

## ZERO-FREE REGION FOR POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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ABSTRACT. In this paper we prove some extension of the Eneström-Kakeya theorem which says that if  $P(z) = \sum_{i=1}^n a_i z^i$  be a polynomial of degree  $n$  such that  $0 \leq a_n \leq a_{n-1} \leq \dots \leq a_1 \leq a_0$  then  $P(z)$  does not vanish in  $|z| \leq 1$ . By relaxing the hypothesis of this result in several ways one can obtain zero-free regions for polynomials with restricted coefficients and there by present some interesting generalizations and extensions of the Eneström-Kakeya Theorem.

### 1. INTRODUCTION

The well known Results namely Eneström-Kakeya theorem [1, 2] in the theory of the distribution of zeros of polynomials is the following.

**Theorem 1.** *Let  $P(z) = \sum_{i=0}^n a_i z^i$  be a polynomial of degree  $n$  such that  $0 < a_0 \leq a_1 \leq \dots \leq a_{n-1} \leq a_n$ , then all the zeros of  $P(z)$  lie in  $|z| \leq 1$ .*

Applying the above result to the polynomial  $z^n P(\frac{1}{z})$ , we get the following result:

**Theorem 2.** *If  $P(z) = \sum_{i=0}^n a_i z^i$  be a polynomial of degree  $n$  such that  $0 < a_n \leq a_{n-1} \leq \dots \leq a_1 \leq a_0$ , then  $P(z)$  does not vanish in  $|z| < 1$ .*

In the literature [3, 4, 5, 6, 7, 8], there exist several extensions and generalizations of the Eneström-Kakeya theorem.

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results:

**Theorem 3.** *Let  $P(z) = \sum_{i=0}^n a_i z^i$  be a polynomial of degree  $n \geq 2$  with real coefficients such that for some  $k \geq 1, \rho \geq 0$  and*

*i)  $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n + \rho$  if  $n$  is even,*

*OR*

*ii)  $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n - \rho$  if  $n$  is odd.*

*Then*

*i)  $P(z)$  does not vanish in the disk*

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + M_1} \text{ if } n \text{ is even}$$

$$\text{where } M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})],$$

*OR*

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ii)  $P(z)$  does not vanish in the disk

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| - a_n + 2\rho + M_2} \text{ if } n \text{ is odd}$$

$$\text{where } M_2 = 2[(a_2 + a_4 + \dots + a_{n-3} + a_{n-1}) - (a_1 + a_3 + \dots + a_{n-4} + a_{n-2})].$$

**Corollary 1.** Let  $P(z) = \sum_{i=0}^n a_i z^i$  be a polynomial of degree  $n \geq 2$  with real coefficients such that

i)  $a_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n$  if  $n$  is even,

OR

ii)  $a_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n$  if  $n$  is odd.

Then

i)  $P(z)$  does not vanish in the disk

$$|z| < \frac{|a_0|}{a_0 + |a_n| + a_n + M_1} \text{ if } n \text{ is even}$$

$$\text{where } M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})],$$

OR

ii)  $P(z)$  does not vanish in the disk

$$|z| < \frac{|a_0|}{a_0 + |a_n| - a_n + M_2} \text{ if } n \text{ is odd}$$

$$\text{where } M_2 = 2[(a_2 + a_4 + \dots + a_{n-3} + a_{n-1}) - (a_1 + a_3 + \dots + a_{n-4} + a_{n-2})].$$

**Corollary 2.** Let  $P(z) = \sum_{i=0}^n a_i z^i$  be a polynomial of degree  $n \geq 2$  with positive real coefficients such that for some  $k \geq 1$ ,  $\rho \geq 0$  and

i)  $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n + \rho$  if  $n$  is even,

OR

ii)  $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n - \rho$  if  $n$  is odd.

Then

i)  $P(z)$  does not vanish in the disk

$$|z + k - 1| < \frac{a_0}{ka_0 + 2a_n + 2\rho + M_1} \text{ if } n \text{ is even,}$$

$$\text{where } M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})],$$

OR

ii)  $P(z)$  does not vanish in the disk

$$|z + k - 1| < \frac{a_0}{ka_0 + 2\rho + M_2} \text{ if } n \text{ is odd}$$

$$\text{where } M_2 = 2[(a_2 + a_4 + \dots + a_{n-3} + a_{n-1}) - (a_1 + a_3 + \dots + a_{n-4} + a_{n-2})].$$

**Remark 1.** By taking  $\rho = 0$ ,  $r = 1$  in Theorem 1, it reduces to Corollary 1.

**Remark 2.** By taking  $a_i > 0$ , for  $i = 0, 1, \dots, n$  in Theorem 1, it reduces to Corollary 2.

**Theorem 4.** Let  $P(z) = \sum_{i=0}^n a_i z^i$  be a polynomial of degree  $n \geq 2$  with real coefficients such that for some  $0 < r \leq 1$ ,  $\rho \geq 0$  and

i)  $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n - \rho$  if  $n$  is even,

OR

ii)  $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n + \rho$  if  $n$  is odd.

Then

i)  $P(z)$  does not vanish in the disk

$$|z| < \frac{|a_0|}{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1} \text{ if } n \text{ is even}$$

$$\text{where } M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})],$$

OR

ii)  $P(z)$  does not vanish in the disk

$$|z| < \frac{|a_0|}{2\rho + |a_0| + |a_n| + a_n - r(|a_0| + a_0) - M_2} \text{ if } n \text{ is odd}$$

$$\text{where } M_2 = 2[(a_2 + a_4 + \dots + a_{n-3} + a_{n-1}) - (a_1 + a_3 + \dots + a_{n-4} + a_{n-2})].$$

**Corollary 3.** Let  $P(z) = \sum_{i=0}^n a_i z^i$  be a polynomial of degree  $n \geq 2$  with real coefficients such that

i)  $a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n$  if  $n$  is even,

OR

ii)  $a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n$  if  $n$  is odd.

Then

i)  $P(z)$  does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| - a_n - a_0 - M_1} \text{ if } n \text{ is even}$$

$$\text{where } M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})],$$

OR

ii)  $P(z)$  does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + a_n - a_0 - M_2} \text{ if } n \text{ is odd}$$

$$\text{where } M_2 = 2[(a_2 + a_4 + \dots + a_{n-3} + a_{n-1}) - (a_1 + a_3 + \dots + a_{n-4} + a_{n-2})].$$

**Corollary 4.** Let  $P(z) = \sum_{i=0}^n a_i z^i$  be a polynomial of degree  $n \geq 2$  with positive real coefficients such that for some  $0 < r \leq 1$ ,  $\rho \geq 0$  and

i)  $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n - \rho$  if  $n$  is even,

OR

ii)  $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n + \rho$  if  $n$  is odd.

Then

i)  $P(z)$  does not vanish in the disk

$$|z| < \frac{a_0}{2\rho + (1-2r)a_0 - M_1} \text{ if } n \text{ is even}$$

$$\text{where } M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})],$$

OR

ii)  $P(z)$  does not vanish in the disk

$$|z| < \frac{a_0}{2\rho + 2a_n + (1-2r)a_0 - M_2} \text{ if } n \text{ is odd}$$

$$\text{where } M_2 = 2[(a_2 + a_4 + \dots + a_{n-3} + a_{n-1}) - (a_1 + a_3 + \dots + a_{n-4} + a_{n-2})].$$

**Remark 3.** By taking  $\rho = 0$ ,  $r = 1$  in Theorem 2, it reduces to Corollary 3.

**Remark 4.** By taking  $a_i > 0$ , for  $i = 0, 1, \dots, n$  in Theorem 2, it reduces to Corollary 4.

## 2. PROOFS OF THE THEOREMS

*Proof of Theorem 1.* Let  $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  be a polynomial of degree  $n$ . Consider the polynomials  $J(z) = z^n P(\frac{1}{z})$  and  $R(z) = (z-1)J(z)$  so that

$$\begin{aligned} R(z) &= (z-1)[a_0 z^n + a_1 z^{n-1} + \dots + a_{m-1} z^{m-1} + a_m z^m + a_{m+1} z^{m+1} + \dots + a_{n-1} z + a_n] \\ &= a_0 z^n (z+k-1) - \{ (ka_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + (a_2 - a_3) z^{n-2} \\ &\quad + \dots + (a_{n-3} - a_{n-2}) z^3 + (a_{n-2} - a_{n-1}) z^2 + (a_{n-1} - a_n) z + a_n \}. \end{aligned}$$

Also if  $|z| > 1$  then  $\frac{1}{|z|^{n-i}} < 1$  for  $i = 0, 1, \dots, n-1$ . Now

$$\begin{aligned} |R(z)| &\geq |a_0| |z|^n |z+k-1| - \{ |ka_0 - a_1| |z|^n + |a_1 - a_2| |z|^{n-1} + |a_2 - a_3| |z|^{n-2} \\ &\quad + \dots + |a_{n-3} - a_{n-2}| |z|^3 + |a_{n-2} - a_{n-1}| |z|^2 + |a_{n-1} - a_n| |z| + |a_n| \} \\ &\geq |a_0| |z|^n \left[ |z+k-1| - \frac{1}{|a_0|} \left\{ |ka_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_2 - a_3|}{|z|^3} \right. \right. \\ &\quad \left. \left. + \dots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \right\} \right] \\ &\geq |a_0| |z|^n \left[ |z+k-1| - \frac{1}{|a_0|} \{ |ka_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| \right. \\ &\quad \left. + \dots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} + \rho - a_n - \rho| + |a_n| \} \right] \\ &\geq |a_0| |z|^n \left[ |z+k-1| - \frac{1}{|a_0|} \{ (ka_0 - a_1) + (a_2 - a_1) + (a_2 - a_3) + (a_4 - a_3) + \dots \right. \\ &\quad \left. + (a_{n-2} - a_{n-3}) + (a_{n-2} - a_{n-1}) + (a_n + \rho - a_{n-1}) + \rho + |a_n| \} \right] \text{ if } n \text{ is even} \\ &= |a_0| |z|^n \left[ |z+k-1| - \frac{1}{|a_0|} \{ ka_0 + |a_n| + a_n + 2\rho + M_1 \} \right] \end{aligned}$$

where  $M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})]$ .

$$\Rightarrow |R(z)| > 0 \text{ if } |z+k-1| > \frac{1}{|a_0|} \{ ka_0 + |a_n| + a_n + 2\rho + M_1 \}.$$

This shows that all the zeros of  $R(z)$  whose modulus is greater than 1 lie in the closed disk  $|z+k-1| \leq \frac{1}{|a_0|} \{ ka_0 + |a_n| + a_n + 2\rho + M_1 \}$ .

But those zeros of  $R(z)$  whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of  $R(z)$  and hence the zeros of  $J(z)$  lie in

$$|z+k-1| \leq \frac{1}{|a_0|} \{ ka_0 + |a_n| + a_n + 2\rho + M_1 \}.$$

Since  $P(z) = z^n J(\frac{1}{z})$ , by replacing  $z$  by  $\frac{1}{z}$ , we get that if  $n$  is even, all the zeros of  $P(z)$  lie in

$$|z+k-1| \geq \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + M_1}.$$

Hence, if  $n$  is even,  $P(z)$  does not vanish in the disk

$$|z+k-1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + M_1},$$

where  $M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})]$ .

Similarly we can prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is if  $n$  is odd then  $P(z)$  does not vanish in the disk

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| - a_n + 2\rho + M_2},$$

where  $M_2 = 2[(a_2 + a_4 + \dots + a_{n-3} + a_{n-1}) - (a_1 + a_3 + \dots + a_{n-4} + a_{n-2})]$ .

This completes the proof of the Theorem 1. □

*Proof of Theorem 2.* Let  $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  be a polynomial of degree  $n$ . Consider the polynomials  $J(z) = z^n P(\frac{1}{z})$  and  $R(z) = (z - 1)J(z)$  so that

$$\begin{aligned} R(z) &= (z - 1)[a_0 z^n + a_1 z^{n-1} + \dots + a_{m-1} z^{m-1} + a_m z^m + a_{m+1} z^{m+1} + \dots + a_{n-1} z + a_n] \\ &= a_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + (a_2 - a_3)z^{n-2} \\ &\quad + \dots + (a_{n-3} - a_{n-2})z^3 + (a_{n-2} - a_{n-1})z^2 + (a_{n-1} - a_n)z + a_n\}. \end{aligned}$$

Also if  $|z| > 1$  then  $\frac{1}{|z|^{n-i}} < 1$  for  $i = 0, 1, \dots, n - 1$ . Now

$$\begin{aligned} |R(z)| &\geq |a_0||z|^{n+1} - \{|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + |a_2 - a_3||z|^{n-2} \\ &\quad + \dots + |a_{n-3} - a_{n-2}||z|^3 + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n|z + |a_n|\} \\ &\geq |a_0||z|^n \left[ |z| - \frac{1}{|a_0|} \left\{ |a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_2 - a_3|}{|z|^3} \right. \right. \\ &\quad \left. \left. + \dots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \right\} \right] \\ &\geq |a_0||z|^n \left[ |z| - \frac{1}{|a_0|} \left\{ |a_0 - ra_0 + ra_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| \right. \right. \\ &\quad \left. \left. + \dots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} + \rho - a_n - \rho| + |a_n| \right\} \right] \\ &\geq |a_0||z|^n \left[ |z| - \frac{1}{|a_0|} \left\{ (a_1 - ra_0) + (1 - r)|a_0| + (a_1 - a_2) + (a_3 - a_2) + (a_3 - a_4) + \right. \right. \\ &\quad \left. \left. \dots + (a_{n-3} - a_{n-2}) + (a_{n-1} - a_{n-2}) + (a_{n-1} + \rho - a_n) + \rho + |a_n| \right\} \right] \text{ if } n \text{ is even} \\ &= |a_0||z|^n \left[ |z| - \frac{1}{|a_0|} \left\{ 2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1 \right\} \right] \end{aligned}$$

where  $M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})]$ .

$$\Rightarrow |R(z)| > 0 \text{ if } |z| > \frac{1}{|a_0|} \{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1\}.$$

This shows that all the zeros of  $R(z)$  whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|} \{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1\}.$$

But those zeros of  $R(z)$  whose modulus is less than or equal to 1 already lie in the above disk.

Therefore, it follows that all the zeros of  $R(z)$  and hence the zeros of  $J(z)$  lie in

$$|z| \leq \frac{1}{|a_0|} \{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1\}.$$

Since  $P(z) = z^n J(\frac{1}{z})$ , by replacing  $z$  by  $\frac{1}{z}$ , we get that if  $n$  is even, all the zeros of  $P(z)$  lie in

$$|z| \geq \frac{|a_0|}{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1}.$$

Hence if  $n$  is even,  $P(z)$  does not vanish in the disk

$$|z| < \frac{|a_0|}{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1}.$$

where  $M_1 = 2[(a_2 + a_4 + \dots + a_{n-4} + a_{n-2}) - (a_1 + a_3 + \dots + a_{n-3} + a_{n-1})]$ .

Similarly we can prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is if  $n$  is odd then  $P(z)$  does not vanish in the disk

$$|z| < \frac{|a_0|}{2\rho + |a_0| + |a_n| + a_n - r(|a_0| + a_0) - M_2}.$$

where  $M_2 = 2[(a_2 + a_4 + \dots + a_{n-3} + a_{n-1}) - (a_1 + a_3 + \dots + a_{n-4} + a_{n-2})]$ .

This completes the proof of the Theorem 2.  $\square$

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