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ZERO-FREE REGION FOR POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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Abstract. In this paper we prove some extension of the Eneström-Kakeya theorem which says that if $P(z) = \sum_{i=1}^{n} a_i z^i$ be a polynomial of degree n such that $0 \leq a_n \leq a_{n-1} \leq ... \leq a_1 \leq a_0$ then $P(z)$ does not vanish in $|z| \leq 1$. By relaxing the hypothesis of this result in several ways one can obtain zero-free regions for polynomials with restricted coefficients and there by present some interesting generalizations and extensions of the Eneström-Kakeya Theorem.

1. Introduction

The well known Results namely Eneström-Kakeya theorem [1, 2] in the theory of the distribution of zeros of polynomials is the following.

Theorem 1. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0 < a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$, then all the zeros of P(z) lie in $|z| \leq 1$.*

Applying the above result to the polynomial $z^n P(\frac{1}{z})$, we get the following result:

Theorem 2. *If $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0 < a_n \leq a_{n-1} \leq ... \leq a_1 \leq a_0$, then $P(z)$ does not vanish in $|z| < 1$.*

In the literature [3, 4, 5, 6, 7, 8], there exist several extensions and generalizations of the Eneström-Kakeya theorem.

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results:

Theorem 3. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ with real coefficients such that for some $k \geq 1$, $\rho \geq 0$ and*

i) $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq ... \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n + \rho$ if n is even,

OR

ii) $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq ... \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n - \rho$ if n is odd.

Then

i) $P(z)$ does not vanish in the disk

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + M_1} \text{ if } n \text{ is even}$$

where $M_1 = 2\big[(a_2 + a_4 + ... + a_{n-4} + a_{n-2}) - (a_1 + a_3 + ... + a_{n-3} + a_{n-1})\big]$,

OR

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ii) $P(z)$ *does not vanish in the disk*

$$|z+k-1| \ < \ \frac{|a_0|}{ka_0+|a_n|-a_n+2\rho+M_2} \text{ if } n \text{ is odd}$$

where $M_2 = 2\big[(a_2+a_4+...+a_{n-3}+a_{n-1})-(a_1+a_3+...+a_{n-4}+a_{n-2})\big]$.

Corollary 1. *Let* $P(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that*

i) $a_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq ... \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n$ *if* n *is even,*

OR

ii) $a_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq ... \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n$ *if* n *is odd.*

Then

i) $P(z)$ *does not vanish in the disk*

$$|z| \ < \ \frac{|a_0|}{a_0+|a_n|+a_n+M_1} \text{ if } n \text{ is even}$$

where $M_1 = 2\big[(a_2+a_4+...+a_{n-4}+a_{n-2})-(a_1+a_3+...+a_{n-3}+a_{n-1})\big]$,

OR

ii) $P(z)$ *does not vanish in the disk*

$$|z| \ < \ \frac{|a_0|}{a_0+|a_n|-a_n+M_2} \text{ if } n \text{ is odd}$$

where $M_2 = 2\big[(a_2+a_4+...+a_{n-3}+a_{n-1})-(a_1+a_3+...+a_{n-4}+a_{n-2})\big]$.

Corollary 2. *Let* $P(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with positive real coefficients such that for some* $k \geq 1$, $\rho \geq 0$ *and*

i) $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq ... \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n+\rho$ *if* n *is even,*

OR

ii) $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq ... \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n-\rho$ *if* n *is odd.*

Then

i) $P(z)$ *does not vanish in the disk*

$$|z+k-1| \ < \ \frac{a_0}{ka_0+2a_n+2\rho+M_1} \text{ if } n \text{ is even,}$$

where $M_1 = 2\big[(a_2+a_4+...+a_{n-4}+a_{n-2})-(a_1+a_3+...+a_{n-3}+a_{n-1})\big]$,

OR

ii) $P(z)$ *does not vanish in the disk*

$$|z+k-1| \ < \ \frac{a_0}{ka_0+2\rho+M_2} \text{ if } n \text{ is odd}$$

where $M_2 = 2\big[(a_2+a_4+...+a_{n-3}+a_{n-1})-(a_1+a_3+...+a_{n-4}+a_{n-2})\big]$.

Remark 1. *By taking* $\rho = 0$, $r = 1$ *in Theorem 1, it is reduces to Corollary 1.*

Remark 2. *By taking* $a_i > 0$, *for* $i = 0, 1, ..., n$ *in Theorem 1, it reduces to Corollary 2.*

Theorem 4. *Let* $P(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that for some* $0 < r \leq 1$, $\rho \geq 0$ *and*

i) $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq ... \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n-\rho$ *if* n *is even,*

OR

ii) $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq ... \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n+\rho$ *if* n *is odd.*

Then

i) $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|a_0|}{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1} \text{ if } n \text{ is even}$$

where $M_1 = 2\big[(a_2 + a_4 + ... + a_{n-4} + a_{n-2}) - (a_1 + a_3 + ... + a_{n-3} + a_{n-1})\big]$,

OR

ii) $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|a_0|}{2\rho + |a_0| + |a_n| + a_n - r(|a_0| + a_0) - M_2} \text{ if } n \text{ is odd}$$

where $M_2 = 2\big[(a_2 + a_4 + ... + a_{n-3} + a_{n-1}) - (a_1 + a_3 + ... + a_{n-4} + a_{n-2})\big]$.

Corollary 3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that*

i) $a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq ... \geq a_{n-4} \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n$ *if* n *is even,*

OR

ii) $a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq ... \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n$ *if* n *is odd.*

Then

i) $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|a_0|}{|a_n| - a_n - a_0 - M_1} \text{ if } n \text{ is even}$$

where $M_1 = 2\big[(a_2 + a_4 + ... + a_{n-4} + a_{n-2}) - (a_1 + a_3 + ... + a_{n-3} + a_{n-1})\big]$,

OR

ii) $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|a_0|}{|a_n| + a_n - a_0 - M_2} \text{ if } n \text{ is odd}$$

where $M_2 = 2\big[(a_2 + a_4 + ... + a_{n-3} + a_{n-1}) - (a_1 + a_3 + ... + a_{n-4} + a_{n-2})\big]$.

Corollary 4. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with positive real coefficients such that for some* $0 < r \leq 1$, $\rho \geq 0$ *and*

i) $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq ... \leq a_{n-3} \geq a_{n-2} \leq a_{n-1} \geq a_n - \rho$ *if* n *is even,*

OR

ii) $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq ... \leq a_{n-4} \geq a_{n-3} \leq a_{n-2} \geq a_{n-1} \leq a_n + \rho$ *if* n *is odd.*

Then

i) $P(z)$ *does not vanish in the disk*

$$|z| < \frac{a_0}{2\rho + (1 - 2r)a_0 - M_1} \text{ if } n \text{ is even}$$

where $M_1 = 2\big[(a_2 + a_4 + ... + a_{n-4} + a_{n-2}) - (a_1 + a_3 + ... + a_{n-3} + a_{n-1})\big]$,

OR

ii) $P(z)$ *does not vanish in the disk*

$$|z| < \frac{a_0}{2\rho + 2a_n + (1 - 2r)a_0 - M_2} \text{ if } n \text{ is odd}$$

where $M_2 = 2\big[(a_2 + a_4 + ... + a_{n-3} + a_{n-1}) - (a_1 + a_3 + ... + a_{n-4} + a_{n-2})\big]$.

Remark 3. *By taking* $\rho = 0$, $r = 1$ *in Theorem 2, it reduces to Corollary 3.*

Remark 4. *By taking $a_i > 0$, for $i = 0, 1, ..., n$ in Theorem 2, it reduces to Corollary 4.*

2. Proofs of the Theorems

Proof of Theorem 1. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degree n. Consider the polynomials $J(z) = z^n P(\frac{1}{z})$ and $R(z) = (z-1)J(z)$ so that

$$\begin{aligned}R(z) =&(z-1)[a_0 z^n + a_1 z^{n-1} + ... + a_{m-1}z^{m-1} + a_m z^m + a_{m+1}z^{m+1} + ... + a_{n-1}z + a_n]\\ =&a_0 z^n(z+k-1) - \big\{(ka_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + (a_2 - a_3)z^{n-2}\\ &+ ... + (a_{n-3} - a_{n-2})z^3 + (a_{n-2} - a_{n-1})z^2 + (a_{n-1} - a_n)z + a_n\big\}.\end{aligned}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1$ for $i = 0, 1, ..., n-1$. Now

$$\begin{aligned}|R(z)| \geq\ & |a_0||z|^n|z+k-1| - \big\{|ka_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + |a_2 - a_3||z|^{n-2}\\ &+ ... + |a_{n-3} - a_{n-2}||z|^3 + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n|z + |a_n|\big\}\\ \geq\ & |a_0||z|^n\Bigg[|z+k-1| - \frac{1}{|a_0|}\big\{|ka_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_2 - a_3|}{|z|^3}\\ &+ ... + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\big\}\Bigg]\\ \geq\ & |a_0||z|^n\Bigg[|z+k-1| - \frac{1}{|a_0|}\big\{|ka_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4|\\ &+ ... + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} + \rho - a_n - \rho| + |a_n|\big\}\Bigg]\\ \geq\ & |a_0||z|^n\Bigg[|z+k-1| - \frac{1}{|a_0|}\{(ka_0 - a_1) + (a_2 - a_1) + (a_2 - a_3) + (a_4 - a_3) + ...\\ &+ (a_{n-2} - a_{n-3}) + (a_{n-2} - a_{n-1}) + (a_n + \rho - a_{n-1}) + \rho + |a_n|\}\Bigg] \text{ if } n \text{ is even}\\ =&|a_0||z|^n\Bigg[|z+k-1| - \frac{1}{|a_0|}\big\{ka_0 + |a_n| + a_n + 2\rho + M_1\big\}\Bigg]\end{aligned}$$

where $M_1 = 2\big[(a_2 + a_4 + ... + a_{n-4} + a_{n-2}) - (a_1 + a_3 + ... + a_{n-3} + a_{n-1})\big]$.

$$\Rightarrow |R(z)|\ >\ 0\ if\ |z+k-1|\ >\ \frac{1}{|a_0|}\big\{ka_0 + |a_n| + a_n + 2\rho + M_1\big\}.$$

This shows that all the zeros of $R(z)$ whose modulus is greater than 1 lie in the closed disk $|z+k-1| \leq \frac{1}{|a_0|}\big\{ka_0 + |a_n| + a_n + 2\rho + M_1\big\}$.

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of $R(z)$ and hence the zeros of $J(z)$ lie in

$$|z+k-1| \leq \frac{1}{|a_0|}\big\{ka_0 + |a_n| + a_n + 2\rho + M_1\big\}.$$

Since $P(z) = z^n J(\frac{1}{z})$, by replacing z by $\frac{1}{z}$, we get that if n is even, all the zeros of $P(z)$ lie in

$$|z+k-1| \geq \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + M_1}.$$

Hence, if n is even, $P(z)$ does not vanish in the disk

$$|z+k-1|\ <\ \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + M_1},$$

where $M_1 = 2\big[(a_2 + a_4 + ... + a_{n-4} + a_{n-2}) - (a_1 + a_3 + ... + a_{n-3} + a_{n-1})\big]$.

Similarly we can prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is if n is odd then $P(z)$ does not vanish in the disk

$$|z + k - 1| \ < \ \frac{|a_0|}{ka_0 + |a_n| - a_n + 2\rho + M_2},$$

where $M_2 = 2\big[(a_2 + a_4 + ... + a_{n-3} + a_{n-1}) - (a_1 + a_3 + ... + a_{n-4} + a_{n-2})\big]$.

This completes the proof of the Theorem 1. □

Proof of Theorem 2. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degree n. Consider the polynomials $J(z) = z^n P(\frac{1}{z})$ and $R(z) = (z-1)J(z)$ so that

$$\begin{aligned} R(z) =&(z-1)[a_0 z^n + a_1 z^{n-1} + ... + a_{m-1} z^{m-1} + a_m z^m + a_{m+1} z^{m+1} + ... + a_{n-1} z + a_n] \\ =&a_0 z^{n+1} - \big\{(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + (a_2 - a_3) z^{n-2} \\ &+ ... + (a_{n-3} - a_{n-2}) z^3 + (a_{n-2} - a_{n-1}) z^2 + (a_{n-1} - a_n) z + a_n\big\}. \end{aligned}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1$ for $i = 0, 1, ..., n-1$. Now

$$\begin{aligned} |R(z)| \geq\ & |a_0||z|^{n+1} - \big\{|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + |a_2 - a_3||z|^{n-2} \\ & + ... + |a_{n-3} - a_{n-2}||z|^3 + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n|z + |a_n|\big\} \\ \geq\ & |a_0||z|^n \Bigg[|z| - \frac{1}{|a_0|}\big\{|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_2 - a_3|}{|z|^3} \\ & + ... + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\big\}\Bigg] \\ \geq\ & |a_0||z|^n \Bigg[|z| - \frac{1}{|a_0|}\big\{|a_0 - ra_0 + ra_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| \\ & + ... + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} + \rho - a_n - \rho| + |a_n|\big\}\Bigg] \\ \geq\ & |a_0||z|^n \Bigg[|z| - \frac{1}{|a_0|}\{(a_1 - ra_0) + (1-r)|a_0| + (a_1 - a_2) + (a_3 - a_2) + (a_3 - a_4) + \\ & ... + (a_{n-3} - a_{n-2}) + (a_{n-1} - a_{n-2}) + (a_{n-1} + \rho - a_n) + \rho + |a_n|\}\Bigg] \text{ if } n \text{ is even} \\ =&|a_0||z|^n \Bigg[|z| - \frac{1}{|a_0|}\big\{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1\big\}\Bigg] \end{aligned}$$

where $M_1 = 2\big[(a_2 + a_4 + ... + a_{n-4} + a_{n-2}) - (a_1 + a_3 + ... + a_{n-3} + a_{n-1})\big]$.

$\Rightarrow |R(z)| \ > \ 0$ if $|z| \ > \ \frac{1}{|a_0|}\big\{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1\big\}$.

This shows that all the zeros of $R(z)$ whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\big\{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1\big\}.$$

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk.

Therefore, it follows that all the zeros of $R(z)$ and hence the zeros of $J(z)$ lie in

$$|z| \leq \frac{1}{|a_0|}\big\{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1\big\}.$$

Since $P(z) = z^n J(\frac{1}{z})$, by replacing z by $\frac{1}{z}$, we get that if n is even, all the zeros of $P(z)$ lie in

$$|z| \geq \frac{|a_0|}{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1}.$$

Hence if n is even, $P(z)$ does not vanish in the disk

$$|z| \ < \ \frac{|a_0|}{2\rho + |a_0| + |a_n| - a_n - r(|a_0| + a_0) - M_1}.$$

where $M_1 = 2\big[(a_2 + a_4 + ... + a_{n-4} + a_{n-2}) - (a_1 + a_3 + ... + a_{n-3} + a_{n-1})\big]$.

Similarly we can prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is if n is odd then $P(z)$ does not vanish in the disk

$$|z| \ < \ \frac{|a_0|}{2\rho + |a_0| + |a_n| + a_n - r(|a_0| + a_0) - M_2}.$$

where $M_2 = 2\big[(a_2 + a_4 + ... + a_{n-3} + a_{n-1}) - (a_1 + a_3 + ... + a_{n-4} + a_{n-2})\big]$.

This completes the proof of the Theorem 2. □

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