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EXISTENCE RESULTS FOR MULTI-POINT BOUNDARY VALUE PROBLEMS OF NONLINEAR FRACTIONAL q-DIFFERENCE EQUATIONS

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Abstract. In this paper, by using the Schauder fixed point theorem and the Banach contraction principle, we investigate existence and uniqueness of solutions for a class of multi-point boundary value problems of nonlinear fractional q-difference equations. As applications, two examples are presented to show the effectiveness of the obtained results.

1. Introduction

Since q-difference equations can describe physical phenomena much better and more accurately, the theory of q-difference equations has received attention, for example, see [1, 2, 3]. In [4, 5], Al-Salam and Agarwal proposed the fractional q-difference calculus. What is more, the fractional q-difference calculus plays an important role in quantum calculus. Recently, boundary value problems for nonlinear fractional q-difference equations have been addressed extensively by many scholars. For the development of fractional q-difference equations, see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and the references therein. For instance, Graef and Kong [16, 17] considered the existence of positive solutions for boundary value problems with fractional q-derivatives in terms of different ranges of λ, respectively. By means of the Banach contraction mapping principle and Scheafer fixed point theorem, Yang [18] and Li et al. [19] investigated the existence and uniqueness of solutions of anti-periodic boundary value problems for fractional q-difference equations. In [20, 21], by applying some standard fixed point theorems, some existence results for sequential q-fractional integrodifferential equations with q-antiperiodic boundary conditions and nonlocal four-point boundary conditions are obtained, respectively.

In this work, motivated by papers [12, 22], we investigate the existence and uniqueness of solutions for the multi-point boundary value problems of nonlinear fractional q-difference equations of the form

$$\begin{aligned} &D_q^\alpha u(t) = f(t, u(t), D_q^\beta u(t)), \quad t \in (0,1), \\ &y(0) = 0, \quad D_q^\beta u(1) - \sum_{i=1}^{m-2} a_i D_q^\beta u(\xi_i) = u_0, \end{aligned} \tag{1}$$

where $0 < \beta \leq 1 < \alpha \leq 2$, $0 = \xi_0 < \xi_1 < \cdots < \xi_{m-2} < \xi_{m-1} = 1$ $(i = 1, 2, \dots, m-2)$, $a_i \geq 0$ with $\gamma = \sum_{i=1}^{m-2} a_i \xi_i^{\alpha-\beta-1} < 1$ and D_q^α represents the standard Riemann-Liouville fractional q-derivative of order α. The nonlinear function f is assumed to satisfy certain conditions, which will be specified later. Here we point out that the nonlinearity still depends on the lower derivative. In this paper, we will establish the existence and

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uniqueness of solutions for a class of nonlinear multi-point boundary value problems (1) by using the Schauder fixed point theorem and the Banach contraction principle. At last, we give two examples to show the effectiveness of the desired results.

2. Preliminaries

For the convenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory. These details can be found in the recent literature; see [23] and references therein.

Definition 1 ([23])**.** Let $\alpha \geq 0$, $0 < q < 1$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$I_q^\alpha f(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, x \in [0,1],$$

where $\Gamma_q(\alpha) = (1-q)^{(\alpha-1)}(1-q)^{1-\alpha}$, $0 < q < 1$, and satisfies the relation: $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$, with

$$[\alpha]_q = \frac{q^\alpha - 1}{q - 1}, \quad (1-q)^{(0)} = 1, \quad (1-q)^{(n)} = \prod_{k=0}^{n-1} (1 - q^{k+1}), \quad n \in \mathbb{N}.$$

More generally, if $\alpha \in \mathbb{R}$, then $(1-q)^{(\alpha)} = \prod_{n=0}^{\infty} \big((1-q^{n+1})/(1-q^{1+\alpha+n})\big)$.

For $0 < q < 1$,the q-derivative of a real valued function f is here defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ x \neq 0, \quad \text{and } D_q f(0) = \lim_{x \to 0} D_q f(x),$$

and q-derivatives of higher order by

$$D_q^0 f(x) = f(x) \quad \text{and} \quad D_q^n f(x) = D_q D_q^{n-1} f(x), \quad n \in \mathbb{N}.$$

Definition 2 ([23])**.** The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $D_q^0 f(x) = f(x)$ and

$$D_q^\alpha f(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 1 ([23])**.** *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then the next formulas hold: (1) $I_q^\beta I_q^\alpha f(x) = I_q^{\alpha+\beta} f(x)$, (2) $D_q^\alpha I_q^\alpha f(x) = f(x)$.*

Lemma 2 ([13])**.** *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$I_q^\alpha D_q^p f(x) = D_q^p I_q^\alpha f(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} D_q^k f(0).$$

Lemma 3. *Let $h \in C[0,1]$. Then for $0 < \beta \leq 1 < \alpha \leq 2$, the unique solution of the linear problem*

$$\begin{aligned} &D_q^\alpha u(t) = h(t), \quad t \in (0,1), \\ &y(0) = 0, \quad D_q^\beta u(1) - \sum_{i=1}^{m-2} a_i D_q^\beta u(\xi_i) = u_0, \end{aligned} \tag{2}$$

is given by

$$u(t) = \int_0^1 G(t,qs) h(s) d_q s + \frac{\Gamma_q(\alpha-\beta) u_0}{\Gamma_q(\alpha)(1-\gamma)} t^{\alpha-1}, \tag{3}$$

where $G(t,s)$ is the Green function and is given by

$$G(t,s)=\begin{cases}\dfrac{(t-s)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\dfrac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma)}\Big(\sum_{i=1}^{m-2}a_i(\xi_i-s)^{(\alpha-\beta-1)}-(1-s)^{(\alpha-\beta-1)}\Big),\\ \qquad\qquad s\le t,\ \xi_{i-1}<s\le\xi_i,\ i=1,2,\dots,m-1,\\ \dfrac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma)}\Big(\sum_{i=1}^{m-2}a_i(\xi_i-s)^{(\alpha-\beta-1)}-(1-s)^{(\alpha-\beta-1)}\Big),\\ \qquad\qquad t\le s,\ \xi_{i-1}<s\le\xi_i,\ i=1,2,\dots,m-1.\end{cases}$$

Proof. Let $u(t)$ be a solution of (2). In view of Lemma 1 and 2, (2) is equivalent to the integral equation

$$u(t)=I_q^\alpha h(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2},\quad \text{for } c_1,c_2\in\mathbb{R}. \tag{4}$$

The boundary condition $u(0)=0$ implies that $c_2=0$. From (4), we can get

$$D_q^\beta u(t)=I_q^{\alpha-\beta}h(t)+\frac{\Gamma_q(\alpha)t^{\alpha-\beta-1}}{\Gamma_q(\alpha-\beta)}c_1. \tag{5}$$

Using the boundary condition $D_q^\beta u(1)-\sum_{i=1}^{m-2}a_iD_q^\beta u(\xi_i)=u_0$ in (5), we obtain

$$c_1=\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)(1-\gamma)}\Big(\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}h(\xi_i)-I_q^{\alpha-\beta}h(1)+u_0\Big).$$

Therefore the unique solution of problem (2) is given by

$$u(t)=I_q^\alpha h(t)+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)(1-\gamma)}\Big(\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}h(\xi_i)-I_q^{\alpha-\beta}h(1)+u_0\Big)t^{\alpha-1}. \tag{6}$$

Furthermore, (6) can be rewritten as (3). This completes the proof. □

Let $I=[0,1]$, and $C(I)$ be the space of continuous functions defined on I. The space $\mathscr{B}=\{u:u\in C(I),D_q^\sigma u\in C(I),0<\sigma\le1\}$ equipped with the norm $\|u\|_{\mathscr{B}}=\max_{t\in I}|u(t)|+\max_{t\in I}|D_q^\sigma u(t)|$ is a Banach space

Lemma 4. *Let $f:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous. Then $u\in\mathscr{B}$ is a solution of* (1) *if and only if $u\in\mathscr{B}$ is a solution of the integral equation*

$$u(t)=\int_0^1G(t,qs)f(s,u(s),D_q^\beta u(s))d_qs+\frac{\Gamma_q(\alpha-\beta)u_0}{\Gamma_q(\alpha)(1-\gamma)}t^{\alpha-1}. \tag{7}$$

Proof. Let $u\in\mathscr{B}$ be a solution of (1). By the method used to prove Lemma 3, we can prove that $u\in\mathscr{B}$ is a solution of the integral equation (7). Conversely, let $u\in\mathscr{B}$ be a solution of the integral equation (7). We denote the right hand side of (6) by $w(t)$, that is,

$$\begin{aligned}w(t)=&I_q^\alpha f(t,u(t),D_q^\beta u(t))+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)(1-\gamma)}\Big(\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}f(\xi_i,u(\xi_i),D_q^\beta u(\xi_i))\\&-I_q^{\alpha-\beta}f(1,u(1),D_q^\beta u(1))+u_0\Big)t^{\alpha-1}.\end{aligned}$$

Using the relation $D_q^\alpha I_q^\alpha f(t)=f(t)$, we can have $D_q^\alpha w(t)=f(t,u(t),D_q^\beta u(t))$, that is $D_q^\alpha u(t)=f(t,u(t),D_q^\beta u(t))$. On the other hand, obviously, $y(0)=0$. Now, in view of the relation $D_q^\beta I_q^\alpha f(t) = D_q^\beta I_q^\alpha I_q^{\alpha-\beta} f(t) = I_q^{\alpha-\beta} f(t)$, we obtain

$$D_q^\beta u(1) = \frac{1}{1-\gamma}\left(\sum_{i=1}^{m-2} a_i I_q^{\alpha-\beta} f(\xi_i, u(\xi_i), D_q^\beta u(\xi_i)) - I_q^{\alpha-\beta} f(1, u(1), D_q^\beta u(1))\gamma + u_0\right),$$

$$\begin{aligned}\sum_{i=1}^{m-2} a_i D_q^\beta u(\xi_i) =& \frac{1}{1-\gamma}\Bigg(\sum_{i=1}^{m-2} a_i I_q^{\alpha-\beta} f(\xi_i, u(\xi_i), D_q^\beta u(\xi_i)) \\ &- (I_q^{\alpha-\beta} f(1, u(1), D_q^\beta u(1)) - u_0)\gamma\Bigg) = D_q^\beta u(1) + u_0,\end{aligned}$$

which implies that $u \in \mathscr{B}$ is a solution of the boundary value problem (1). This completes the proof. □

3. Main results

In order to establish the existence and uniqueness results for the nonlinear boundary value problem (1), we give some assumptions on the function f.

- (H_1) $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.
- (H_2) There exists a nonnegative function $\phi \in L(I)$ such that $f(t,u,v) \le \phi(t) + c_1|u|^{\sigma_1} + c_2|v|^{\sigma_2}$, where $c_1, c_2 \in \mathbb{R}$ are nonnegative constants and $0 < \sigma_1, \sigma_2 < 1$.
- (H_3) There exists a nonnegative function $\phi \in L(I)$ such that $f(t,u,v) \le \phi(t) + c_1|u|^{\sigma_1} + c_2|v|^{\sigma_2}$, where $c_1, c_2 \in \mathbb{R}$ are nonnegative constants and $\sigma_1, \sigma_2 > 1$.
- (H_4) There exists a constant $k > 0$ such that $|f(t,u,v) - f(t,\overline{u},\overline{v})| \le k(|u-\overline{u}| + |v-\overline{v}|)$, for each $t \in I$ and all $u, v, \overline{u}, \overline{v} \in \mathbb{R}$.

For the sake of convenience, we define the following constants:

$$\begin{aligned}\mu =& \max_{t\in I}\int_0^1 |G(t,qs)\phi(s)| d_q s + \frac{\Gamma_q(\alpha-\beta)|u_0|}{\Gamma_q(\alpha)(1-\gamma)} \\ &+ \frac{1}{1-\gamma}\left((2-\gamma) I_q^{\alpha-\beta}|\phi(1)| + \sum_{i=1}^{m-2} a_i I_q^{\alpha-\beta}|\phi(\xi_i)| + |u_0|\right), \\ \nu =& \frac{1}{\Gamma_q(\alpha+1)} + \frac{\left|\sum_{i=1}^{m-2} a_i \xi_i^{(\alpha-\beta)} - 1\right|}{\Gamma_q(\alpha)[\alpha-\beta]_q(1-\gamma)} + \frac{\left|\sum_{i=1}^{m-2} a_i \xi_i^{(\alpha-\beta)} - \gamma\right|}{(1-\gamma)\Gamma_q(\alpha-\beta+1)}.\end{aligned}$$

Theorem 1. *Assume that* (H_1) *and* (H_2) *hold. Then the boundary value problem* (1) *has a solution.*

Proof. Define an operator $\mathscr{T} : \mathscr{B} \to \mathscr{B}$ by

$$(\mathscr{T}u)(t) = \int_0^1 G(t,qs) f(s, u(s), D_q^\beta u(s)) d_q s + \frac{\Gamma_q(\alpha-\beta)u_0}{\Gamma_q(\alpha)(1-\gamma)} t^{\alpha-1}.$$

By Lemma 4, fixed points of the operator $\mathscr{T}$ are solutions of the boundary value problem (1). In view of the continuity of f and G, the operator $\mathscr{T}$ is continuous. Let us choose $R \ge \max\{3\mu, (3c_1\nu)^{\frac{1}{1-\sigma_1}}, (3c_2\nu)^{\frac{1}{1-\sigma_2}}\}$ and define a ball $\mathscr{M} = \{u \in \mathscr{B} : \|u\|_{\mathscr{B}} \le R, t \in I\}$.

Firstly, we will prove that $\mathscr{T}:\mathscr{M}\to\mathscr{M}$. For every $u\in\mathscr{M}$, we have

$$
\begin{aligned}
|(\mathscr{T}u)(t)| =& \left|\int_0^1 G(t,qs)f(s,u(s),D_q^\beta u(s))d_qs+\frac{\Gamma_q(\alpha-\beta)u_0}{\Gamma_q(\alpha)(1-\gamma)}t^{\alpha-1}\right|\\
\le& \int_0^1|G(t,qs)\phi(s)|d_qs+\frac{\Gamma_q(\alpha-\beta)|u_0|}{\Gamma_q(\alpha)(1-\gamma)}+(c_1R^{\sigma_1}+c_2R^{\sigma_2})\int_0^1|G(t,qs)|d_qs\\
\le& \int_0^1|G(t,qs)\phi(s)|d_qs+\frac{\Gamma_q(\alpha-\beta)|u_0|}{\Gamma_q(\alpha)(1-\gamma)}+(c_1R^{\sigma_1}+c_2R^{\sigma_2})\\
&\times\left(\frac{1}{\Gamma_q(\alpha+1)}+\frac{\left|\sum\limits_{i=1}^{m-2}a_i\xi_i^{(\alpha-\beta)}-1\right|}{\Gamma_q(\alpha)[\alpha-\beta]_q(1-\gamma)}\right).
\end{aligned}
$$

In view of relation $D_q^\beta I_q^\alpha=I_q^{\beta-\alpha}$ and (6), we have the following estimate:

$$
\begin{aligned}
|D_q^\beta(\mathscr{T}u)(t)| =& \Bigg|I_q^{\alpha-\beta}f(t,u(t),D_q^\beta u(t))+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)(1-\gamma)}\\
&\times\left(\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}f(\xi_i,u(\xi_i),D_q^\beta u(\xi_i))-I_q^{\alpha-\beta}f(1,u(1),D_q^\beta u(1))+u_0\right)D_q^\beta t^{\alpha-1}\Bigg|\\
\le& I_q^{\alpha-\beta}|\phi(t)|+\frac{1}{1-\gamma}\left(\left|\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}|\phi(\xi_i)|-I_q^{\alpha-\beta}|\phi(1)|\right|+|u_0|\right)\\
&+(c_1R^{\sigma_1}+c_2R^{\sigma_2})\left|I_q^{\alpha-\beta}|\phi(1)|+\frac{1}{1-\gamma}\left(\sum_{i=1}^{m-2}a_i(I_q^{\alpha-\beta}1)(\xi_i)-I_q^{\alpha-\beta}1(1)\right)\right|\\
\le& \frac{1}{1-\gamma}\left((2-\gamma)I_q^{\alpha-\beta}|\phi(1)|+\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}|\phi(\xi_i)|+|u_0|\right)\\
&+(c_1R^{\sigma_1}+c_2R^{\sigma_2})\frac{\left|\sum\limits_{i=1}^{m-2}a_i\xi_i^{(\alpha-\beta)}-\gamma\right|}{(1-\gamma)\Gamma_q(\alpha-\beta+1)}
\end{aligned}
$$

Therefore, $\|(\mathscr{T}u)(t)\|_{\mathscr{B}}\le\mu+(c_1R^{\sigma_1}+c_2R^{\sigma_2})\nu\le R/3+R/3+R/3=R$. Thus, we have $\mathscr{T}:\mathscr{M}\to\mathscr{M}$.

Now we show that $\mathscr{T}$ is completely continuous. For this, let

$$N=\max_{t\in I}|f(t,u(t),D_q^\beta u(t))|+1$$

for $u\in\mathscr{M}$ and $t_1,t_2\in I$ be such that $t_1<t_2$. Then we get

$$
\begin{aligned}
&|(\mathscr{T}u)(t_2)-(\mathscr{T}u)(t_1)|\\
&\quad\le |I_q^\alpha f(t_2,u(t_2),D_q^\beta u(t_2))-I_q^\alpha f(t_1,u(t_1),D_q^\beta u(t_1))|+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)(1-\gamma)}\\
&\quad\times\left(\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}f(\xi_i,u(\xi_i),D_q^\beta u(\xi_i))-I_q^{\alpha-\beta}f(1,u(1),D_q^\beta u(1))+u_0\right)(t_2^{\alpha-1}-t_1^{\alpha-1})\\
&\quad\le\frac{N}{\Gamma_q(\alpha+1)}(t_2^\alpha-t_1^\alpha)+\frac{N\left(\sum_{i=1}^{m-2}a_i\xi_i^{(\alpha-\beta)}+\Gamma_q(\alpha-\beta)|u_0|+1\right)}{\Gamma_q(\alpha)(1-\gamma)[\alpha-\beta]_q}(t_2^{\alpha-1}-t_1^{\alpha-1}).
\end{aligned}
$$

$$
\begin{aligned}
&|D_q^{\beta}(\mathscr{T}u)(t_2)-D_q^{\beta}(\mathscr{T}u)(t_1)|\\
&\quad\le |I_q^{\alpha-\beta}f(t_2,u(t_2),D_q^{\beta}u(t_2))-I_q^{\alpha-\beta}f(t_1,u(t_1),D_q^{\beta}u(t_1))|+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)(1-\gamma)}\\
&\qquad\times\Bigg(\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}f(\xi_i,u(\xi_i),D_q^{\beta}u(\xi_i))-I_q^{\alpha-\beta}f(1,u(1),D_q^{\beta}u(1))+u_0\Bigg)\\
&\qquad\times(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1})\le\frac{N\Big(2-\gamma+\sum_{i=1}^{m-2}a_i\xi_i^{(\alpha-\beta)}+|u_0|\Big)}{(1-\gamma)\Gamma_q(\alpha-\beta+1)}(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}).
\end{aligned}
$$

Now using the fact that the functions $t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}$, $t_2^{\alpha-1}-t_1^{\alpha-1}$ and $t_2^{\alpha}-t_1^{\alpha}$ are uniformly continuous on I, we conclude that $\mathscr{T}\mathscr{M}$ is equicontinuous. Also $\mathscr{T}\mathscr{M}$ is a uniformly bounded set. We have $\mathscr{T}\mathscr{M}\subset\mathscr{M}$ By the Arzela-Ascoli theorem, $\mathscr{T}:\mathscr{M}\to\mathscr{M}$ is completely continuous. Hence the Schauder fixed point theorem implies the existence of a solution in $\mathscr{M}$ for the boundary value problem (1). This completes the proof. □

Theorem 2. *Assume that* (H_1) *and* (H_3) *hold. Then the boundary value problem* (1) *has a solution.*

Proof. The proof is similar to that of Theorem 1, so it is omitted. □

Theorem 3. *Assume that* (H_1) *and* (H_4) *hold. If* $k<(\frac{1}{\Gamma_q(\alpha+1)}+\rho)^{-1}$, *then the boundary value problem* (1) *has a unique solution, where*

$$
\rho=\frac{(\Gamma_q(\alpha-\beta)+\Gamma_q(\alpha))(2-\gamma+\sum_{i=1}^{m-2}a_i\xi_i^{(\alpha-\beta)})}{\Gamma_q(\alpha)\Gamma_q(\alpha-\beta+1)(1-\gamma)}.
$$

Proof. By assumption (H_4), we have following estimates:

$$
\begin{aligned}
|(\mathscr{T}u)(t)-(\mathscr{T}\overline{u})(t)|&\le\int_0^1|G(t,qs)|f(s,u(s),D_q^{\beta}u(s))-f(s,\overline{u}(s),D_q^{\beta}\overline{u}(s))|d_qs\\
&\le\left(\frac{1}{\Gamma_q(\alpha+1)}+\frac{\sum_{i=1}^{m-2}a_i\xi_i^{(\alpha-\beta)}+1}{\Gamma_q(\alpha)(1-\gamma)[\alpha-\beta]_q}\right)\|u-\overline{u}\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|D_q^{\beta}(\mathscr{T}u)(t)-D_q^{\beta}(\mathscr{T}\overline{u})(t)|\\
&\quad\le I_q^{\alpha-\beta}|f(t_2,u(t),D_q^{\beta}u(t))-f(t_2,\overline{u}(t),D_q^{\beta}\overline{u}(t))|+\frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)(1-\gamma)}\\
&\qquad\times\Bigg(\sum_{i=1}^{m-2}a_iI_q^{\alpha-\beta}|f(\xi_i,u(\xi_i),D_q^{\beta}u(\xi_i))-f(\xi_i,\overline{u}(\xi_i),D_q^{\beta}\overline{u}(\xi_i))|\\
&\qquad-I_q^{\alpha-\beta}|f(1,u(1),D_q^{\beta}u(1))-f(1,\overline{u}(1),D_q^{\beta}\overline{u}(1))|\Bigg)t^{\alpha-\beta-1}\\
&\quad\le\frac{k\Big(2-\gamma+\sum_{i=1}^{m-2}a_i\xi_i^{(\alpha-\beta)}\Big)}{(1-\gamma)\Gamma_q(\alpha-\beta+1)}\|u-\overline{u}\|.
\end{aligned}
$$

Hence, it follows that $\|(\mathscr{T}u)(t)-(\mathscr{T}\overline{u})(t)\|\le\eta\|u-\overline{u}\|$, where $\eta=k(\frac{1}{\Gamma_q(\alpha+1)}+\rho)<1$. Hence, by the contraction mapping principle the boundary value problem (1) has a unique solution. □

4. Two examples

In this section, we give two examples to show the effectiveness of the obtained results.

Example 1. Consider the boundary value problem

$$D_q^\alpha u(t) = \frac{\lambda_1 t}{1+t^2} + \frac{\lambda_2 \sin 2t}{\sqrt{\pi + |u(t)|}}|u(t)|^{\sigma_1} + \frac{\lambda_3 \cos 3t}{\sqrt{3 + |D_q^\beta u(t)|}}|D_q^\beta u(t)|^{\sigma_2}, \quad t \in (0,1),$$
$$y(0) = 0, \quad D_q^\beta u(1) = \frac{1}{2}D_q^\beta u(\frac{1}{4}) + \frac{1}{4}D_q^\beta u(\frac{1}{2}) + \frac{1}{4}D_q^\beta u(\frac{3}{4}) + 2, \tag{8}$$

where $0 < \beta \leq 1 < \alpha \leq 2$, $0 \leq \lambda_i \in \mathbb{R}$. It is easy to see that $\gamma = \sum_{i=1}^3 a_i \xi_i^{\alpha-\beta-1} = \frac{1}{2}(\frac{1}{4})^{\alpha-\beta-1} + \frac{1}{4}(\frac{1}{2})^{\alpha-\beta-1} + \frac{1}{4}(\frac{3}{4})^{\alpha-\beta-1} < (\frac{3}{4})^{\alpha-\beta-1}$. For the nonlinearity $f(t,u,v) = \frac{\lambda_1 t}{1+t^2} + \frac{\lambda_2 \sin 2t}{\sqrt{\pi+|u|}}|u|^{\sigma_1} + \frac{\lambda_3 \cos 3t}{\sqrt{3+|v|}}|v|^{\sigma_2}$, $t \in (0,1)$, we obtain that $f(t,u,v) < \phi(t) + c_1|u|^{\sigma_1} + c_2|v|^{\sigma_2}$, where $\phi(t) = \frac{\lambda_1 t}{1+t^2}$, $c_1 = \frac{\lambda_2}{\sqrt{\pi}}$, $c_1 = \frac{\lambda_3}{\sqrt{3}}$. For $0 < \sigma_1, \sigma_2 < 2$, the assumption (H_2) holds and for $\sigma_1, \sigma_2 > 1$, the assumption (H_3) holds. Therefore, by Theorems 1 and 2, the boundary value problem (8) has a solution.

Example 2. Consider the boundary value problem

$$D_{0.5}^{\frac{3}{2}} u(t) = \frac{e^{-2t}(|u(t)| + |D_{0.5}^{\frac{1}{2}} u(t)|)}{(13 + 23e^{-t})(1 + |u(t)| + |D_{0.5}^{\frac{1}{2}} u(t)|)}, \quad t \in (0,1),$$
$$y(0) = 0, \quad D_{0.5}^\beta u(1) = \frac{3}{7}D_{0.5}^{\frac{1}{2}} u(\frac{2}{5}) + \frac{1}{3}D_{0.5}^{\frac{1}{2}} u(\frac{3}{5}) + \frac{4}{21}D_{0.5}^{\frac{1}{2}} u(\frac{4}{5}) + 1. \tag{9}$$

By simple computation, we have $\gamma = \sum_{i=1}^3 a_i \xi_i^{\alpha-\beta-1} = \frac{20}{21}$. For the nonlinearity $f(t,u,v) = \frac{e^{-2t}(u+v)}{(13+23e^{-t})(1+u+v)}$, $t \in (0,1)$. For $u, v, \overline{u}, \overline{v}$, we have $|f(t,u,v) - f(t,u,v)| \leq \frac{1}{36}(|u - \overline{u}| + |v - \overline{v}|)$. Hence (H_4) is satisfied. By computation assumption, $\eta \approx 0.9376 < 1$. Therefore, by Theorems 3, the boundary value problem (9) has a unique solution.

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