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COMPARATIVE ANALYSIS OF ESTIMATION METHODS OF THE REAL GROSS VALUE ADDED, IN ROMANIA THROUGH COBB-DOUGLAS PRODUCTION FUNCTION

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Abstract. This article describes a comparative analysis of the methods for estimating Cobb-Douglas production function. In the paper, the three models analyzed, in which two are static and one is dynamic, are solved by the linearization method and the logarithmic transcendental method. The data series which occur in patterns, are given by the real gross value added, regarded as output variable, and the tangible assets, respectively the average number of employees, regarded as input variables. The parameters of the models are determined using the least squares method (LSM), using Eviews. The comparative analysis of models refers both to capacity by estimate with small errors and to verify the statistical tests and ease of implementing these methods.

1. Introduction

Over time, they was developed many theoretical and empirical studies and analysis regarding the Cobb-Douglas production function. Starting from the function form proposed by Cobb and Douglas in [9], and from the model approximated by the two authors through this function, many researchers have tried to make improvements or develop generalizations of this function and of the models approximated by this. Thus, in [10], Felipe and Gerard performs a retrospective to determinations and development of the models approximated by the Cobb-Douglas functions and analyzes by econometric point of view a re-evaluation of the data series used by Cobb-Douglas in 1928, in the original work [5], for the period (1899-1922). On the other hand, the two authors, bring arguments regarding at the various appreciations and criticisms against some researchers, who helped develop and apply of the production models approximated by these functions.

Given the strong assumptions that the Cobb-Douglas production function imposes on the underlying technology, Christensen et al. [6] proposed a more flexible generalization of the Cobb-Douglas function, the Trans-log (transcendental logarithmic) production function. In 1976, Cobb and Douglas, publish a new article, in which testing the series by the empirical values, with help of the models approximated by the production functions introduced by the two authors. A year later, Meeusen and Broeck, in [11], describe the efficiency of the estimations Cobb-Douglas production functions with composed errors.

Theories, analysis and more recent publications, regarding this topic can be found in Batese and Coelli [3], Antrás [2], Bons and Söderbom [4], Constantin [8] or Czekaj and Henningsenu [7]. Currently, there are no production-related economic theory, in which to be not implemented and the Cobb-Douglas function.

In this article, are made comparisons in terms of quality of three types of econometric models, approximated by the Cobb-Douglas production function, for the case if we want

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to estimate the real gross value added in Romania for a period of 19 years (1995 -2013). The three models analyzed, in which two are static and one is dynamic, are solved by the linearization method and the logarithmic transcendental method. The data series which occur in patterns, are given by the real gross value added, regarded as output variable, and the tangible assets, respectively the average number of employees, regarded as input variables. The parameters of the models are determined using the least squares method (LSM), using Eviews.

2. The representation forms of the functions Cobb - Douglas

The general form of Cobb-Douglas production function is represented by the relation:

$$Q(x_1, x_2, ..., x_n) = a \cdot x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot ... \cdot x_n^{\alpha_n} \tag{1}$$

where $Q : R_+^n \to R_+$ is the production function, $a > 0$ is the scale parameter, $x_i \in [0, \infty)$, $i = \overline{1, n}$ are the production factors, $\alpha_1, \alpha_2, ..., \alpha_n$ are the real parameters.

The Cobb-Douglas simplified function is defined by the following relation:

$$Q(K, L) = a \cdot K^\alpha \cdot L^\beta \tag{2}$$

where $Q : R_+^2 \to R_+$ is the production function, $a > 0$ is the scale parameter, K is is the production factor expressed through capital, L is the production factor expressed through label, α, β are the real parameters.

In relation (2), the two parameters α and β represent the partial elasticities in relation with every factor of the production process.

The elasticity of production in relation to the capital, signify the percentage increase of the production, at the variation with an percentage of the utilization level of capital and is calculated using the relation:

$$E_{Q/K} = \frac{\frac{\partial Q(K,L)}{\partial K}}{\frac{Q(K,L)}{K}} = \alpha \tag{3}$$

The elasticity of production in relation to the label, signify the percentage increase of the production, at the variation with an percentage of the utilization level of label and is calculated using the relation:

$$E_{Q/L} = \frac{\frac{\partial Q(K,L)}{\partial L}}{\frac{Q(K,L)}{L}} = \beta \tag{4}$$

The total elasticity of the production is calculated as sum of those two elasticities, namely:

$$E_Q = E_{Q/K} + E_{Q/L} = \alpha + \beta \tag{5}$$

In characterizing the economic process, we can identify the following three situations:

1. The production process with yield ascending scale, for which the total production elasticity is greater than one, namely $\alpha + \beta > 1$. In this case, an specified increase of the production factors, namely of the labor and of the capital, lead to an increase of the production function, but in a greater proportion.

2. The production process with yield constant scale, for which the total production elasticity is constant, namely $\alpha + \beta = 1$. In this case, an increase of the production factors, lead to an increase of the production function in the same proportion.

3. The production process with yield descending scale, for which the total production elasticity is lesser than one, namely $\alpha + \beta < 1$. In this case, an specified increase of the production factors, namely of the labor and of the capital, lead to an increase of the production function, but in a smaller proportion.

Intensive form of the Cobb-Douglas production function.We considering the case in which the production function is with yield constant scale, namely $\alpha + \beta = 1$. If we introduce in the expression of the production function (2), the parameter $\beta = 1 - \alpha$, then we have:

$$Q(K, L) = a \cdot K^{\alpha} \cdot L^{1-\alpha} \tag{6}$$

The relation (6) it can be written as

$$\frac{Q(K, L)}{L} = a \cdot \left(\frac{K}{L}\right)^{\alpha} \tag{7}$$

or

$$q = a \cdot k^{\alpha} \tag{8}$$

In relation (8) is denoted with $q = Q/L$ the labor productivity, and with $k = K/L$.

The generalized Cobb-Douglas production function is defined by the following relation:

$$Q(K, L) = a \cdot K^{1-\alpha} \cdot L^{\alpha} \cdot e^{-\beta k} \tag{9}$$

where in addition to the notations introduced earlier, $k = K/L$.

Quasi Cobb-Douglas production function is defined by the following relation:

$$Q(K, L) = a \cdot K^{1-\alpha} \cdot L^{\alpha} \cdot e^{-\frac{\beta}{\gamma} \cdot k^{\gamma}} \tag{10}$$

where γ is a positive parameter.

In economic practice, in addition to forms of representation introduced above, there are other forms of representation of the Cobb-Douglas production function (see Stroe [12]), but are not addressed in this article.

3. The estimation methods of the multiplicative model parameters approximated by Cobb-Douglas function

Multiplicative model has the following general form of representation:

$$y_t = a \cdot x_{1t}^{\alpha_1} \cdot x_{2t}^{\alpha_2} \cdot ... \cdot x_{nt}^{\alpha_n} \cdot e^{\varepsilon_t} \tag{11}$$

where y represents the output from the model , $x_1, x_2, ..., x_n$ represents inputs in the model, a is the scale parameter, and ε is a residual variable, that has a normal distribution type, by the null value mean and variance σ^2.

In the following, we describe three forms by representation of the multiplicative model in which the time variable explicitly appears or not.

Model 1 - The static model in which the Cobb-Douglas production function is without technical progress, is defined by the following relationship in which not appear explicitly the time variable:

$$y_t = a \cdot K_t^{\alpha} \cdot L_t^{\beta} \cdot \varepsilon_t \tag{12}$$

where y is the output, a is the scale parameter, K and L are the inputs in the model, α and β are the real parameters, ε is the residual variable.

Model 2 - The static model in which the Cobb-Douglas production function is written as intensive form (7) is defined by the following relationship in which not appear explicitly the time variable:

$$\left(\frac{y_t}{L_t}\right) = a \cdot \left(\frac{K_t}{L_t}\right)^{\alpha} \cdot \varepsilon_t \tag{13}$$

where y is the output, a is the scale parameter, K and L are the inputs in the model, α and β are the real parameters, ε is the residual variable.

Model 3 - The dynamic model in which the Cobb-Douglas production function is with technical progress, is defined by the following relationship in which appear explicitly the time variable:

$$y_t = a \cdot K_t^{\alpha} \cdot L_t^{\beta} \cdot e^{ct} \cdot \varepsilon_t \tag{14}$$

where y represents the output, K and L represents the inputs in the model, a, α and β are the real parameters, c is the econometric expresses by the influence of the technical progress, t is the time variable, ε is the residual variable.

The parameters of the three models defined above, can be estimated using two methods, namely linearization method and the method Translog, which we describe below.

A. The linearization method

In this case, we linearized the function defined by equation (12) by logarithms. In these circumstances, the relation (12) will be written under the following form:

$$ln\ y_t = ln\ a + \alpha \cdot ln\ K_t + \beta \cdot ln\ L_t + ln\ \varepsilon_t \tag{15}$$

If we use the following notation

$$ln\ y_t = y_t^*,\ ln\ a = A,\ ln\ K_t = K_t^*,\ ln\ L_t = L_t^*,\ ln\ \varepsilon_t = \varepsilon_t^* \tag{16}$$

then the above expression may be written as:

$$y_t^* = A + \alpha \cdot K_t^* + \beta \cdot L_t^* + \varepsilon_t^* \tag{17}$$

The parameters of the model defined in the relation (17), A, α and β can be determined using least squares method (LSM). After applying the LSM method, we obtain the following system:

$$\begin{cases} nA + \alpha \sum\limits_{t=1}^{n} K_t^* + \beta \sum\limits_{t=1}^{n} L_t^* = \sum\limits_{t=1}^{n} y_t^* \\ A \sum\limits_{t=1}^{n} K_t^* + \alpha \sum\limits_{t=1}^{n} {K_t^*}^2 + \beta \sum\limits_{t=1}^{n} K_t^* L_t^* = \sum\limits_{t=1}^{n} K_t^* y_t^* \\ A \sum\limits_{t=1}^{n} L_t^* + \alpha \sum\limits_{t=1}^{n} K_t^* L_t^* + \beta \sum\limits_{t=1}^{n} {L_t^*}^2 = \sum\limits_{t=1}^{n} L_t^* y_t^* \end{cases} \tag{18}$$

After solving this system, we can determine the parameters specified above.

B. Translog method (transcedental logarithmic)

In this method we apply Taylor's formula quadratic, in the point $(1, 1)$, to the function defined in the relation (12), more precisely:

$$\begin{aligned} f(x,y) = f(a,b) + \left(\frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b) \right) + \\ + \tfrac{1}{2} \left(\frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b) \right)^2 \end{aligned} \tag{19}$$

The model described by a production function without technical progress, defined in (12), can be represented from a Translog function by the form (see Andrei and Bourbonnais [1]):

$$lny_t = \alpha_0 + \alpha_1 \cdot lnK_t + \beta_1 \cdot lnL_t + \alpha_2 \cdot (lnK_t)^2 + \beta_2 \cdot (lnL_t)^2 + \gamma_1 \cdot lnK_t \cdot lnL_t + ln\varepsilon_t \quad (20)$$

If we use the following notation

$$ln\ y_t = y_t^*,\ ln\ K_t = K_t^*,\ ln\ L_t = L_t^*,\ ln\ \varepsilon_t = \varepsilon_t^* \quad (21)$$

then the above expression may be written as:

$$y_t^* = \alpha_0 + \alpha_1 \cdot K_t^* + \beta_1 \cdot L_t^* + \alpha_2 \cdot {K_t^*}^2 + \beta_2 \cdot {L_t^*}^2 + \gamma_1 \cdot K_t^* L_t^* + \varepsilon_t^* \quad (22)$$

Like in the above model, the model parameters Translog, $\alpha_0, \alpha_1, \beta_0, \beta_1$ and γ_1 can be determined with the LSM method.

Remarks

- The static model defined by the Cobb-Douglas function, written as intensive form, defined in relation (13), can be linearized by using the following logarithm expressions:

$$ln\left(\frac{y_t}{L_t}\right) = lnA + \alpha \cdot ln\left(\frac{K_t}{L_t}\right) + ln\varepsilon_t \quad (23)$$

  the parameters A and α can be determined by LSM method.
- The dynamic model in which the Cobb-Douglas function is with technical progress, defined in relation (14) can be linearized by logarithm with the following expression:

$$lny_t = lnA + \alpha \cdot lnK_t + \beta \cdot lnL_t + ct + \varepsilon_t \quad (24)$$

  the parameters A, α, β and c can be determined by LSM method.

4. Comparative analysis of estimation methods for Cobb-Douglas production function

To make comparisons between models specified above, we consider the following macroeconomic measures. The output variable Y of the three models analyzed, is the real gross value added and input variables are real fixed capital (tangible assets or fixed assets) K, and the average number of employees on the activities of the national economy L.

The real parameters a, α, β and c from the model, can be determined by LSM method. The data series for the three measures [16], [17], [19] were expressed in real prices.

The gross value added and the tangible assets it was expressed in constant prices (2000 = 100), following processing of the Annual Report of the National Institute of Statistics, with help of GDP deflator [15]. For the period 1995-1997, the tangible assets data, it was taken from the site on the "Dynamics of the structure of the Romanian economy in the EU pre-accession period," see [18]. Comparisons between the three models are made for Romania, and the analyzed period is 19 years (1995-2013).

As seen from the above, we specify that, the analyzed models are nonlinear models of type MISO (Multiple input - Single output) (see Stoicuta [13]). For determine the parameters of the analyzed models, we will use Eviews [14], this program being specified to the analyzes econometric models. As seen in the table above, the best results are obtained in Model 1 - Translog method. This conclusion can be seen in both the high value of R-squared (0.981) and as well in the obtained results of the three criteria, specific to the information theory, these having the lowest values between the analyzed models.

Schvartz criterion has most appropriate value by zero, this criterion showing that Model 1 has the best performance. This model also has the square sum of errors with

Years	Real gross value added [mil. lei]	Tangible assets [mil. lei] 2000=100	The average number of employees [mii. pers.]
1995	694.325,06	2.016.692,39	6160
1996	592.810,45	1.543.578,43	5938
1997	412.414,02	802.625,66	5597
1998	335.255,53	307.504,25	5368
1999	340.579,58	341.940,59	4760
2000	336.959,00	268.919,00	4623
2001	321.952,68	288.796,89	4618
2002	307.935,99	257.720,65	4567
2003	318.495,27	208.330,37	4590
2004	370.475,24	263.426,72	4468
2005	38.806,48	29.484,84	4558
2006	42.793,54	31.158,12	4667
2007	47.195,37	43.248,91	4885
2008	56.571,76	37.948,22	5046
2009	48.223,56	26.036,29	4774
2010	47.407,18	23.551,52	4376
2011	47.713,44	34.081,87	4348
2012	47.834,68	28.639,64	4442
2013	50.184,41	22.957,28	4443

Figure 1. Data series for the three analyzed indicators

the lowest value (0.421), showing another reason that Model 1 - Translog method, is the most indicated to apply in this case.

Regarding the F statistic, comparing the calculated value of this statistic for the three analyzed models, with the tabulated value of this statistic $\chi^2 = 40.79$, we see that $F_{calc} > \chi^2$, in each case. This shows that, for a significance level of 1 per cent, we can say that, between the real values of the series and the estimated values of the variable y, there is a significant dependency.

On the other hand, is observed that as both Model 1 and Model 3, solved by the liniarization method, have the total sum of the elasticity, subunit, ie $E = \alpha + \beta = -2,02 < 1$. This shows that both the static model and the dynamic model, give the same results, in this case. The real gross value added, is growing a lesser extent in this situation, compared to an increase in a certain proportion of tangible assets and average number of employees. Regarding the Durbin-Watson statistic for Model 1 and 3, we have a value close to 2. This shows that the hypothesis by which the errors series no shows the correlation of the first order, is accepted. The exception to this rule, makes Model 2, in which the condition $d_2 < DW < 4 - D_1$ is not satisfied, where $D_1 = 1.02$ and $d_2 = 1.54$ are values of this statistics, for a materiality threshold of 5 percent.

In the case in which we consider the total elasticity equal to unity, ie $\alpha + \beta = 1$, Model 2 - Linearization method has the worst results of the all considered models, this resulting

MODEL 1 – LINEARIZATION METHOD

$$\ln y_t = A + \alpha \cdot \ln K_t + \beta \ln L_t + \varepsilon_t, \quad t = 1,19$$

Coeff.	R^2	F stat.	Durbin-Watson stat.	Akaike criterion	Schwartz criterion	Hannan – Quinn criterion	Sum squared resid
A=26,22 α =0,85 β =-2,87	0,974	302,35	1,83	-0,37	-0,22	-0,34	0,562

MODEL 1 – TRANSLOG METHOD

$$\ln y_t = \alpha_0 + \alpha_1 \cdot \ln K_t + \beta_1 \cdot \ln L_t + \alpha_2 \cdot (\ln K_t)^2 + \beta_2 \cdot (\ln L_t)^2 + \gamma_1 \cdot \ln K_t \cdot \ln L_t + \varepsilon_t, \quad t = 1,19$$

Coeff.	R^2	F stat.	Durbin-Watson stat.	Akaike criterion	Schwartz criterion	Hannan – Quinn criterion	Sum squared resid
α_0 =729,64 α_1 =11,51 β_1 = -184,39 α_2 = -0,02 β_2 =11,59 γ_1 = -1,20	0,981	131,95	1,81	-0,34	-0,04	-0,29	0,421

MODEL 2 – LINEARIZATION METHOD

$$\ln y_t = A + \alpha \cdot \ln K_t + (1-\alpha) \cdot \ln L_t + \varepsilon_t, \quad t = 1,19$$

Coeff.	R^2	F stat.	Durbin-Watson stat.	Akaike criterion	Schwartz criterion	Hannan – Quinn criterion	Sum squared resid
A=1,10 α =0,70	0,931	230,84	0,72	0,51	0,61	0,52	1,495

MODEL 3 – LINEARIZATION METHOD

$$\ln y_t = A + \alpha \cdot \ln K_t + \beta \ln L_t + c \cdot t + \varepsilon_t, \quad t = 1,19$$

Coeff.	R^2	F stat.	Durbin-Watson stat.	Akaike criterion	Schwartz criterion	Hannan – Quinn criterion	Sum squared resid
A=26,25 α =0,85 β =-2,87 C=-0,0003 α =0,70	0,974	188,97	1,83	-0,26	-0,06	-0,23	0,562

FIGURE 2. Cobb-Douglas Regression (analyzed period 1995-2013, parameter estimation - least squares method LSM)

from the fact that in reality the model has the total elasticity, subunit, as shown in Model 1 and Model 3 - Linearization method.

5. Conclusion

In comparative analysis between the three models described in this paper, we conclude that the best results are obtained in the Model 1 -Translog method. This affirmation denotes both from the high value of the coefficient of determination (R-squared) and of

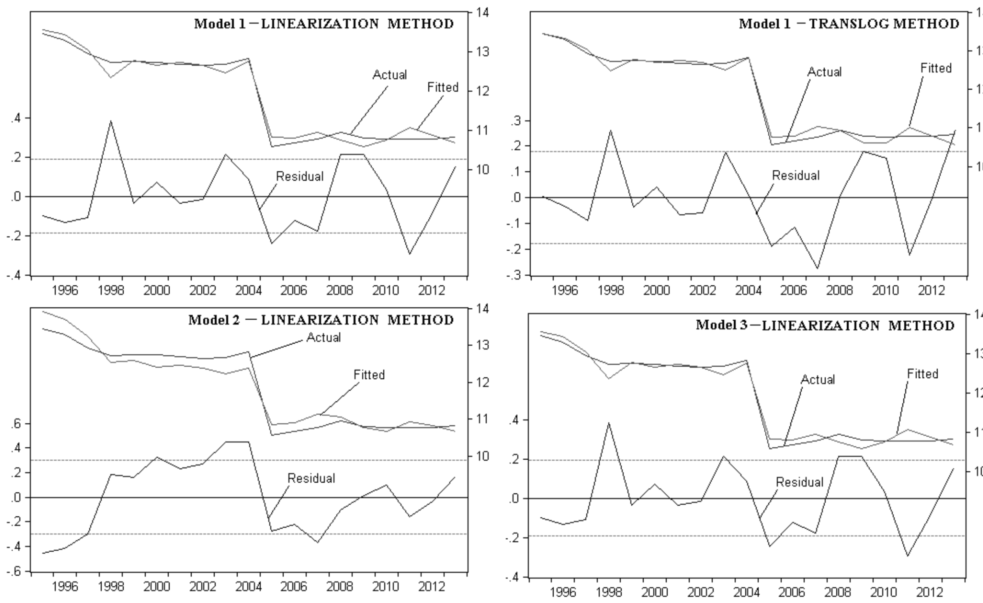

Figure 3. The variation over time of the real gross value added of the Romania (Actual), in tandem with the time variation of the three models analyzed (Fitted), with highlighting the residue (Residual)

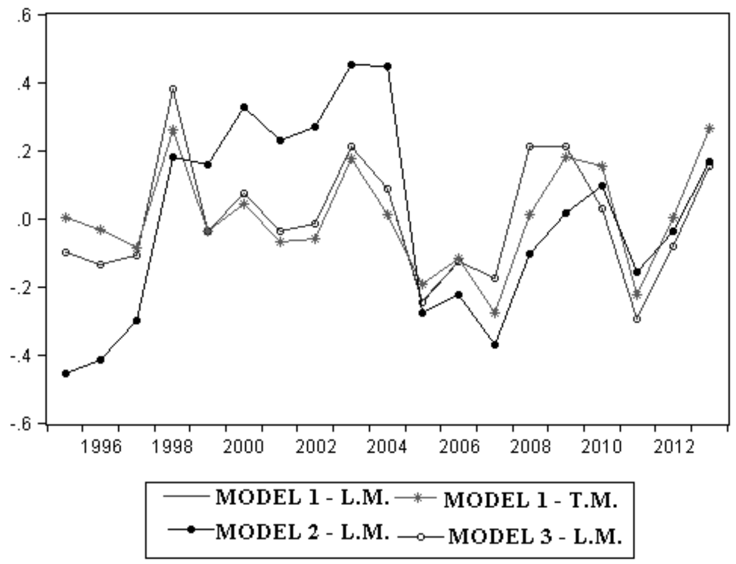

Figure 4. The adjustment errors for the three methods

the results of the tests obtained in this model and applied in order to assess the quality of the estimators. Also, in this model, deviations between the empirical and adjusted values are the lowest compared to other analyzed models.

The models and methods proposed in this paper can be applied and adapted to other conditions in România, in order to achieve forecasts and their impact on the economic growth. On the other hand, the comparison of the three models is recommended so in the argumentation of the analysis and for obtaining the some quality forecasts.

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