1, 2, \ldots$ in order to accomodate a wide variety of emerging functionals in the study of coefficients of mappings of the unit disk. Now we say:

Definition 8. *(see [2]) Let λ_i, $i = 1, 2, \ldots, t$ be nonnegative real numbers. Then for integers $n \geq 1$ and $t \geq 1$, we define the t-th Hankel determinants with Fekete-Szegö parameter λ_i, that is $H_t^{\lambda_1,\lambda_2,\ldots,\lambda_t}(n)$, as*

$$H_t^{\lambda_1,\lambda_2,\ldots,\lambda_t}(n) = \begin{vmatrix} \lambda_1 a_n & \lambda_2 a_{n+1} & \cdots & \lambda_t a_{n+t-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+t-1} & \cdots & \cdots & a_{n+2(t-1)} \end{vmatrix}$$

and we also say:

Definition 9. *(see [2]) Let λ_j, $j = 1, 2, \ldots, n$ be nonnegative real numbers. Then for integers $n \geq 1$ and $t \geq 1$, we define the $B_t^{\lambda_1,\lambda_2,\ldots,\lambda_n}(n)$ determinants as*

$$B_t^{\lambda_1,\lambda_2,\cdots,\lambda_n}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & \lambda_1 a_{n+t-1} \\ a_{n+t} & a_{n+t+1} & \cdots & \lambda_2 a_{n+2t-1} \\ a_{n+2t} & a_{n+2t+1} & \cdots & \lambda_3 a_{n+3t-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+t(t-1)} & \cdots & \cdots & \lambda_n a_{n+t^2-1} \end{vmatrix}.$$

For $\lambda_j = 1$, $j = 1, 2, \ldots, (t-1)(n-1)$, we simply write $H_t^{\lambda}(n)$ and $B_t^{\lambda}(n)$ in place of $H_t^{1,1,\ldots,1,\lambda_n}(n)$ and $B_t^{1,1,\ldots,1,\lambda_n}(n)$ respectively, and we quickly note that for real numbers, γ, α and β we have:

$$H_2^{\gamma}(1) = \begin{vmatrix} 1 & \gamma a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - \gamma a_2^2,$$

$$H_2^{\alpha}(2) = \begin{vmatrix} a_2 & \alpha a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - \alpha a_3^2$$

and

$$B_2^{\beta}(1) = \begin{vmatrix} 1 & a_2 \\ a_3 & \beta a_4 \end{vmatrix} = a_2 a_3 - \beta a_4.$$

In this paper, we get upper bound for the functional

$$|H_3(1)| \leq |a_3| \left|a_2 a_4 - a_3^2\right| + |a_4| \left|a_2 a_3 - a_4\right| + |a_5| \left|a_3 - a_2^2\right|$$

for functions f belongs to the class $R(q)$.

We first state some preliminary lemmas which shall be used in our proof.

2. Preliminary Lemmas

Let P be the family of functions p analytic in E for which $\Re(p(z)) > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots ,$$

for $z \in E$.

Lemma 1. [12] *If $p \in P$ then $|c_n| \leq 2$ for each n $(n \in \mathbb{N})$.*

Lemma 2. [4] *If $p \in P$ then*

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ \\ 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \end{aligned}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 3. *(see [7]) If $p \in P$, then*

$$\left|c_2 - \upsilon c_1^2\right| \leq \begin{cases} -4\upsilon + 2; & if \quad \upsilon \leq 0 \\ \\ 2; & if \quad 0 \leq \upsilon \leq 1 \\ \\ 4\upsilon - 2; & if \quad \upsilon \geq 1 \end{cases} .$$

3. Main Results

Thus we shall in this paper obtain the best possible bounds on $H_2^{\gamma}(1)$ and $B_2^{\beta}(1)$ to conclude our investigation.

Theorem 1. *Let $f \in R(q)$. Then for real number γ,*

$$|H_2^{\gamma}(1)| \leq 2 \max \left\{ \frac{1}{[3]_q}, \left| \frac{2\gamma}{[2]_q^2} - \frac{1}{[3]_q} \right| \right\}.$$

Proof. It is well known that if $f \in R(q)$, then

$$a_2 = \frac{c_1}{[2]_q}, \quad a_3 = \frac{c_2}{[3]_q} \text{ and } \quad a_4 = \frac{c_3}{[4]_q}. \tag{4}$$

Then, we can establish that

$$\begin{aligned} |H_2^{\gamma}(1)| &= \left|a_3 - \gamma a_2^2\right| = \left| \frac{c_2}{[3]_q} - \gamma \frac{c_1^2}{[2]_q^2} \right| \\ &= \frac{1}{[3]_q} \left|c_2 - \upsilon c_1^2\right| \end{aligned}$$

where $\upsilon = \frac{\gamma[3]_q}{[2]_q^2}$.

Next, according to Lemma 3, we write

$$\left|a_3 - \gamma a_2^2\right| \leq \begin{cases} \frac{2}{[3]_q} - \frac{4\gamma}{[2]_q^2}; & \gamma \leq 0 \\ \\ \frac{2}{[3]_q}; & 0 \leq \gamma \leq \frac{[2]_q^2}{[3]_q} \\ \\ \frac{4\gamma}{[2]_q^2} - \frac{2}{[3]_q}; & \gamma \geq \frac{[2]_q^2}{[3]_q} \end{cases} .$$

This evidently completes the proof of the above Theorem. □

By setting $\gamma = 1$ in Theorem 1 we get the following consequence.

Corollary 1. *Let* $f \in R(q)$. *Then,*

$$|H_2(1)| \le \frac{2}{[3]_q}. \tag{5}$$

Theorem 2. *Let* $f \in R(q)$. *Then for real number* β,

$$\left|B_2^{\beta}(1)\right| \le 2\max\left\{\frac{\beta}{[4]_q}, \left|\frac{2}{[2]_q[3]_q} - \frac{\beta}{[4]_q}\right|\right\}.$$

Proof. By using (4), we obtain

$$\left|B_2^{\beta}(1)\right| = |a_2a_3 - \beta a_4| = \left|\frac{c_1c_2}{[2]_q[3]_q} - \beta\frac{c_3}{[4]_q}\right|.$$

Now substituting for c_2 and c_3 using Lemma 2, we have

$$\begin{aligned}\left|B_2^{\beta}(1)\right| &= \left|\left(\frac{1}{2[2]_q[3]_q} - \frac{\beta}{4[4]_q}\right)c_1^3 + \left(\frac{c_1}{2[2]_q[3]_q} - \frac{\beta c_1}{2[4]_q}\right)\left(4-c_1^2\right)x\right.\\ &\quad \left.+\frac{\beta}{4[4]_q}c_1\left(4-c_1^2\right)x^2 - \frac{\beta}{2[4]_q}\left(4-c_1^2\right)\left(1-|x|^2\right)z\right|.\end{aligned}$$

Using the facts $|z| < 1$ and $|pa + pb| \le |p||a| + |q||b|$; $(c+a) \ge (c-a)$ where p, q, a, b are real numbers, and also letting $|c_1| = c \in [0,2]$, for $\rho = |x| \le 1$, we obtain

$$\begin{aligned}\left|B_2^{\beta}(1)\right| &\le \frac{1}{2}\left|\frac{1}{[2]_q[3]_q} - \frac{\beta}{2[4]_q}\right|c^3 + \frac{1}{2}\left|\frac{1}{[2]_q[3]_q} - \frac{\beta}{[4]_q}\right|\left(4-c^2\right)c\rho\\ &\quad +\frac{\beta}{4[4]_q}c\left(4-c^2\right)\rho^2 + \frac{\beta}{2[4]_q}\left(4-c^2\right)\left(1-\rho^2\right).\end{aligned} \tag{6}$$

First observe that if $\beta < \frac{[4]_q}{[2]_q[3]_q}$, then (6) yields

$$\begin{aligned}\left|B_2^{\beta}(1)\right| &\le \frac{1}{2}\left(\frac{1}{[2]_q[3]_q} - \frac{\beta}{2[4]_q}\right)c^3 + \frac{1}{2}\left(\frac{1}{[2]_q[3]_q} - \frac{\beta}{[4]_q}\right)\left(4-c^2\right)c\rho\\ &\quad +\frac{\beta}{4[4]_q}\left(4-c^2\right)(c-2)\rho^2 + \frac{\beta}{2[4]_q}\left(4-c^2\right) \qquad (7)\\ &= F(\rho).\end{aligned}$$

Since $\beta < \frac{[4]_q}{[2]_q[3]_q}$, the extreme point of $F(\rho)$ is $\rho = \frac{c}{2-c}\left(\frac{[4]_q}{[2]_q[3]_q\beta} - 1\right)$ (with $c \in [0,2]$ in this case since $\rho \in [0,1]$). Now let

$$G_1(c) = F(0) = \frac{1}{2}\left(\frac{1}{[2]_q[3]_q} - \frac{\beta}{2[4]_q}\right)c^3 - \frac{\beta}{2[4]_q}c^2 + \frac{2\beta}{[4]_q},$$

$$G_2(c) = F(1) = \left(\frac{2}{[2]_q[3]_q} - \frac{\beta}{[4]_q}\right)c$$

and

$$G_3(c) = F\left(\frac{c}{2-c}\left(\frac{[4]_q}{[2]_q[3]_q\beta}-1\right)\right) = \frac{[4]_q}{4[2]_q^2[3]_q^2\beta}c^3 + \left(\frac{[4]_q}{2[2]_q^2[3]_q^2\beta} - \frac{1}{[2]_q[3]_q}\right)c^2 + \frac{2\beta}{[4]_q}.$$

By elementary calculus, we find that $G_1(c) \le G_2(c) \le G_2(2) = \frac{4}{[2]_q[3]_q} - \frac{2\beta}{[4]_q}$ while $G_3(c) \le G_3(2) = \frac{4[4]_q}{[2]_q^2[3]_q^2\beta} - \frac{4}{[2]_q[3]_q} + \frac{2\beta}{[4]_q}$. Hence for $\beta < \frac{[4]_q}{[2]_q[3]_q}$ the maximum of the functional is $\frac{4}{[2]_q[3]_q} - \frac{2\beta}{[4]_q}$. That is,

$$\left|B_2^\beta(1)\right| \le \frac{4}{[2]_q[3]_q} - \frac{2\beta}{[4]_q} \quad \text{for } \beta < \frac{[4]_q}{[2]_q[3]_q}.$$

Next we consider the case $\frac{[4]_q}{[2]_q[3]_q} < \beta < \frac{2[4]_q}{[2]_q[3]_q}$. Then we have

$$\begin{aligned}\left|B_2^\beta(1)\right| &\le \frac{1}{2}\left(\frac{1}{[2]_q[3]_q} - \frac{\beta}{2[4]_q}\right)c^3 + \frac{1}{2}\left(\frac{\beta}{[4]_q} - \frac{1}{[2]_q[3]_q}\right)\left(4-c^2\right)c\rho \\ &\quad + \frac{\beta}{4[4]_q}\left(4-c^2\right)(c-2)\rho^2 + \frac{\beta}{2[4]_q}\left(4-c^2\right) \\ &= F(\rho).\end{aligned}$$

Since $\frac{[4]_q}{[2]_q[3]_q} < \beta < \frac{2[4]_q}{[2]_q[3]_q}$, the extreme point of $F(\rho)$ is $\rho = \frac{c}{2-c}\left(1 - \frac{[4]_q}{[2]_q[3]_q\beta}\right)$ (with $c \in [0,2]$ in this case since $\rho \in [0,1]$). Now let

$$G_1(c) = F(0) = \frac{1}{2}\left(\frac{1}{[2]_q[3]_q} - \frac{\beta}{2[4]_q}\right)c^3 - \frac{\beta}{2[4]_q}c^2 + \frac{2\beta}{[4]_q},$$

$$G_2(c) = F(1) = \left(\frac{1}{[2]_q[3]_q} - \frac{\beta}{[4]_q}\right)c^3 + \left(\frac{3\beta}{[4]_q} - \frac{2}{[2]_q[3]_q}\right)c$$

and

$$G_3(c) = F\left(\frac{c}{2-c}\left(1-\frac{[4]_q}{[2]_q[3]_q\beta}\right)\right) = \frac{[4]_q}{4[2]_q[3]_q\beta}c^3 + \left(\frac{[4]_q}{2[2]_q[3]_q\beta} - \frac{1}{[2]_q[3]_q}\right)c^2 + \frac{2\beta}{[4]_q}.$$

Hence for $\frac{[4]_q}{[2]_q[3]_q} < \beta < \frac{2[4]_q}{[2]_q[3]_q}$, the maximum of the functional is $\frac{2\beta}{[4]_q}$. That is,

$$\left|B_2^\beta(1)\right| \le \frac{2\beta}{[4]_q} \quad \text{for } \frac{[4]_q}{[2]_q[3]_q} < \beta < \frac{2[4]_q}{[2]_q[3]_q}.$$

Finally, if $\beta > \frac{2[4]_q}{[2]_q[3]_q}$ we write (7) as

$$\begin{aligned}\left|B_2^\beta(1)\right| &\le \left(\frac{\beta}{4[4]_q} - \frac{1}{2[2]_q[3]_q}\right)c^3 + \left(\frac{\beta}{[4]_q} - \frac{1}{[2]_q[3]_q}\right)\frac{c}{2}\left(4-c^2\right)\rho \\ &\quad + \frac{\beta(c-2)}{4[4]_q}\left(4-c^2\right)\rho^2 + \frac{\beta}{2[4]_q}\left(4-c^2\right) \\ &= F(\rho).\end{aligned}$$

Since $\beta > \frac{2[4]_q}{[2]_q[3]_q}$, the extreme point of $F(\rho)$ is $\rho = \frac{c}{c-2}\left(\frac{[4]_q}{[2]_q[3]_q\beta} - 1\right)$ (with $c \in [0,2]$ in this case since $\rho \in [0,1]$). Next, let

$$G_1(c) = F(0) = \left(\frac{\beta}{4[4]_q} - \frac{1}{2[2]_q[3]_q}\right)c^3 - \frac{\beta}{2[4]_q}c^2 + \frac{2\beta}{[4]_q},$$

$$\begin{aligned} G_2(c) &= F(1) = \left(\frac{\beta}{4[4]_q} - \frac{1}{2[2]_q[3]_q}\right)c^3 + \left(\frac{\beta}{[4]_q} - \frac{1}{[2]_q[3]_q}\right)\frac{c}{2}\left(4-c^2\right) \\ &\quad + \frac{\beta(c-2)}{4[4]_q}\left(4-c^2\right) + \frac{\beta}{2[4]_q}\left(4-c^2\right) \end{aligned}$$

and

$$\begin{aligned} G_3(c) &= F\left(\frac{c}{c-2}\left(\frac{[4]_q}{[2]_q[3]_q\beta} - 1\right)\right) \\ &= \left(\frac{[4]_q}{4[2]_q^2[3]_q^2\beta} - \frac{1}{[2]_q[3]_q} + \frac{\beta}{2[4]_q}\right)c^3 + \left(\frac{[4]_q}{2[2]_q^2[3]_q^2\beta} - \frac{1}{[2]_q[3]_q}\right)c^2 + \frac{2\beta}{[4]_q}. \end{aligned}$$

Hence for $\beta > \frac{2[4]_q}{[2]_q[3]_q}$, the maximum of the functional is $\frac{2\beta}{[4]_q} - \frac{4}{[2]_q[3]_q}$. That is,

$$\left|B_2^{\beta}(1)\right| \leq \frac{2\beta}{[4]_q} - \frac{4}{[2]_q[3]_q} \text{ for } \beta > \frac{2[4]_q}{[2]_q[3]_q}.$$

Hence we have

$$\left|B_2^{\beta}(1)\right| \leq \begin{cases} \frac{4}{[2]_q[3]_q} - \frac{2\beta}{[4]_q}; & \beta < \frac{[4]_q}{[2]_q[3]_q} \\ \frac{2\beta}{[4]_q}; & \frac{[4]_q}{[2]_q[3]_q} < \beta < \frac{2[4]_q}{[2]_q[3]_q} \\ \frac{2\beta}{[4]_q} - \frac{4}{[2]_q[3]_q}; & \beta > \frac{2[4]_q}{[2]_q[3]_q} \end{cases}.$$

This completes the proof. □

By setting $\beta = 1$ in Theorem 2 we get the following consequence.

Corollary 2. *Let $f \in R(q)$. Then,*

$$|B_2(1)| \leq \frac{2}{[4]_q}. \tag{8}$$

Theorem 3. *Let $f \in R(q)$. Then*

$$|H_2(2)| \leq \frac{4}{[3]_q^2}.$$

Proof. Following from (4), we have

$$|H_2(2)| = \left|a_2a_4 - a_3^2\right| = \left|\frac{c_1c_3}{[2]_q[4]_q} - \frac{c_2^2}{[3]_q^2}\right|. \tag{9}$$

Next, according to Lemma 2 and (9), we obtain

$$\begin{aligned}|H_2(2)| &= \frac{1}{[2]_q[4]_q}\left|\frac{1}{4}\left(1-\frac{[2]_q[4]_q}{[3]_q^2}\right)c_1^4+\left(1-\frac{[2]_q[4]_q}{[3]_q^2}\right)\frac{c_1^2\left(4-c_1^2\right)x}{2}\right.\\ &\left.+\frac{c_1\left(1-|x|^2\right)\left(4-c_1^2\right)z}{2}-\frac{c_1^2\left(4-c_1^2\right)x^2}{4}-\frac{[2]_q[4]_q}{[3]_q^2}\frac{\left(4-c_1^2\right)^2x^2}{4}\right|.\end{aligned}$$

Letting $|c_1|=c$, assuming without restriction that $c\in[0,2]$ and setting $\rho=|x|\le 1$, we have

$$\begin{aligned}|H_2(2)| &\le \frac{1}{[2]_q[4]_q}\left\{\frac{1}{4}\left(1-\frac{[2]_q[4]_q}{[3]_q^2}\right)c^4+\frac{c\left(4-c^2\right)}{2}+\left(1-\frac{[2]_q[4]_q}{[3]_q^2}\right)\frac{c^2\left(4-c^2\right)\rho}{2}\right.\\ &\left.+\frac{c(c-2)\left(4-c^2\right)\rho^2}{4}+\frac{[2]_q[4]_q}{[3]_q^2}\frac{\left(4-c^2\right)^2\rho^2}{4}\right\}\\ &= F(c,\rho).\end{aligned}$$

In the following, we obtain

$$\frac{\partial F}{\partial\rho}=\frac{1}{[2]_q[4]_q}\left\{\left(1-\frac{[2]_q[4]_q}{[3]_q^2}\right)\frac{c^2\left(4-c^2\right)}{2}+\frac{c(c-2)\left(4-c^2\right)\rho}{2}+\frac{[2]_q[4]_q}{[3]_q^2}\frac{\left(4-c^2\right)^2\rho}{2}\right\},$$

that is, $F(c,\rho)$ is increasing on $[0,1]$ so that $F(c,\rho)\le F(c,1)$. Thus we have

$$\begin{aligned}F(c,1) &= \frac{1}{[2]_q[4]_q}\left\{-\frac{1}{2}\left(1-\frac{[2]_q[4]_q}{[3]_q^2}\right)c^4+\left(3-\frac{4[2]_q[4]_q}{[3]_q^2}\right)c^2+\frac{4[2]_q[4]_q}{[3]_q^2}\right\}\\ &= G(c).\end{aligned}\tag{10}$$

Then, by elementary calculation using (10), we have

$$G'(c)=\frac{2c}{[2]_q[4]_q}\left[\left(3-\frac{4[2]_q[4]_q}{[3]_q^2}\right)-\left(1-\frac{[2]_q[4]_q}{[3]_q^2}\right)c^2\right]<0.$$

Since $G(c)\le G(0)$ for $c\in[0,2]$, we get $\max G(c)=G(0)$. □

From (5), (8), Lemma 1 and Theorem 3, we get the following Theorem.

Theorem 4. *Let $f(z)$ given by (1) be in the class $R(q)$, then*

$$|H_3(1)|\le\frac{8}{[3]_q^3}+\frac{4}{[4]_q^2}+\frac{4}{[3]_q[5]_q}.$$

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Uludag University
Department of Mathematics
16059, Bursa, Turkey
E-mail address: karahuseyinzeliha@gmail.com

Uludag University
Department of Mathematics
16059, Bursa, Turkey
E-mail address: sahsene@uludag.edu.tr

Uludag University
Department of Mathematics
16059, Bursa, Turkey
E-mail address: syalcin@uludag.edu.tr