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SOME RESULTS ON PSEUDO PARALLEL IMMERSION

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Abstract. We study the properties of pseudo parallel immersions of Kahler manifolds. We also find the bound for scalar curvature under these conditions.

1. Introduction

We study pseudo parallel immersions with first normal bundle in space forms. The concept of pseudo-parallel immersions onto a space form $Q^N(c)$ of constant curvature c was introduced in [3]. A. Yildiz and C. Murathan in [2] study the properties of the second fundamental form of Kaehlerian submanifolds of a complex m-dimensional Kaehlerian manifold $\tilde{M}^m$ under Ricci Generalized Pseudo-parallel condition. The condition of pseudo-parallel immersion is being well explored. Recently in [4], a new geometrical view of this condition is discussed.

2. Preliminaries

Let $\tilde{M}(c)$ be a non-flat complex space form endowed with the metric g of constant holomorphic sectional curvature c. We define endomorphisms $R(X,Y)$ and $X \wedge_B Y$ of $\chi(M)$, the Lie algebra of vector fields on M by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y, \tag{2}$$

where $X, Y, Z \in \chi(M)$ and B is a symmetric $(0,2)-$ tensor.

Let $f : M^n \to \tilde{M}^m$ be a pseudo parallel immersion. Then the first normal space at $x \in M^n$ is defined as

$$N_1(x) = span\{h(X,Y) : X, Y \in T_x M\} \tag{3}$$

where $h(X,Y)$ is the second fundamental form on M^n.

3. Pseudo parallel submanifolds

Lemma 1. [1] *Let M^n be a complex n-dimensional Kaehlerian submanifold of a complex m-dimensional Kaehlerian manifold $\tilde{M}^m$. Then*

$$\frac{1}{n} \parallel h \parallel^4 \leq \sum_{\alpha,\beta=1}^{m-n} \parallel [A_\alpha, A_\beta] \parallel^2 \leq \parallel h \parallel^4, \tag{4}$$

$$\frac{1}{2(m-n)} \parallel h \parallel^4 \leq \sum_{\alpha,\beta=1}^{m-n} (TrA_\alpha A_\beta)^2 \leq \frac{1}{2} \parallel h \parallel^4, \tag{5}$$

where A_α, A_β are the Weingarten operators.

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Lemma 2. [3] *Let $f : M^n \to Q^N(c), n \geq 3$ be a pseudo parallel immersion of a complete manifold with dim $N_1(x) = \frac{1}{2}n(n+1)$. Then*

$$\sum_{\alpha,\beta=1}^{m-n} \| [A_\alpha, A_\beta] \|^2 = n^2 \| H \|^4 + 2(n-1)(n+2)(K+L)^2 \tag{6}$$

$$\text{and} \quad \sum_{\alpha,\beta=1}^{m-n} (TrA_\alpha A_\beta)^2 = 2n(n-1)(n+2)(K+L)^2. \tag{7}$$

Theorem 1. [3] *Let $f : M^n \to Q^N(c), n \geq 3$, be a pseudo parallel immersion. If dim $N_1(x) = \frac{1}{2}n(n+1)$ for all $x \in M^n$, then M has constant sectional curvature K and $K + \phi > 0$.*

Theorem 2. *Let $f : M^n \to \tilde{M}^m(c)$, $n \geq 3$ be a pseudo parallel immersion. Let $N_1(x) = span\{\alpha(X,Y) : X, Y \in T_xM\}$ be such that $dimN_1(x) = \frac{n(n+1)}{2}$, where M^n is a Kahlerian submanifold. Then*

$$n < \frac{2m}{3}. \tag{8}$$

Proof. Substituting (6) in (4) and (7) in (5), we get

$$\frac{1}{n} \| h \|^4 \leq 2n(n-1)(n+2)(K+L)^2 \leq \| h \|^4 \tag{9}$$

$$\frac{1}{2(m-n)} \| h \|^4 \leq 2(n-1)(n+2)(K+L)^2 \leq \frac{\| h \|^4}{2}. \tag{10}$$

Multiplying (9) throughout by $\frac{1}{2(m-n)}$ and comparing with (10) we get the required result. □

Theorem 3. *Let $\tilde{M}(c)$ be a complex m-dimensional space form of constant holomorphic sectional curvature c and M^n be a complex n-dimensional Kaehlerian submanifold of $\tilde{M}^n(c)$. If M^n is Ricci generalised pseudo-parallel and dim $N_1(x) = \frac{n(n+1)}{2}$, $n \geq 3$ then*

$$K = \frac{4(n+1)}{c(n+2) - 2L\tau} - L. \tag{11}$$

Proof. From the condition of Ricci generalized pseudo parallel immersion, we have

$$\bar{R}(e_l, e_k).h = L[(e_l \wedge_s e_k)]h, \tag{12}$$

where

$$[(e_l \wedge_S e_k)h](e_i, e_j) = -h((e_l \wedge_S e_k)e_i, e_j) - h(e_i, (e_l \wedge_S e_k)e_j) \tag{13}$$

for $1 \leq i, j, k, l \leq n$ and

$$(\bar{R}(e_l, e_k).h)(e_i, e_j) = (\bar{\nabla}_{e_l}\bar{\nabla}_{e_k}h)(e_i, e_j) - (\bar{\nabla}_{e_k}\bar{\nabla}_{e_l}h)(e_i, e_j). \tag{14}$$

But $(e_l \wedge_S e_k)e_i = S(e_k, e_i)e_l - S(e_l, e_i)e_k$. Hence equation (13) becomes,

$$\begin{aligned}[(e_l \wedge_S e_k)h](e_i, e_j) = &- S(e_k, e_i)h(e_l, e_j) + h(e_j, e_k)S(e_l, e_i) - h(e_l, e_i)S(e_k, e_j) \\ &+ h(e_k, e_i)S(e_l, e_j).\end{aligned} \tag{15}$$

Substituting (13) and (14) in (12) and simplifying we get,

$$(\bar{\nabla}_{e_l}\bar{\nabla}_{e_k}h)(e_i, e_j) - (\bar{\nabla}_{e_k}\bar{\nabla}_{e_l}h)(e_i, e_j) = -L\{S_{ki}h^\alpha_{lj} - S_{li}h^\alpha_{kj} + S_{kj}h^\alpha_{il} - S_{lj}h^\alpha_{ki}\}. \tag{16}$$

Hence

$$h^\alpha_{ijkl} = h^\alpha_{ijlk} - L\{S_{ki}h^\alpha_{lj} - S_{li}h^\alpha_{kj} + S_{kj}h^\alpha_{il} - S_{lj}h^\alpha_{ki}\}, \tag{17}$$

where $S(e_i, e_j) = S_{ij}$ and $1 \leq i, j, k, l \leq n$, $1 \leq \alpha \leq p$.

The Laplacian $\Delta h^\alpha_{ij} = \sum_{i,j,k=1}^n h^\alpha_{ijkk}$ and $\| h \|^2 = \sum_{i,j,k=1}^n \sum_{\alpha=1}^p (h^\alpha_{ij})^2$. Then we have $\| h \|^2 = \sum_{i,j,k=1}^n \sum_{\alpha=1}^p h^\alpha_{ij} h^\alpha_{ijkk} + \| \bar{\nabla} h \|^2$.

It is shown [2] by direct calculations and using a result of Yano and Kon [1]

$$\frac{1}{2}\Delta(\| h \|^2) = \| \nabla h \|^2 - \sum_{\alpha,\beta=1}^{p} \{[Tr(A_\alpha \circ A_\beta)]^2 + \| [A_\alpha, A_\beta] \|^2\} + \frac{1}{2}(n+2)c \| h \|^2 \quad (18)$$

that

$$(L\tau - \frac{1}{2}(n+2)c) \| h \|^2 + \sum_{\alpha,\beta=1}^{p} \{[Tr(A_\alpha \circ A_\beta)]^2 + \| [A_\alpha, A_\beta] \|^2\} = 0. \quad (19)$$

Using equations (6) and (7) in (19) we get

$$(L\tau - \frac{1}{2}(n+2)c) \| h \|^2 + 2(n+1)(n-1)(n+2)(K+L)^2 = 0. \quad (20)$$

But $\| h \|^2 = (n-1)(n+2)(K+L)$. Therefore equation (20) becomes

$$(L\tau - \frac{1}{2}(n+2)c)(n-1)(n+2)(K+L) + 2(n+1)(n-1)(n+2)(K+L)^2 = 0. \quad (21)$$

That is

$$[(K+L)(L\tau - \frac{1}{2}(n+2)c) + 2(n+1)](n-1)(n+2)(K+L) = 0. \quad (22)$$

Hence

$$(K+L)(L\tau - \frac{1}{2}(n+2)c) + 2(n+1) = 0 \quad (23)$$

from which we get the required result. □

Corollary 1. *Let $\tilde{M}(c)$ be a complex m-dimensional space form of constant holomorphic sectional curvature c and M^n be a complex n-dimensional Kaehlerian submanifold of $\tilde{M}^n(c)$. If M^n is Ricci generalised pseudo-parallel and dim $N_1(x) = \frac{n(n+1)}{2}$, $n \geq 3$ then the scalar curvature $\tau \leq \frac{c(n+2)}{2L}$.*

Proof. From theorem (1), $K+L > 0$. Hence from equation (23), we have $\frac{c(n+2)}{2} - L\tau > 0$. i.e. $\frac{c(n+2)}{2} > L\tau$ or $\tau < \frac{c(n+2)}{2L}$. □

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