TJMM
9 (2017), No. 1, 05-24

INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL WITH APPLICATIONS FOR SELFADJOINT OPERATORS AND QUANTUM DIVERGENCE MEASURES

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Abstract. Some inequalities for Riemann-Stieltjes integral that provide weighted reverses of Jensen's inequality are obtained. Applications for functions of selfadjoint operators and quantum f-divergence measures are also given.

1. Introduction

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $\operatorname{Sp}(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [21, p. 257]):

$$\langle f(A) x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y\rangle), \tag{1}$$

and

$$\|f(A) x\|^2 = \int_{m-0}^{M} |f(\lambda)|^2\, d\|E_\lambda x\|^2,$$

for any $x, y \in H.$

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ while } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

The following result that provides an operator version for the Jensen inequality is well known [28] (see also [20, p. 5]):

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m, M]$, then*

$$h(\langle Ax, x\rangle) \le \langle h(A) x, x\rangle \tag{2}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the following Hölder-McCarthy inequality:

Theorem 2 (Hölder-McCarthy, 1967, [25])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then for all $x \in H$ with $\|x\| = 1$,*

(i) $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ *for all* $r > 1$;

2010 *Mathematics Subject Classification.* 26D15, 47A63, 47A99.
Key words and phrases. Riemann-Stieltjes integral, Reverse Jensen's inequality, Selfadjoint bounded linear operators, Functions of operators, Trace of operators, Quantum divergence measures.

(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ *for all* $0 < r < 1$;
(iii) If A *is invertible, then* $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ *for all* $r < 0$.

The following reverse of (2) inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [20, p. 57]:

Theorem 3. *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *If* h *is a convex function on* $[m, M]$, *then*

$$\langle h(A) x, x\rangle \le \frac{M - \langle Ax, x\rangle}{M - m} h(m) + \frac{\langle Ax, x\rangle - m}{M - m} h(M) \tag{3}$$

for each $x \in H$ *with* $\|x\| = 1$.

We can state the following result concerning the weighted Jensen's inequality for continuous functions of selfadjoint operators [17]:

Theorem 4. *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $\operatorname{Sp}(A) \subseteq [a, b]$ *for some scalars* a, b *with* $a < b$. *If* $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous convex function on the interval* $[m, M]$, $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function on the interval* $[a, b]$ *and with the property that*

$$m \le f(t) \le M \text{ for any } t \in [a, b] \tag{4}$$

and $w : [a, b] \to [0, \infty)$ *is continuos on* $[a, b]$, *then*

$$\begin{aligned}
&\Phi\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) \\
&\le \frac{\langle w(A) (\Phi \circ f)(A) x, x\rangle}{\langle w(A) x, x\rangle} \\
&\le \frac{\Phi(m)\left(M - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) + \Phi(M)\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - m\right)}{M - m},
\end{aligned} \tag{5}$$

for any $x \in H$ *with* $\|x\| = 1$ *and* $\langle w(A) x, x\rangle \ne 0$.

We proved the above result by employing the following weighted inequality for the Riemann-Stieltjes integral [17]:

Theorem 5. *Let* $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *be a continuous convex function on the interval* $[m, M]$, $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *be a continuous function on the interval* $[a, b]$ *and with the property (4) and* $w : [a, b] \to [0, \infty)$ *be continuos on* $[a, b]$. *Then for each monotonic nondecreasing function* $u : [a, b] \to \mathbb{R}$ *such that* $\int_a^b w(t)\,du(t) > 0$ *we have the inequalities*

$$\begin{aligned}
&\Phi\left(\frac{\int_a^b w(t) f(t)\,du(t)}{\int_a^b w(t)\,du(t)}\right) \\
&\le \frac{\int_a^b w(t) (\Phi \circ f)(t)\,du(t)}{\int_a^b w(t)\,du(t)} \\
&\le \frac{\Phi(m)\left(M - \frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)}\right) + \Phi(M)\left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} - m\right)}{M - m}.
\end{aligned} \tag{6}$$

Following [18], we introduce the following class of complex valued functions:

Definition 1. *A function* $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{C}$ *is called square-convex on* $[m, M]$ *if the associated function* $\varphi : [m, M] \to [0, \infty)$, $\varphi(t) = |\Phi(t)|^2$ *is convex on* $[m, M]$.

A simple example of such a function is the concave power function $\Phi : [m, M] \subset [0, \infty) \to [0, \infty)$, $\Phi(t) = t^r$ with $r \in \left[\frac{1}{2}, 1\right]$. Also, if $h : [m, M] \to [0, \infty)$ is convex then the complex valued function $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{C}$ given by $\Phi(t) = h^{1/2}(t) e^{it}$ is square-convex on $[m, M]$.

Consider the function $f(t) = \ln(t+1)$. We observe that it is concave and positive on $(0, \infty)$ and if we define $\varphi(t) = [\ln(t+1)]^2$, then we have that

$$\varphi''(t) = \frac{2}{(t+1)^2}[1 - \ln(t+1)], \ t > -1,$$

showing that f is square-convex on the interval $[0, e-1]$.

Another example for trigonometric functions is for instance $f(t) = \cos t, t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. The function $\varphi(t) = \cos^2 t$ has the second derivative $\varphi''(t) = -2\cos(2t)$ which is positive for $t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. Therefore f is square-convex on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

It is obvious that if $\Phi(t) = h(t) + i\ell(t)$ and h and ℓ are square-convex on an interval, then Φ is square-convex on that interval.

However, there are square-convex functions for which the components are not. Consider, for instance, the complex valued function $\Phi(t) = (\ln t)^{1/2} + i[g(t)]^{1/2}$, defined for $t \geq 1$ and a twice differentiable nonnegative function $g : [1, \infty) \to [0, \infty)$. Then

$$\varphi(t) = |\Phi(t)|^2 = \ln t + g(t), \ t \geq 1$$

and

$$\varphi''(t) = g''(t) - \frac{1}{t^2}, \ t \geq 1.$$

If we take g such that

$$g''(t) \geq \frac{1}{t^2}, \ t \geq 1$$

then Φ is square-convex on $[1, \infty)$.

As a simple particular example, if we take $g(t) = \frac{1}{6}t^3$, then

$$\varphi''(t) = t - \frac{1}{t^2} = \frac{t^3 - 1}{t^2} \geq 0$$

showing that Φ is square-convex on $[1, \infty)$.

Theorem 6. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [a, b]$ for some scalars a, b with $a < b$. If $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{C}$ is a continuous square-convex function on the interval $[m, M]$, $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function on the interval $[a, b]$ and with the property that*

$$m \leq f(t) \leq M \text{ for any } t \in [a, b] \tag{7}$$

and $w : [a, b] \to [0, \infty)$ is continuos on $[a, b]$, then

$$\begin{aligned} &\left|\Phi\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)\right| \\ &\leq \left[\frac{\left\langle w(A)\left(|\Phi|^2 \circ f\right)(A) x, x\right\rangle}{\langle w(A) x, x\rangle}\right]^{1/2} \\ &\leq \left[\frac{|\Phi(m)|^2\left(M - \frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right) + |\Phi(M)|^2\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle} - m\right)}{M - m}\right]^{1/2}, \end{aligned} \tag{8}$$

for any $x \in H$ with $\|x\| = 1$ and $\langle w(A) x, x\rangle \neq 0$.

The proof follows from Theorem 7 applied for the function $|\Phi(\cdot)|^2$ that is continuous convex on $[m, M]$. The details are omitted.

Remark 1. *If $w(t) = 1$, then we get from (8) the following simpler result*

$$\begin{aligned} & |\Phi(\langle f(A)x, x\rangle)| \\ & \leq \|(\Phi \circ f)(A)x\| \\ & \leq \left[\frac{|\Phi(m)|^2 \langle (M1_H - f(A))x, x\rangle + |\Phi(M)|^2 \langle (f(A) - 1_H m)x, x\rangle}{M - m}\right]^{1/2}, \end{aligned} \tag{9}$$

for any $x \in H$ with $\|x\| = 1$.

2. Some Reverses for Riemann-Stieltjes Integral

It is well known that, if $u : [a, b] \to \mathbb{R}$ is continuous and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_a^b u(t)\,dv(t)$ exists and we have the inequality

$$\left|\int_a^b u(t)\,dv(t)\right| \leq \int_a^b |u(t)|\,dv(t). \tag{10}$$

We observe that for any constant $\lambda \in \mathbb{R}$ and positive continuous weight function $w : [a, b] \to [0, \infty)$ we have the *Sonin type identity* (see [35], or [27, p. 246]),

$$\begin{aligned} & \frac{\int_a^b u(t)h(t)w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} - \frac{\int_a^b u(t)w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} \frac{\int_a^b h(t)w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} \\ & = \frac{1}{\int_a^b w(t)\,dv(t)} \int_a^b [u(t) - \lambda]\left[h(t) - \frac{\int_a^b h(s)w(s)\,dv(s)}{\int_a^b w(s)\,dv(s)}\right] w(t)\,dv(t), \end{aligned} \tag{11}$$

where $u, h : [a, b] \to \mathbb{R}$ are continuous and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing.

If u is bounded below and above by the constants m, M, namely

$$m \leq u(x) \leq M \text{ for any } x \in [a, b], \tag{12}$$

then, obviously

$$\left|u(x) - \frac{m + M}{2}\right| \leq \frac{1}{2}(M - m) \text{ for any } x \in [a, b]$$

and by taking the modulus in (11) and utilizing the property (10) we get the following *Grüss' type inequality*

$$\begin{aligned} & \left|\frac{\int_a^b u(t)h(t)w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} - \frac{\int_a^b u(t)w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} \frac{\int_a^b h(t)w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)}\right| \\ & \leq \frac{1}{2}(M - m)\frac{1}{\int_a^b w(t)\,dv(t)} \\ & \times \int_a^b \left|h(t) - \frac{1}{\int_a^b w(s)\,dv(s)} \int_a^b h(s)w(s)\,dv(s)\right| w(t)\,dv(t). \end{aligned} \tag{13}$$

If we use the Cauchy-Bunyakovsky-Schwarz inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators

$$\left|\frac{\int_a^b g(t)w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)}\right| \leq \left[\frac{\int_a^b g^2(t)w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)}\right]^{1/2} \tag{14}$$

where $g, w : [a, b] \to \mathbb{R}$ are continuous, w is nonnegative on $[a, b]$ and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then we also have

$$\begin{aligned}
&\frac{1}{\int_a^b w(t)\,dv(t)} \int_a^b \left| h(t) - \frac{\int_a^b h(s)\,w(s)\,dv(s)}{\int_a^b w(s)\,dv(s)} \right| w(t)\,dv(t) \qquad (15)\\
&\leq \left[\frac{1}{\int_a^b w(t)\,dv(t)} \int_a^b \left(h(t) - \frac{\int_a^b h(s)\,w(s)\,dv(s)}{\int_a^b w(s)\,dv(s)} \right)^2 w(t)\,dv(t) \right]^{1/2} \\
&= \frac{\int_a^b h^2(t)\,w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} - \left(\frac{\int_a^b h(s)\,w(s)\,dv(s)}{\int_a^b w(s)\,dv(s)} \right)^2,
\end{aligned}$$

for $h : [a, b] \to \mathbb{R}$ continuous.

Now, if we use (13) and (15) for the function $h = u$ satisfying the condition (12) then we get the string of inequalities:

$$\begin{aligned}
&\frac{\int_a^b u^2(t)\,w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} - \left(\frac{\int_a^b u(s)\,w(s)\,dv(s)}{\int_a^b w(s)\,dv(s)} \right)^2 \qquad (16)\\
&\leq \frac{1}{2}(M - m) \\
&\times \frac{1}{\int_a^b w(t)\,dv(t)} \int_a^b \left| u(t) - \frac{\int_a^b u(s)\,w(s)\,dv(s)}{\int_a^b w(s)\,dv(s)} \right| w(t)\,dv(t) \\
&\leq \frac{1}{2}(M - m) \\
&\times \left[\frac{\int_a^b u^2(t)\,w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} - \left(\frac{\int_a^b u(s)\,w(s)\,dv(s)}{\int_a^b w(s)\,dv(s)} \right)^2 \right]^{1/2},
\end{aligned}$$

which also implies that

$$\left[\frac{\int_a^b u^2(t)\,w(t)\,dv(t)}{\int_a^b w(t)\,dv(t)} - \left(\frac{\int_a^b u(s)\,w(s)\,dv(s)}{\int_a^b w(s)\,dv(s)} \right)^2 \right]^{1/2} \leq \frac{1}{2}(M - m). \qquad (17)$$

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points $\alpha, \beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

Theorem 7. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $a, b \in \mathbb{R}$, $a < b$ with $[a, b] \subset \mathring{I}$, $\mathring{I}$ the interior of I. If $f, w : [a, b] \to \mathbb{R}$ are continuous, $w(t) \geq 0$ for any $t \in [a, b]$, $g : [a, b] \to \mathbb{R}$ is monotonic nondecreasing with $g(b) > g(a)$,*

$$m \leq f(x) \leq M \text{ for any } x \in [a, b] \qquad (18)$$

and

$$\overline{f}_{w,g} := \frac{1}{\int_a^b w(x)\,dg(x)} \int_a^b w(x)\,f(x)\,dg(x) \in [m, M]$$

then, by assuming that $\overline{f}_{w,g} \neq m, M$, *we have*

$$\frac{1}{\int_a^b w(x)\,dg(x)} \left| \int_a^b \left| \Phi(f(x)) - \Phi\left(\overline{f}_{w,g}\right) \right| sgn\left[f(x) - \overline{f}_{w,g}\right] w(x)\,dg(x) \right| \tag{19}$$
$$\leq \frac{1}{\int_a^b w(x)\,dg(x)} \int_a^b \Phi(f(x))\,w(x)\,dg(x) - \Phi\left(\overline{f}_{w,g}\right)$$
$$\leq \frac{1}{2}\left(\left[\overline{f}_{w,g}, M; \Phi\right] - \left[m, \overline{f}_{w,g}; \Phi\right]\right) D_{w,g}(f)$$
$$\leq \frac{1}{2}\left(\left[\overline{f}_{w,g}, M; \Phi\right] - \left[m, \overline{f}_{w,g}; \Phi\right]\right) D_{w,g,2}(f)$$
$$\leq \frac{1}{4}\left(\left[\overline{f}_{w,g}, M; \Phi\right] - \left[m, \overline{f}_{w,g}; \Phi\right]\right)(M - m),$$

where

$$D_{w,g}(f) = \frac{1}{\int_a^b w(y)\,dg(y)} \int_a^b \left| f(y) - \overline{f}_{w,g} \right| w(y)\,dg(y)$$

and

$$D_{w,g,2}^2(f) = \frac{1}{\int_a^b w(y)\,dg(y)} \int_a^b \left(f(y) - \overline{f}_{w,g}\right)^2 w(y)\,dg(y)$$
$$= \frac{1}{\int_a^b w(y)\,dg(y)} \int_a^b f^2(y)\,w(y)\,dg(y) - \left(\frac{1}{\int_a^b w(y)\,dg(y)} \int_a^b f(y)\,w(y)\,dg(y)\right)^2.$$

The constant $\frac{1}{2}$ *in the second inequality from (19) is best possible.*

Proof. We observe that all the Riemann-Stieltjes integrals above exist since the integrands are continuous and the integrator is monotonic nondecreasing on $[a, b]$.

Since the class of differentiable convex functions on $\mathring{I}$ is dense in the class of all continuous convex functions on $\mathring{I}$, we may assume without loosing the generality that Φ is differentiable on $\mathring{I}$.

For $\alpha \in \mathring{I}$ observe that the *divided difference* function $\Phi_\alpha : I \setminus \{\alpha\} \to \mathbb{R}$,

$$\Phi_\alpha(t) := [\alpha, t; \Phi] := \frac{\Phi(t) - \Phi(\alpha)}{t - \alpha}$$

is monotonic nondecreasing and continuous on $I \setminus \{\alpha\}$.

We can extend it by continuity in α on choosing

$$\Phi_\alpha(\alpha) := \lim_{t \to \alpha} \frac{\Phi(t) - \Phi(\alpha)}{t - \alpha} = \Phi'(\alpha).$$

For f as considered in the statement of the theorem we can assume that it is not constant, since for that case the inequality (19) is trivially satisfied.

For $\overline{f}_{w,g} \in (m, M)$, we consider now the function defined on $[a, b]$ by

$$\Phi_{\overline{f}_{w,g}}(x) := \frac{\Phi(f(x)) - \Phi\left(\overline{f}_{w,g}\right)}{f(x) - \overline{f}_{w,g}}.$$

We will show that $\Phi_{\overline{f}_{w,g}}$ and $h := f - \overline{f}_{w,g}$ are synchronous on $[a, b]$.

Let $x, y \in [a, b]$ with $f(x), f(y) \neq \overline{f}_{w,g}$. Assume that $f(x) \geq f(y)$, then

$$\begin{aligned} \Phi_{\overline{f}_{w,g}}(x) &= \frac{\Phi(f(x)) - \Phi(\overline{f}_{w,g})}{f(x) - \overline{f}_{w,g}} \\ &\geq \frac{\Phi(f(y)) - \Phi(\overline{f}_{w,g})}{f(y) - \overline{f}_{w,g}} = \Phi_{\overline{f}_{w,g}}(y) \end{aligned} \tag{20}$$

and

$$h(x) \geq h(y), \tag{21}$$

which shows that

$$\left[\Phi_{\overline{f}_{w,g}}(x) - \Phi_{\overline{f}_{w,g}}(y)\right][h(x) - h(y)] \geq 0. \tag{22}$$

If $f(x) < f(y)$, then the inequalities (20) and (21) reverse but the inequality (22) still holds true.

Utilising the continuity property of the modulus we have

$$\begin{aligned} &\left|\left[\left|\Phi_{\overline{f}_{w,g}}(x)\right| - \left|\Phi_{\overline{f}_{w,g}}(y)\right|\right][h(x) - h(y)]\right| \\ &\leq \left|\left[\Phi_{\overline{f}_{w,g}}(x) - \Phi_{\overline{f}_{w,g}}(y)\right][h(x) - h(y)]\right| \\ &= \left[\Phi_{\overline{f}_{w,g}}(x) - \Phi_{\overline{f}_{w,g}}(y)\right][h(x) - h(y)] \end{aligned}$$

for $x, y \in [a, b]$.

Multiplying with $w(x), w(y) \geq 0$ and integrating over $g(x)$ and $g(y)$ we have

$$\begin{aligned} &\left|\int_a^b \int_a^b \left[\left|\Phi_{\overline{f}_{w,g}}(x)\right| - \left|\Phi_{\overline{f}_{w,g}}(y)\right|\right][h(x) - h(y)] w(x) w(y) dg(x) dg(y)\right| \\ &\leq \int_a^b \int_a^b \left[\Phi_{\overline{f}_{w,g}}(x) - \Phi_{\overline{f}_{w,g}}(y)\right][h(x) - h(y)] w(x) w(y) dg(x) dg(y). \end{aligned} \tag{23}$$

A simple calculation shows that

$$\begin{aligned} &\frac{1}{2}\int_a^b \int_a^b \left[\left|\Phi_{\overline{f}_{w,g}}(x)\right| - \left|\Phi_{\overline{f}_{w,g}}(y)\right|\right] \\ &\times [h(x) - h(y)] w(x) w(y) dg(x) dg(y) \\ &= \int_a^b \left|\Phi_{\overline{f}_{w,g}}(x)\right| h(x) w(x) dg(x) \\ &- \int_a^b \left|\Phi_{\overline{f}_{w,g}}(x)\right| w(x) dg(x) \int_a^b w(x) h(x) dg(x) \\ &= \int_a^b \left|\frac{\Phi(f(x)) - \Phi(\overline{f}_{w,g})}{f(x) - \overline{f}_{w,g}}\right| \left[f(x) - \overline{f}_{w,g}\right] w(x) dg(x) \\ &= \int_a^b \left|\Phi(f(x)) - \Phi(\overline{f}_{w,g})\right| sgn\left[f(x) - \overline{f}_{w,g}\right] w(x) dg(x) \end{aligned} \tag{24}$$

and

$$\frac{1}{2}\int_a^b\int_a^b\left[\Phi_{\overline{f}_{w,g}}(x)-\Phi_{\overline{f}_{w,g}}(y)\right] \tag{25}$$
$$\times\left[h(x)-h(y)\right]w(x)w(y)\,dg(x)\,dg(y)$$
$$=\int_a^b\Phi_{\overline{f}_{w,g}}(x)\,h(x)\,w(x)\,dg(x)$$
$$-\int_a^b\Phi_{\overline{f}_{w,g}}(x)\,w(x)\,dg(x)\int_a^b h(x)\,w(x)\,dg(x)$$
$$=\int_a^b\frac{\Phi(f(x))-\Phi\left(\overline{f}_{w,g}\right)}{f(x)-\overline{f}_{w,g}}\left[f(x)-\overline{f}_{w,g}\right]w(x)\,dg(x)$$
$$=\int_a^b\left[\Phi(f(x))-\Phi\left(\overline{f}_{w,g}\right)\right]w(x)\,dg(x)$$
$$=\int_a^b\Phi(f(x))\,w(x)\,dg(x)-\Phi\left(\overline{f}_{w,g}\right)\int_a^b w(x)\,dg(x).$$

On making use of the identities (24) and (25) we obtain from (23) the first inequality in (19).

Now, since f satisfies the condition (18) then we have that

$$\left[m,\overline{f}_{w,g};\Phi\right]=\frac{\Phi\left(\overline{f}_{w,g}\right)-\Phi(m)}{\overline{f}_{w,g}-m}\leq\Phi_{\overline{f}_{w,g}}(x) \tag{26}$$
$$\leq\frac{\Phi(M)-\Phi\left(\overline{f}_{w,g}\right)}{M-\overline{f}_{w,g}}=\left[\overline{f}_{w,g},M;\Phi\right]$$

for $x\in[a,b]$.

Applying now the Grüss' type inequality (13) and taking into account the second part of the equality in (24) we have that

$$\int_a^b\Phi(f(x))\,w(x)\,dg(x)-\Phi\left(\overline{f}_{w,g}\right)\int_a^b w(x)\,dg(x)$$
$$\leq\frac{1}{2}\left(\left[\overline{f}_{w,g},M;\Phi\right]-\left[m,\overline{f}_{w,g};\Phi\right]\right)\int_a^b w(x)\left|f(x)-\overline{f}_{w,g}\right|dg(x),$$

which proves the second inequality in (19).

The other two bounds are obvious from the comments at the beginning of the section.

It is obvious that from (19) we get the following reverse of the first *Hermite-Hadamard inequality* for the convex function $\Phi:[a,b]\to\mathbb{R}$

$$\frac{1}{b-a}\int_a^b\Phi(t)\,dt-\Phi\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left(\left[\frac{a+b}{2},b;\Phi\right]-\left[a,\frac{a+b}{2};\Phi\right]\right)D_w(e) \tag{27}$$

where $e(t)=t, t\in[a,b]$.

Since a simple calculation shows that

$$\frac{1}{2}\left(\left[\frac{a+b}{2},b;\Phi\right]-\left[a,\frac{a+b}{2};\Phi\right]\right)=\frac{2}{b-a}\left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]$$

and

$$D_w(e)=\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{1}{4}(b-a),$$

and we get from (27) that

$$0 \leq \frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right]. \tag{28}$$

To prove the sharpness of the constant $\frac{1}{2}$ in the second inequality from (19) we need now only to show that the equality case in (28) is realized.

If we take, for instance $\Phi(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$, then we observe that Φ is convex and we get in both sides of (28) the same quantity $\frac{1}{4}(b-a)$. □

Corollary 1. *With the assumptions in Theorem 7 and if the lateral derivatives* $\Phi'_+(m)$ *and* $\Phi'_-(M)$ *are finite, then we have the inequalities*

$$\begin{aligned}
0 &\leq \frac{1}{\int_a^b w(x)\,dg(x)}\int_a^b \Phi(f(x))\,w(x)\,dg(x) - \Phi\left(\overline{f}_{w,g}\right) \quad (29)\\
&\leq \frac{1}{2}\left(\left[\overline{f}_{w,g}, M;\Phi\right] - \left[m,\overline{f}_{w,g};\Phi\right]\right) D_{w,g}(f)\\
&\leq \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] D_{w,g}(f) \leq \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] D_{w,g,2}(f)\\
&\leq \frac{1}{4}\left[\Phi'_-(M) - \Phi'_+(m)\right](M-m).
\end{aligned}$$

The constant $\frac{1}{2}$ *in the second and third inequality from (29) is best possible.*

Proof. We need to prove only the third inequality.

By the convexity of Φ we have the gradient inequalities

$$\frac{\Phi(M) - \Phi\left(\overline{f}_{w,g}\right)}{M - \overline{f}_{w,g}} \leq \Phi'_-(M)$$

and

$$\frac{\Phi\left(\overline{f}_{w,g}\right) - \Phi(m)}{\overline{f}_{w,g} - m} \geq \Phi'_+(m).$$

These imply that

$$\left[\overline{f}_{w,g}, M;\Phi\right] - \left[m,\overline{f}_{w,g};\Phi\right] \leq \Phi'_-(M) - \Phi'_+(m)$$

and the proof is concluded.

We observe that from (29) we get the following reverse of the Hermite-Hadamard inequality for the convex function $\Phi : [a,b] \to \mathbb{R}$ having finite lateral derivative $\Phi'_+(a)$ and $\Phi'_-(b)$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right) \quad (30)\\
&\leq \frac{1}{2}\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right] \leq \frac{1}{8}\left[\Phi'_-(b) - \Phi'_+(a)\right](b-a).
\end{aligned}$$

We observe that the convex function $\Phi(t) = \left|t - \frac{a+b}{2}\right|$ has finite lateral derivatives

$$\Phi'_-(b) = 1 \text{ and } \Phi'_+(a) = -1$$

and replacing this function in (30) we get in all terms the same quantity $\frac{1}{4}(b-a)$.

This proves that the constant $\frac{1}{2}$ in the second and third inequality from (29) is best possible. □

3. Inequalities for Selfadjoint Operators

We have the following Grüss' type weighted inequality:

Theorem 8. *Let A be a selfadjoint operator on the Hilbert space H and assume that* $\operatorname{Sp}(A) \subseteq [a,b]$ *for some scalars a, b with $a < b$. Assume that $u, h : \mathbb{R} \to \mathbb{R}$ are continuous function on the interval $[a,b]$ and with the property that*

$$m \leq u(t) \leq M,\ n \leq h(t) \leq N \text{ for any } t \in [a,b] \tag{31}$$

for some real numbers m, M, n, N and $w : [a,b] \to [0,\infty)$ is continuos on $[a,b]$, then

$$\begin{aligned} &\left| \frac{\langle u(A) h(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \frac{\langle u(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} \frac{\langle h(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} \right| \\ &\leq \frac{1}{2}(M-m) \frac{1}{\langle w(A) x, x\rangle} \left\langle \left| h(A) - \frac{\langle h(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} 1_H \right| w(A) x, x \right\rangle \\ &\leq \frac{1}{2}(M-m) \left[\frac{\langle h^2(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle h(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} \right)^2 \right]^{1/2} \\ &\leq \frac{1}{4}(M-m)(N-n) \end{aligned} \tag{32}$$

for any $x \in H$ with $\|x\| = 1$ and $\langle w(A) x, x\rangle \neq 0$.

Proof. Let $\{E_\lambda\}_\lambda$ be the spectral family of the selfadjoint operator A and $x \in H$ with $\|x\| = 1$ and $\langle w(A) x, x\rangle \neq 0$. Assume that $\varepsilon > 0$. If we use the inequalities (13) and (16) for the monotonic function $v(t) = \langle E_t x, x\rangle$ on the interval $[a-\varepsilon, b]$ we have

$$\begin{aligned} &\left| \frac{\int_{a-\varepsilon}^b u(t) h(t) w(t) d\langle E_t x, x\rangle}{\int_{a-\varepsilon}^b w(t) d\langle E_t x, x\rangle} \right. \\ &\left. - \frac{\int_{a-\varepsilon}^b u(t) w(t) d\langle E_t x, x\rangle}{\int_{a-\varepsilon}^b w(t) d\langle E_t x, x\rangle} \frac{\int_{a-\varepsilon}^b h(t) w(t) d\langle E_t x, x\rangle}{\int_{a-\varepsilon}^b w(t) d\langle E_t x, x\rangle} \right| \\ &\leq \frac{1}{2}(M-m) \frac{1}{\int_{a-\varepsilon}^b w(t) d\langle E_t x, x\rangle} \\ &\times \int_{a-\varepsilon}^b \left| h(t) - \frac{1}{\int_{a-\varepsilon}^b w(s) d\langle E_s x, x\rangle} \int_{a-\varepsilon}^b h(s) w(s) d\langle E_s x, x\rangle \right| \\ &\times w(t) dv\langle E_t x, x\rangle \\ &\leq \frac{1}{2}(M-m) \\ &\times \left[\frac{\int_{a-\varepsilon}^b h^2(t) w(t) d\langle E_t x, x\rangle}{\int_{a-\varepsilon}^b w(t) d\langle E_t x, x\rangle} - \left(\frac{\int_{a-\varepsilon}^b h(t) w(t) d\langle E_t x, x\rangle}{\int_{a-\varepsilon}^b w(t) d\langle E_t x, x\rangle} \right)^2 \right]^{1/2} \\ &\leq \frac{1}{4}(M-m)(N-n). \end{aligned} \tag{33}$$

Letting $\varepsilon \to 0+$ in (33) and utilizing the spectral representation theorem (1) we deduce the desired result (32). □

Remark 2. *The unweighted case, namely when $w(t) = 1$ is as follows:*

$$
\begin{aligned}
&|\langle u(A)\, h(A)\, x, x\rangle - \langle u(A)\, x, x\rangle \langle h(A)\, x, x\rangle| \qquad (34)\\
&\le \frac{1}{2}(M-m)\,\langle |h(A) - \langle h(A)\, x, x\rangle 1_H|\, x, x\rangle\\
&\le \frac{1}{2}(M-m)\left[\left\langle h^2(A)\, x, x\right\rangle - \langle h(A)\, x, x\rangle^2\right]^{1/2}\\
&\le \frac{1}{4}(M-m)(N-n)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

Example 1. *1) Let A be a selfadjoint operator on the Hilbert space H and assume that* $\mathrm{Sp}(A) \subseteq [\gamma, \Gamma]$ *for some scalars γ, Γ with $\gamma < \Gamma$. If $\gamma > 0$, then for any $p, q, r > 0$ we have the power inequalities*

$$
\begin{aligned}
&\left|\frac{\langle A^{p+q+r}x, x\rangle}{\langle A^r x, x\rangle} - \frac{\langle A^{p+r}x, x\rangle}{\langle A^r x, x\rangle}\frac{\langle A^{q+r}x, x\rangle}{\langle A^r x, x\rangle}\right| \qquad (35)\\
&\le \frac{1}{2}(\Gamma^p - \gamma^p)\frac{1}{\langle A^r x, x\rangle}\left\langle \left|A^q - \frac{\langle A^{q+r}x, x\rangle}{\langle A^r x, x\rangle}1_H\right| A^r x, x\right\rangle\\
&\le \frac{1}{2}(\Gamma^p - \gamma^p)\left[\frac{\langle A^{2q+r}x, x\rangle}{\langle A^r x, x\rangle} - \left(\frac{\langle A^{q+r}x, x\rangle}{\langle A^r x, x\rangle}\right)^2\right]^{1/2}\\
&\le \frac{1}{4}(\Gamma^p - \gamma^p)(\Gamma^q - \gamma^q)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

2) With the assumptions of 1) we have

$$
\begin{aligned}
&\left|\frac{\langle A^{q+r}\ln Ax, x\rangle}{\langle A^r x, x\rangle} - \frac{\langle A^{q}\ln Ax, x\rangle}{\langle A^r x, x\rangle}\frac{\langle A^{q+r}x, x\rangle}{\langle A^r x, x\rangle}\right| \qquad (36)\\
&\le \frac{1}{\langle A^r x, x\rangle}\left\langle \left|A^q - \frac{\langle A^{q+r}x, x\rangle}{\langle A^r x, x\rangle}1_H\right| A^r x, x\right\rangle \ln\left(\sqrt{\frac{\Gamma}{\gamma}}\right)\\
&\le \left[\frac{\langle A^{2q+r}x, x\rangle}{\langle A^r x, x\rangle} - \left(\frac{\langle A^{q+r}x, x\rangle}{\langle A^r x, x\rangle}\right)^2\right]^{1/2} \ln\left(\sqrt{\frac{\Gamma}{\gamma}}\right)\\
&\le \frac{1}{2}(\Gamma^q - \gamma^q)\ln\left(\sqrt{\frac{\Gamma}{\gamma}}\right)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

3) With the assumptions of 1) we have

$$
\begin{aligned}
&0 \le \frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle^2} - 1 \qquad (37)\\
&\le \frac{\Gamma - \gamma}{2\Gamma\gamma}\frac{1}{\langle Ax, x\rangle}\left\langle \left|A - \frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle}1_H\right| Ax, x\right\rangle\\
&\le \frac{\Gamma - \gamma}{2\Gamma\gamma}\left[\frac{\left\langle A^3 x, x\right\rangle}{\langle Ax, x\rangle} - \left(\frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle}\right)^2\right]^{1/2} \le \frac{(\Gamma - \gamma)^2}{4\Gamma\gamma}
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

4) With the assumptions of 1) we have

$$\left|\frac{\langle Ax,x\rangle}{\langle A^{-1}x,x\rangle}-\frac{1}{\langle A^{-1}x,x\rangle^{2}}\right| \tag{38}$$
$$\leq\frac{1}{2}\left(\Gamma-\gamma\right)\frac{1}{\langle A^{-1}x,x\rangle}\left\langle\left|A-\frac{1}{\langle A^{-1}x,x\rangle}1_{H}\right|A^{-1}x,x\right\rangle$$
$$\leq\frac{1}{2}\left(\Gamma-\gamma\right)\left[\frac{\langle Ax,x\rangle}{\langle A^{-1}x,x\rangle}-\frac{1}{\langle A^{-1}x,x\rangle^{2}}\right]^{1/2}$$
$$\leq\frac{1}{4}\left(\Gamma-\gamma\right)^{2}$$

for any $x\in H$ with $\|x\|=1$.

We have:

Theorem 9. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[a,b]$ for some scalars a,b with $a<b$. If $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a continuous convex function on the interval $[m,M]$, $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is a continuous function on the interval $[a,b]$ and with the property that*

$$m\leq f(t)\leq M \text{ for any } t\in[a,b] \tag{39}$$

and $w:[a,b]\to[0,\infty)$ is continuos on $[a,b]$, then

$$0\leq\frac{\langle w(A)(\Phi\circ f)(A)x,x\rangle}{\langle w(A)x,x\rangle}-\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right) \tag{40}$$
$$\leq\frac{1}{2}\left(\left[\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle},M;\Phi\right]-\left[m,\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};\Phi\right]\right)$$
$$\times\frac{1}{\langle w(A)x,x\rangle}\left\langle\left|f(A)-\frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}1_{H}\right|w(A)x,x\right\rangle$$
$$\leq\frac{1}{2}\left(\left[\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle},M;\Phi\right]-\left[m,\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};\Phi\right]\right)$$
$$\times\left[\frac{\langle f^{2}(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}-\left(\frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^{2}\right]^{1/2}$$
$$\leq\frac{1}{4}\left(\left[\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle},M;\Phi\right]-\left[m,\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};\Phi\right]\right)(M-m)$$

for any $x\in H$ with $\|x\|=1$ and $\langle w(A)x,x\rangle\neq0$.

The proof is similar to the one from Theorem 8 on utilizing the spectral representation theorem and the integral inequality for the Riemann-Stieltjes from Theorem 7 and the details are omitted.

Corollary 2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[a,b]$ for some scalars a,b with $a<b$. If $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{C}$ is a continuous square-convex function on the interval $[m,M]$, $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is a continuous function on the interval $[a,b]$ with the property that (39) is true and $w:[a,b]\to[0,\infty)$ is continuos*

on $[a,b]$, then

$$
\begin{aligned}
0 &\leq \frac{\left\langle w(A)\left|\Phi(f(A))\right|^2 x,x\right\rangle}{\langle w(A)x,x\rangle} - \left|\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\right|^2 \qquad (41)\\
&\leq \frac{1}{2}\left(\left[\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}, M; |\Phi|^2\right] - \left[m, \frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}; |\Phi|^2\right]\right)\\
&\times \frac{1}{\langle w(A)x,x\rangle}\left\langle \left| f(A) - \frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}1_H\right| w(A)x,x\right\rangle\\
&\leq \frac{1}{2}\left(\left[\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}, M; |\Phi|^2\right] - \left[m, \frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}; |\Phi|^2\right]\right)\\
&\times \left[\frac{\left\langle f^2(A)w(A)x,x\right\rangle}{\langle w(A)x,x\rangle} - \left(\frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^2\right]^{1/2}\\
&\leq \frac{1}{4}\left(\left[\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}, M; |\Phi|^2\right] - \left[m, \frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}; |\Phi|^2\right]\right)\\
&\times (M-m)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$ and $\langle w(A)x,x\rangle \neq 0$.

Remark 3. *If we take $f(t) = t$ in (40) then we get*

$$
\begin{aligned}
0 &\leq \frac{\langle w(A)\Phi(A)x,x\rangle}{\langle w(A)x,x\rangle} - \Phi\left(\frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}\right) \qquad (42)\\
&\leq \frac{1}{2}\left(\left[\frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}, M; \Phi\right] - \left[m, \frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}; \Phi\right]\right)\\
&\times \frac{1}{\langle w(A)x,x\rangle}\left\langle \left| A - \frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}1_H\right| w(A)x,x\right\rangle\\
&\leq \frac{1}{2}\left(\left[\frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}, M; \Phi\right] - \left[m, \frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}; \Phi\right]\right)\\
&\times \left[\frac{\left\langle w(A)A^2x,x\right\rangle}{\langle w(A)x,x\rangle} - \left(\frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}\right)^2\right]^{1/2}\\
&\leq \frac{1}{4}\left(\left[\frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}, M; \Phi\right] - \left[m, \frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle}; \Phi\right]\right)\\
&\times (M-m)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$ and $\langle w(A)x,x\rangle \neq 0$, $\frac{\langle w(A)Ax,x\rangle}{\langle w(A)x,x\rangle} \neq m, M$.

In particular, by taking $w(t) = 1$ in (42) we get the simple reverse of Jensen's inequality:

$$
\begin{aligned}
0 &\leq \langle \Phi(A)x,x\rangle - \Phi(\langle Ax,x\rangle) \qquad (43)\\
&\leq \frac{1}{2}\left([\langle Ax,x\rangle, M; \Phi] - [m, \langle Ax,x\rangle; \Phi]\right)\langle |A - \langle Ax,x\rangle 1_H|\, x,x\rangle\\
&\leq \frac{1}{2}\left([\langle Ax,x\rangle, M; \Phi] - [m, \langle Ax,x\rangle; \Phi]\right)\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]^{1/2}\\
&\leq \frac{1}{4}\left([\langle Ax,x\rangle, M; \Phi] - [m, \langle Ax,x\rangle; \Phi]\right)(M-m)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$ and $\langle Ax,x\rangle \neq m, M$.

Similar comments can be made for square-convex functions. However the details are omitted.

4. Applications for Quantum f-Divergence

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{44}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{45}$$

showing that the definition (44) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{46}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

In what follows we assume that $(H, \langle \cdot, \cdot \rangle)$ is a *finite dimensional* complex Hilbert space. On complex Hilbert space $(\mathcal{B}_2(H), \langle \cdot, \cdot \rangle_2)$, where the Hilbert-Schmidt inner product is defined by

$$\langle U, V \rangle_2 := \operatorname{tr}(V^* U), \; U, \; V \in \mathcal{B}_2(H),$$

for $A, \, B \in \mathcal{B}^+(H)$ consider the operators $\mathfrak{L}_A : \mathcal{B}_2(H) \to \mathcal{B}_2(H)$ and $\mathfrak{R}_B : \mathcal{B}_2(H) \to \mathcal{B}_2(H)$ defined by

$$\mathfrak{L}_A T := AT \text{ and } \mathfrak{R}_B T := TB.$$

We observe that they are well defined and since

$$\langle \mathfrak{L}_A T, T \rangle_2 = \langle AT, T \rangle_2 = \operatorname{tr}(T^* AT) = \operatorname{tr}\left(|T^*|^2 A \right) \geq 0$$

and

$$\langle \mathfrak{R}_B T, T \rangle_2 = \langle TB, T \rangle_2 = \operatorname{tr}(T^* TB) = \operatorname{tr}\left(|T|^2 B \right) \geq 0$$

for any $T \in \mathcal{B}_2(H)$, they are also positive in the operator order of $\mathcal{B}(\mathcal{B}_2(H))$, the Banach algebra of all bounded operators on $\mathcal{B}_2(H)$ with the norm $\|\cdot\|_2$ where $\|T\|_2 = \operatorname{tr}\left(|T|^2 \right)$, $T \in \mathcal{B}_2(H)$.

Since $\operatorname{tr}\left(|X^*|^2 \right) = \operatorname{tr}\left(|X|^2 \right)$ for any $X \in \mathcal{B}_2(H)$, then also

$$\begin{aligned} \operatorname{tr}(T^* AT) &= \operatorname{tr}\left(T^* A^{1/2} A^{1/2} T \right) = \operatorname{tr}\left(\left(A^{1/2} T \right)^* A^{1/2} T \right) \\ &= \operatorname{tr}\left(\left| A^{1/2} T \right|^2 \right) = \operatorname{tr}\left(\left| \left(A^{1/2} T \right)^* \right|^2 \right) = \operatorname{tr}\left(\left| T^* A^{1/2} \right|^2 \right) \end{aligned}$$

for $A \geq 0$ and $T \in \mathcal{B}_2(H)$.

We observe that $\mathfrak{L}_A$ and $\mathfrak{R}_B$ are commutative, therefore the product $\mathfrak{L}_A \mathfrak{R}_B$ is a self-adjoint positive operator in $\mathcal{B}(\mathcal{B}_2(H))$ for any positive operators $A, B \in \mathcal{B}(H)$.

For $A, B \in \mathcal{B}^+(H)$ with B invertible, we define the *Araki transform* $\mathfrak{A}_{A,B} : \mathcal{B}_2(H) \to \mathcal{B}_2(H)$ by $\mathfrak{A}_{A,B} := \mathfrak{L}_A \mathfrak{R}_{B^{-1}}$. We observe that for $T \in \mathcal{B}_2(H)$ we have $\mathfrak{A}_{A,B}T = ATB^{-1}$ and

$$\langle \mathfrak{A}_{A,B}T, T\rangle_2 = \langle ATB^{-1}, T\rangle_2 = \operatorname{tr}\left(T^* ATB^{-1}\right).$$

Observe also, by the properties of trace, that

$$\begin{aligned}\operatorname{tr}\left(T^* ATB^{-1}\right) &= \operatorname{tr}\left(B^{-1/2}T^* A^{1/2}A^{1/2}TB^{-1/2}\right)\\ &= \operatorname{tr}\left(\left(A^{1/2}TB^{-1/2}\right)^*\left(A^{1/2}TB^{-1/2}\right)\right) = \operatorname{tr}\left(\left|A^{1/2}TB^{-1/2}\right|^2\right)\end{aligned}$$

giving that

$$\langle \mathfrak{A}_{A,B}T, T\rangle_2 = \operatorname{tr}\left(\left|A^{1/2}TB^{-1/2}\right|^2\right) \geq 0 \tag{47}$$

for any $T \in \mathcal{B}_2(H)$.

We observe that, by the definition of operator order and by (47) we have $r1_{\mathcal{B}_2(H)} \leq \mathfrak{A}_{A,B} \leq R1_{\mathcal{B}_2(H)}$ for some $R \geq r \geq 0$ if and only if

$$r\operatorname{tr}\left(|T|^2\right) \leq \operatorname{tr}\left(\left|A^{1/2}TB^{-1/2}\right|^2\right) \leq R\operatorname{tr}\left(|T|^2\right) \tag{48}$$

for any $T \in \mathcal{B}_2(H)$.

We also notice that a sufficient condition for (48) to hold is that the following inequality in the operator order of $\mathcal{B}(H)$ is satisfied

$$r|T|^2 \leq \left|A^{1/2}TB^{-1/2}\right|^2 \leq R|T|^2 \tag{49}$$

for any $T \in \mathcal{B}_2(H)$.

Let U be a selfadjoint linear operator on a complex Hilbert space $(K; \langle\cdot,\cdot\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(\operatorname{Sp}(U))$ of all *continuous functions* defined on the *spectrum* of U, denoted $\operatorname{Sp}(U)$, and the C^*-algebra $C^*(U)$ generated by U and the identity operator 1_K on K as follows:

For any $f, g \in C(\operatorname{Sp}(U))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi\left(\bar{f}\right) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in\operatorname{Sp}(U)}|f(t)|$;

(iv) $\Phi(f_0) = 1_K$ and $\Phi(f_1) = U$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \operatorname{Sp}(U)$.

With this notation we define

$$f(U) := \Phi(f) \quad \text{for all } f \in C(\operatorname{Sp}(U))$$

and we call it the *continuous functional calculus* for a selfadjoint operator U.

If U is a selfadjoint operator and f is a real valued continuous function on $\operatorname{Sp}(U)$, then $f(t) \geq 0$ for any $t \in \operatorname{Sp}(U)$ implies that $f(U) \geq 0$, i.e. $f(U)$ is a positive operator on K. Moreover, if both f and g are real valued functions on $\operatorname{Sp}(U)$ then the following important property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in \operatorname{Sp}(U) \quad \text{implies that} \quad f(U) \geq g(U) \tag{P}$$

in the operator order of $B(K)$.

Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function. Utilising the continuous functional calculus for the Araki selfadjoint operator $\mathfrak{A}_{Q,P} \in \mathcal{B}(\mathcal{B}_2(H))$ we can define the *quantum f-divergence* for $Q, P \in S_1(H) := \{P \in \mathcal{B}_1(H),\ P \geq 0 \text{ with } \operatorname{tr}(P) = 1\}$ and P

invertible, by

$$S_f(Q,P) := \left\langle f(\mathfrak{A}_{Q,P}) P^{1/2}, P^{1/2} \right\rangle_2 = \operatorname{tr}\left(P^{1/2} f(\mathfrak{A}_{Q,P}) P^{1/2} \right).$$

If we consider the continuous convex function $f : [0,\infty) \to \mathbb{R}$, with $f(0) := 0$ and $f(t) = t\ln t$ for $t > 0$ then for $Q, P \in S_1(H)$ and Q, P invertible we have

$$S_f(Q,P) = \operatorname{tr}[Q(\ln Q - \ln P)] =: U(Q,P),$$

which is the *Umegaki relative entropy*.

If we take the continuous convex function $f : [0,\infty) \to \mathbb{R}$, $f(t) = |t-1|$ for $t \geq 0$ then for $Q, P \in S_1(H)$ with P invertible we have

$$S_f(Q,P) = \operatorname{tr}(|Q-P|) =: V(Q,P),$$

where $V(Q,P)$ is the *variational distance*.

If we take $f : [0,\infty) \to \mathbb{R}$, $f(t) = t^2 - 1$ for $t \geq 0$ then for $Q, P \in S_1(H)$ with P invertible we have

$$S_f(Q,P) = \operatorname{tr}\left(Q^2 P^{-1}\right) - 1 =: \chi^2(Q,P),$$

which is called the χ^2-*distance*

Let $q \in (0,1)$ and define the convex function $f_q : [0,\infty) \to \mathbb{R}$ by $f_q(t) = \frac{1-t^q}{1-q}$. Then

$$S_{f_q}(Q,P) = \frac{1 - \operatorname{tr}\left(Q^q P^{1-q}\right)}{1-q},$$

which is *Tsallis relative entropy*.

If we consider the convex function $f : [0,\infty) \to \mathbb{R}$ by $f(t) = \frac{1}{2}\left(\sqrt{t}-1\right)^2$, then

$$S_f(Q,P) = 1 - \operatorname{tr}\left(Q^{1/2} P^{1/2}\right) =: h^2(Q,P),$$

which is known as *Hellinger discrimination*.

If we take $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\ln t$ then for $Q, P \in S_1(H)$ and Q, P invertible we have

$$S_f(Q,P) = \operatorname{tr}[P(\ln P - \ln Q)] = U(P,Q).$$

In the important case of finite dimensional space H and the generalized inverse P^{-1}, numerous properties of the quantum f-divergence have been obtained in the recent papers [22], [23], [32], [33] and in the references therein.

Theorem 10. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous convex function that is normalized, namely $f(0) = 1$. If $Q, P \in S_1(H)$, with P invertible, and there exists $R \geq 1 \geq r \geq 0$ such that*

$$r \operatorname{tr}\left(|T|^2\right) \leq \operatorname{tr}\left(\left|Q^{1/2} T P^{-1/2}\right|^2\right) \leq R \operatorname{tr}\left(|T|^2\right) \tag{50}$$

for any $T \in \mathcal{B}_2(H)$, then

$$\begin{aligned}(0 \leq) S_f(Q,P) &\leq \frac{(1-r) f(R) + (R-1) f(r)}{2(R-1)(1-r)} V(Q,P) \qquad (51)\\ &\leq \frac{(1-r) f(R) + (R-1) f(r)}{2(R-1)(1-r)} \chi(Q,P)\\ &\leq \frac{(1-r) f(R) + (R-1) f(r)}{2(R-1)(1-r)} (R-r).\end{aligned}$$

Proof. From the inequality (43) applied for the convex function f and the selfadjoint operator $\mathfrak{A}_{Q,P}$ we have

$$\begin{aligned} 0 &\leq \langle f(\mathfrak{A}_{Q,P})T,T\rangle_2 - f\left(\langle \mathfrak{A}_{Q,P}T,T\rangle_2\right) \\ &\leq \frac{1}{2}\left(\left[\langle \mathfrak{A}_{Q,P}T,T\rangle_2, R; f\right] - \left[r, \langle \mathfrak{A}_{Q,P}T,T\rangle_2; f\right]\right) \\ &\times \left\langle \left|\mathfrak{A}_{Q,P} - \langle \mathfrak{A}_{Q,P}T,T\rangle_2 1_{\mathcal{B}_2(H)}\right| T,T\right\rangle_2 \\ &\leq \frac{1}{2}\left(\left[\langle \mathfrak{A}_{Q,P}T,T\rangle_2, R; f\right] - \left[r, \langle \mathfrak{A}_{Q,P}T,T\rangle_2; f\right]\right) \\ &\times \left[\left\langle \mathfrak{A}_{Q,P}^2 T,T\right\rangle_2 - \langle \mathfrak{A}_{Q,P}T,T\rangle_2^2\right]^{1/2} \\ &\leq \frac{1}{4}\left(\left[\langle \mathfrak{A}_{Q,P}T,T\rangle_2, R; f\right] - \left[r, \langle \mathfrak{A}_{Q,P}T,T\rangle_2; f\right]\right)(R-r) \end{aligned} \tag{52}$$

for any $T \in \mathcal{B}_2(H)$.

If we chose in (52) $T = P^{1/2}$ and take into account that

$$\left\langle \mathfrak{A}_{Q,P}P^{1/2}, P^{1/2}\right\rangle_2 = 1,$$

$$\left\langle \left|\mathfrak{A}_{Q,P} - \left\langle \mathfrak{A}_{Q,P}P^{1/2}, P^{1/2}\right\rangle_2 1_{\mathcal{B}_2(H)}\right| P^{1/2}, P^{1/2}\right\rangle_2 = V(Q,P)$$

and

$$\left\langle \mathfrak{A}_{Q,P}^2 T,T\right\rangle_2 - \langle \mathfrak{A}_{Q,P}T,T\rangle_2^2 = \chi^2(Q,P),$$

then by (52) we obtain

$$\begin{aligned} (0 \leq)\, S_f(Q,P) &\leq \frac{1}{2}\left([1,R;f] - [r,1;f]\right)V(Q,P) \\ &\leq \frac{1}{2}\left([1,R;f] - [r,1;f]\right)\chi(Q,P) \\ &\leq \frac{1}{4}\left([1,R;f] - [r,1;f]\right)(R-r), \end{aligned}$$

which is equivalent to the desired result (50). □

Example 2. *1) If we take in (51) $f(t) = t^2 - 1$, then we get*

$$\begin{aligned} (0 \leq)\, \chi^2(Q,P) &\leq \frac{1}{2}(R-r)V(Q,P) \leq \frac{1}{2}(R-r)\chi(Q,P) \\ &\leq \frac{1}{4}(R-r)^2 \end{aligned} \tag{53}$$

for $Q,P \in S_1(H)$, with P invertible and satisfying the condition (50).

2) If we take in (51) $f(t) = t\ln t$, then we get

$$\begin{aligned} (0 \leq)\, U(Q,P) &\leq \frac{1}{2}\ln\left(R^{\frac{R}{R-1}}r^{\frac{r}{1-r}}\right)V(Q,P) \\ &\leq \frac{1}{2}\ln\left(R^{\frac{R}{R-1}}r^{\frac{r}{1-r}}\right)\chi(Q,P) \\ &\leq \frac{1}{4}(R-r)\ln\left(R^{\frac{R}{R-1}}r^{\frac{r}{1-r}}\right) \end{aligned} \tag{54}$$

for $Q,P \in S_1(H)$, with P invertible and satisfying the condition (50).

3) If we take in (51) $f(t)=-\ln t$, then we get

$$
\begin{aligned}
(0\leq)\,U(P,Q) &\leq \frac{1}{2}\ln\left(R^{\frac{1}{1-R}}r^{\frac{1}{r-1}}\right)V(Q,P)\\
&\leq \frac{1}{2}\ln\left(R^{\frac{1}{1-R}}r^{\frac{1}{r-1}}\right)\chi(Q,P)\\
&\leq \frac{1}{4}(R-r)\ln\left(R^{\frac{R}{R-1}}r^{\frac{r}{1-r}}\right)
\end{aligned}
$$

for $Q,P\in S_1(H)$, with P invertible and satisfying the condition (50).

The following result also holds:

Theorem 11. *Let $Q,P\in S_1(H)$, with P invertible, and there exists $R\geq 1\geq r\geq 0$ such that (50) holds for any $T\in\mathcal{B}_2(H)$. Assume that $u,h:[0,\infty)\to\mathbb{R}$ are continuous function on the interval $[0,\infty)$ and with the property that*

$$
m\leq u(t)\leq M,\ n\leq h(t)\leq N \text{ for any } t\in[r,R] \tag{55}
$$

for some real numbers m,M,n,N, then

$$
\begin{aligned}
&|S_{uh}(Q,P)-S_u(Q,P)S_h(Q,P)| \qquad (56)\\
&\leq \frac{1}{2}(M-m)B(Q,P)\leq \frac{1}{2}(M-m)\left[S_{h^2}(Q,P)-S_h^2(Q,P)\right]^{1/2}\\
&\leq \frac{1}{4}(M-m)(N-n)
\end{aligned}
$$

where

$$
B(Q,P)=\left\langle\left|h(\mathfrak{A}_{Q,P})-\left\langle h(\mathfrak{A}_{Q,P})P^{1/2},P^{1/2}\right\rangle_2 1_{\mathcal{B}_2(H)}\right|P^{1/2},P^{1/2}\right\rangle_2.
$$

Proof. Utilising the inequality (32) for the functions u,h and $w(t)=1$ and the selfadjoint operator $\mathfrak{A}_{Q,P}$ we have

$$
\begin{aligned}
&\left|\langle u(\mathfrak{A}_{Q,P})h(\mathfrak{A}_{Q,P})T,T\rangle_2-\langle u(\mathfrak{A}_{Q,P})T,T\rangle_2\langle h(\mathfrak{A}_{Q,P})T,T\rangle_2\right| \qquad (57)\\
&\leq \frac{1}{2}(M-m)\left\langle\left|h(\mathfrak{A}_{Q,P})-\langle h(\mathfrak{A}_{Q,P})T,T\rangle_2 1_{\mathcal{B}_2(H)}\right|T,T\right\rangle_2\\
&\leq \frac{1}{2}(M-m)\left[\left\langle h^2(\mathfrak{A}_{Q,P})T,T\right\rangle_2-\left(\langle h(\mathfrak{A}_{Q,P})T,T\rangle_2\right)^2\right]^{1/2}\\
&\leq \frac{1}{4}(M-m)(N-n)
\end{aligned}
$$

for any $T\in\mathcal{B}_2(H)$.

If we take $T=P^{1/2}$ in (57), then we get the desired result (56). □

On choosing some particular examples for u and h like

$$
\chi^{\alpha}(t):=|t-1|^{\alpha},\ t\in[0,\infty)
$$

or

$$
\Phi_{\alpha}(t):=\frac{|1-t|^{\alpha}}{(t+1)^{\alpha-1}},\ t\in[0,\infty)
$$

for $\alpha\in[1,\infty)$, one can get various inequalities of interest.

The details are omitted

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