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NEW INFLUENCE FUNCTIONS FOR THERMAL DISPLACEMENTS AND STRESSES WITHIN HALF-SPACE

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ABSTRACT. In this study new influence functions for the thermal displacements and stresses caused by a unit point heat source for three-dimensional boundary value problem of thermoelasticity within half-space were obtained. These results are presented in terms of elementary functions. Using the computer program Maple 18, the graphical presentations of thermal displacements and stresses caused by a unit point heat source were constructed.

1. INTRODUCTION

Determination of states of deformations and stresses caused by a internal heat source, temperature and other thermal actions on the body surface is a difficult mathematical problem that requires the creation of special theories. To obtain new integral solutions was required developed a new method, Harmonic Integral Representation Method (HIRM) proposed by V. Şeremet. Using integral solutions can be calculated thermal displacements and stresses. By generalizing Maysel's integral formula [1] and Green's integral formulas, V. Șeremet proposed a new form of these integral formulas [2, 3, 4, 5]:

$$u_i(\xi) = a^{-1} \int\limits_V F(x) U_i(x,\xi) dV(x) - \int\limits_{\Gamma_D} T(y) \frac{\partial U_i(y,\xi)}{\partial n_y} d\Gamma_D(y)$$

$$+ \int\limits_{\Gamma_N} \frac{\partial T(y)}{\partial n_y} U_i(y,\xi) d\Gamma_N(y) + a^{-1} \int\limits_{\Gamma_M} \left[\alpha T(y) + a \frac{\partial T(y)}{\partial n_y} \right] U_i(y,\xi) d\Gamma_M(y); i = 1,2,3, \quad (1)$$

where:

Γ_D, Γ_N and Γ_M are the parts of the body surface $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_M$, which are given: the Dirichlet's boundary conditions (temperature $T(y)$), the Neumann's boundary conditions (heat flux $a\frac{\partial T(y)}{\partial n_y}$) and mixed boundary conditions (heat exchange between exterior medium and surface of the body represented by law $\left[\alpha T(y) + a\frac{\partial T(y)}{\partial n_y}\right]$) are prescribed; a is thermal conductivity; $F(x)$ is the internal heat source; α is the coefficient of convective heat conductivity; $\gamma = \alpha_t(2\mu + 3\lambda)$ is the thermoelastic constant; α_t is the coefficient of the linear thermal expansion, but λ, μ are Lame's constants of elasticity.

The thermal stresses for three-dimensional canonical domains of Cartesian system of coordinates will be calculated by using the following type of Green's integral formula [2]:

$$\sigma_{ij}(\xi) = a^{-1} \int\limits_V F(x) \overline{\sigma}_{ij}(x,\xi) dV(x) - \int\limits_{\Gamma_D} T(y) \frac{\partial \overline{\sigma}_{ij}(y,\xi)}{\partial n_y} d\Gamma_D(y) + \int\limits_{\Gamma_N} \frac{\partial T(y)}{\partial n_y} \overline{\sigma}_{ij}(y,\xi) d\Gamma_N(y)$$

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$$+ a^{-1} \int\limits_{\Gamma_M} \left[\alpha T(y) + a\frac{\partial T(y)}{\partial n_y}\right] \overline{\sigma}_{ij}(y,\xi) d\Gamma_M(y); i,j = 1,2,3, \quad (2)$$

where:
$\overline{\sigma}_{ij}$ represents the influence functions for thermal stresses of a unit point heat source and σ_{ij} represent the thermal stress caused by internal heat source, temperature, heat flux or a heat exchange between exterior medium and surface of the body.

The thermal stresses $\overline{\sigma}_{ij}$ can be determined of Duhamel-Neumann law [6]:

$$\overline{\sigma}_{ij} = \mu(U_{i,j} + U_{j.i}) + \delta_{ij}(\lambda\Theta - \gamma G_T); \Theta = U_{k,k}(x,\xi); i,j,k = 1,2,3, \quad (3)$$

where:
δ_{ij} - Kronecker's symbol, which is equal to 1, if $i = j$ and 0, if $i \neq j$.

2. Thermal stresses $\overline{\sigma}_{ij}$ within half-space caused by a unit point heat source

It is required to determine thermal stresses $\overline{\sigma}_{ij}(x,\xi); i,j = 1,2,3$ of a boundary value problem in the half-space $S(0 \leq x_1 < \infty, -\infty < x_2, x_3 < \infty)$ with thermal boundary condition of Dirichlet type for Green's function:

$$G_T = 0. \quad (4)$$

The mechanical boundary conditions on the marginal plan $\Gamma_{10}(y_1 = 0; -\infty < y_2 < \infty; -\infty < y_3 < \infty)$ are:

$$U_1 = U_2 = U_3 = 0. \quad (5)$$

The mechanical and thermal boundary conditions of a boundary value problems within half-space are showed in the Figure 1.

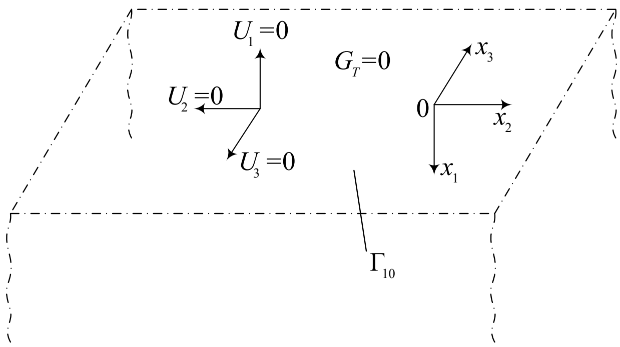

Figure 1. The scheme of the half-space $S(0 \leq x_1 < \infty, -\infty < x_2, x_3 < \infty)$ with the mechanical boundary conditions U_1, U_2, U_3 and the thermal boundary conditions G_T on the marginal plan Γ_{10}.

To solve this problem it is necessary to determine thermoelastic displacements $U_i(x,\xi)$ caused by a unit point heat source.

In the field literature [7], [8] boundary value problems within half-space are solved by using ΘG - convolution method, but with other mechanical and thermal boundary conditions.

In this paper for the first time the thermoelastic displacements $U_i(x,\xi); i = 1,2,3$ and thermal stresses $\overline{\sigma}_{ij}(x,\xi)$ caused by a unit point heat source with boundary conditions (4)

- (5) within half-space using the structural formulas have been proposed. These structural formulas were obtained by HIRM.

2.1. **Determination of thermoelastic displacements** U_i**.** In the half-space $S(0 \leq x_1 < \infty, -\infty < x_2, x_3 < \infty)$, thermoelastic displacements $U_i(x,\xi)$; $x \equiv (x_1,x_2,x_3)$; $\xi \equiv (\xi_1,\xi_2,\xi_3)$ must be calculated. For this must be solved Lame equations:

$$\mu \bigtriangledown_\xi^2 U_i(x,\xi) + (\lambda+\mu)\Theta_{,\xi_i}(x,\xi) - \gamma G_{T,\xi_i}(x,\xi) = 0; i = 1,2,3, \tag{6}$$

with mechanical and thermal boundary conditions on the marginal plan $\Gamma_{10}(y_1 = 0; -\infty < y_2, y_3 < \infty)$:

$$U_1(x,y) = U_2(x,y) = U_3(x,y) = 0; x \in S; G_T(y,\xi) = 0; y \equiv (0,y_2,y_3) \in \Gamma_{10}; \tag{7}$$

To determine the thermal displacements using the structural formulas $U_i(x,\xi)$ and $\Theta(x,\xi)$ which have been demonstrated in theorem 13 of the monograph [2]. In this case thermoelastic displacements take the following form:

$$U_i(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\left[\xi_i G_T(x,\xi) - x_i G_i(x,\xi) - 2x_1\xi_1 B^{-1}\frac{\partial}{\partial \xi_i}W_T(x,\xi)\right], \tag{8}$$

where:

$W_T(x,\xi)$ is regular part of the Green's function $\overline{G}_T(x,\xi)$ - Green's function with reverse boundary condition for marginal plan Γ_{10}: $\dfrac{\partial \overline{G}_T(x,\xi)}{\partial n_y} = -\dfrac{\partial \overline{G}_T(x,\xi)}{\partial y_1} = 0$;

$B = \dfrac{\lambda+3\mu}{\lambda+\mu}$,

and thermoelastic volume dilatation:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu}\left(G_T(x,\xi) + \frac{2\mu}{\lambda+\mu}B^{-1}x_1\frac{\partial}{\partial x_1}W_T(x,\xi)\right). \tag{9}$$

Green's functions $G_T; G_\Theta$ and $G_i; i = 1,2,3$ are connected with the boundary conditions (7) as follows: if on the marginal plan Γ_{10} the thermoelastic displacements are known, then Green's functions are equal to zero:

$$U_1 = U_2 = U_3 = 0; G_T = 0 \Rightarrow G_1 = G_2 = G_3 = G_\Theta = 0. \tag{10}$$

Green's functions $G_T; G_\Theta; G_1; G_2$ and G_3 for the half-space S are extracted from handbook [3] or encyclopedia [9] and these functions will be calculated by the following expressions:

$$G_1 = G_2 = G_3 = G_T = G_\Theta = G^{(1)} = \frac{1}{4\pi}(R^{-1} - R_1^{-1}), \tag{11}$$

where:

$R = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2}$;

$R_1 = \sqrt{(x_1+\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2}$.

Green's function $\overline{G}_T(x,\xi)$ with reverse boundary condition $\overline{G}_{T,1} = 0$ for marginal plan Γ_{10} is calculated from [3, 9]:

$$\overline{G}_T(x,\xi) = \frac{1}{4\pi}(R^{-1} + R_1^{-1}). \tag{12}$$

The regular part of the Green's function $\overline{G}_T(x,\xi)$ (12) is that part which contains inferior index 1, that part of the $\overline{G}_T(x,\xi)$ which are reflected via marginal plan Γ_{10}. So, $W_T(x,\xi)$ of the formulas (8) and (9) is calculated with the following relation:

$$W_T(x,\xi) = \frac{1}{4\pi}(R_1^{-1}). \tag{13}$$

Substituting expressions (11) and (13) in the formula (8) are obtained the final expressions for thermoelastic displacements $U_i(x,\xi)$ within half-space S, which are presented in the following forms:

$$U_1(x,\xi) = \frac{\gamma}{8\pi(\lambda+2\mu)}\left[(\xi_1 - x_1)(R^{-1} - R_1^{-1}) - 2x_1\xi_1 B^{-1}\frac{\partial}{\partial \xi_1}R_1^{-1}\right]; \tag{14}$$

$$U_2(x,\xi) = \frac{\gamma}{8\pi(\lambda+2\mu)}\left[(\xi_2 - x_2)(R^{-1} - R_1^{-1}) - 2x_1\xi_1 B^{-1}\frac{\partial}{\partial \xi_2}R_1^{-1}\right]; \tag{15}$$

$$U_3(x,\xi) = \frac{\gamma}{8\pi(\lambda+2\mu)}\left[(\xi_3 - x_3)(R^{-1} - R_1^{-1}) - 2x_1\xi_1 B^{-1}\frac{\partial}{\partial \xi_3}R_1^{-1}\right]. \tag{16}$$

Graphs of the thermoelastic displacements $U_i(x,\xi)$; $i = 1,2,3$ within half-space S for $0 \le \xi_1 \le 10$, $\xi_2 = 1$, $-10 \le \xi_3 \le 10$ caused by a unit point heat source applied in the point $x_1 = 5m, x_2 = 2m, x_3 = 0$ were constructed using computer program Maple 18. The value of elastic and thermal constants are: the Poisson ration $\nu = 0,3$; modulus of elasticity $E = 2,1 \cdot 10^5 MPa$, and coefficient of linear thermal expansion $\alpha_t = 1,2 \cdot 10^{-5}(K^{-1})$.

Thermoelastic displacements $U_1(x,\xi)$, $U_2(x,\xi)$ and $U_3(x,\xi)$ were constructed using formulas (14)-(16) are presented in the Figure 2, a),b) and c).

Analyzing the Figure 2 graphs one can observe the following:

- the boundary conditions (7) are respected: on the marginal plan Γ_{10}, $U_1(x,\xi) = 0$ (Figure 2, a); $U_2(x,\xi) = 0$ (Figure 2, b); $U_3(x,\xi) = 0$ (Figure 2, c); $\xi \equiv (\xi_1 = 0;$ $-\infty < \xi_2, \xi_3 < \infty)$;
- graph of the thermoelastic displacement $U_2(x,\xi)$ has a local maximum: in the point of application of the unit heat source (Figure 2, b). The others graphs $U_1(x,\xi)$ and $U_3(x,\xi)$ have a discontinuity near this point.
- if $\xi_1 \to \infty$, $\xi_3 \to \pm\infty$, then thermoelastic displacements $U_1(x,\xi)$; $U_2(x,\xi)$ and $U_3(x,\xi) \to 0$.

2.2. **Determination of thermal stresses $\overline{\sigma}_{ij}$.** The thermal stresses $\overline{\sigma}_{ij}(x,\xi)$ are calculated using Duhamel-Neumann law (3), but $\Theta(x,\xi)$ - thermoelastic volume dilatation which is determine by using the formula (9) and has the form:

$$\Theta(x,\xi) = \frac{\gamma}{4\pi(\lambda+2\mu)}\left(R^{-1} - R_1^{-1} + \frac{2\mu}{\lambda+\mu}B^{-1}x_1\frac{\partial}{\partial x_1}R_1^{-1}\right). \tag{17}$$

Substituting Green's function $G_T(x,\xi)$ (11), thermoelastic volume dilatation $\Theta(x,\xi)$ (17) and expressions for thermoelastic displacements $U_i(x,\xi)$; $i = 1,2,3$ (14)-(16) in the Duhamel-Neumann law (3) we obtain the expressions for thermoelastic influence functions for thermal stresses $\overline{\sigma}_{ij}(x,\xi)$:

$$\overline{\sigma}_{11}(x,\xi) = \frac{\gamma\mu}{4\pi(\lambda+2\mu)}\left\{\left[(\xi_1 - x_1)\frac{\partial}{\partial \xi_1} - 1\right]R^{-1} - \left(\xi_1\frac{\partial}{\partial \xi_1} - 1\right)R_1^{-1}\right.$$
$$\left. + 2x_1 B^{-1}\frac{\partial}{\partial \xi_1}R^{-1}\left(1 - 2\xi_1\frac{\partial}{\partial \xi_1}R_1^{-1}\right)\right\}; \tag{18}$$

$$\overline{\sigma}_{12}(x,\xi) = \frac{\gamma\mu}{8\pi(\lambda+2\mu)}\left\{\left[(\xi_1 - x_1)\frac{\partial}{\partial \xi_2} + (\xi_2 - x_2)\frac{\partial}{\partial \xi_1}\right](R^{-1} - R_1^{-1})\right.$$
$$\left. - 4x_1\xi_1 B^{-1}\frac{\partial}{\partial \xi_1}\frac{\partial}{\partial \xi_2}R_1^{-1}\right\}; \tag{19}$$

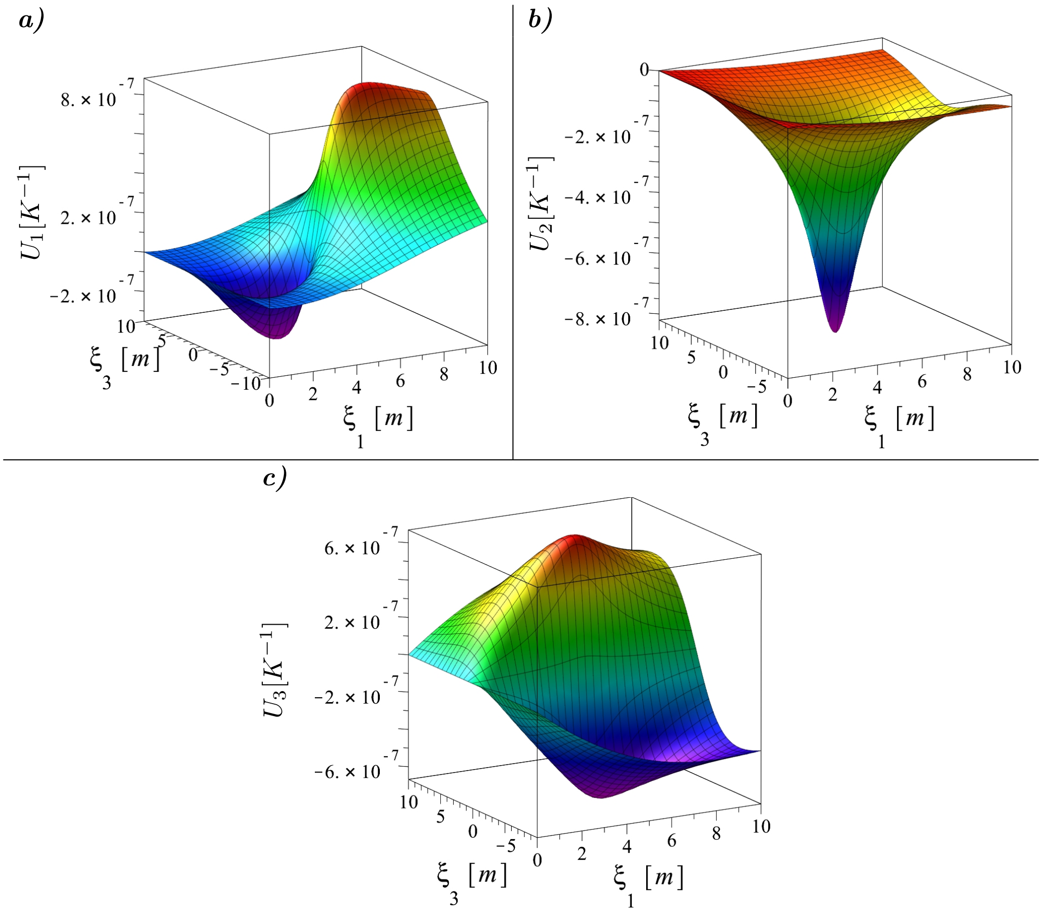

FIGURE 2. Graphs of thermoelastic displacements $U_i(x,\xi)$ in the half-space S for $0 \leq \xi_1 \leq 10$; $\xi_2 = 1$; $-10 \leq \xi_3 \leq 10$ caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 2m, x_3 = 0$.

$$\overline{\sigma}_{22}(x,\xi) = \frac{\gamma\mu}{4\pi(\lambda+2\mu)}\left\{\left[(\xi_2 - x_2)\frac{\partial}{\partial\xi_2} - 1\right]\left(R^{-1} - R_1^{-1}\right)\right.$$
$$\left. +2x_1B^{-1}\left(\frac{\lambda}{\lambda+\mu}\frac{\partial}{\partial\xi_1}R_1^{-1} - \xi_1\frac{\partial^2}{\partial\xi_2^2}R_1^{-1}\right)\right\}; \tag{20}$$

$$\overline{\sigma}_{23}(x,\xi) = \frac{\gamma\mu}{8\pi(\lambda+2\mu)}\left\{\left[(\xi_2 - x_2)\frac{\partial}{\partial\xi_3} + (\xi_3 - x_3)\frac{\partial}{\partial\xi_2}\right]\left(R^{-1} - R_1^{-1}\right)\right.$$
$$\left. -4x_1\xi_1B^{-1}\frac{\partial}{\partial\xi_2}\frac{\partial}{\partial\xi_3}R_1^{-1}\right\}; \tag{21}$$

$$\overline{\sigma}_{33}(x,\xi) = \frac{\gamma\mu}{4\pi(\lambda+2\mu)}\left\{\left[(\xi_3 - x_3)\frac{\partial}{\partial\xi_3} - 1\right]\left(R^{-1} - R_1^{-1}\right)\right.$$
$$\left. +2x_1B^{-1}\left(\frac{\lambda}{\lambda+\mu}\frac{\partial}{\partial\xi_1}R_1^{-1} - \xi_1\frac{\partial^2}{\partial\xi_3^2}R_1^{-1}\right)\right\}; \tag{22}$$

$$\overline{\sigma}_{13}(x,\xi)=\frac{\gamma\mu}{8\pi(\lambda+2\mu)}\left\{\left[(\xi_1-x_1)\frac{\partial}{\partial\xi_3}+(\xi_3-x_3)\frac{\partial}{\partial\xi_1}\right]\left(R^{-1}-R_1^{-1}\right)\right.$$
$$\left.-4x_1\xi_1B^{-1}\frac{\partial}{\partial\xi_1}\frac{\partial}{\partial\xi_3}R_1^{-1}\right\}. \tag{23}$$

Graphs of the thermal stresses $\overline{\sigma}_{ij}(x,\xi)$; $i,j=1,2,3$ within half-space S for $\xi_1=0,5$, $-10\leq\xi_2,\xi_3\leq10$ caused by a unit point heat source applied in the point $x_1=5m$, $x_2=0,x_3=0$ were constructed using computer program Maple 18. The value of elastic and thermal constants were taken the same as for thermoelastic displacements $U_i(x,\xi)$; $i=1,2,3$.

Normal and tangential thermal stresses $\overline{\sigma}_{11}(x,\xi)$, $\overline{\sigma}_{12}(x,\xi)$ were constructed by using the formulas (18) and (19), which are presented in the Figure 3, a), b).

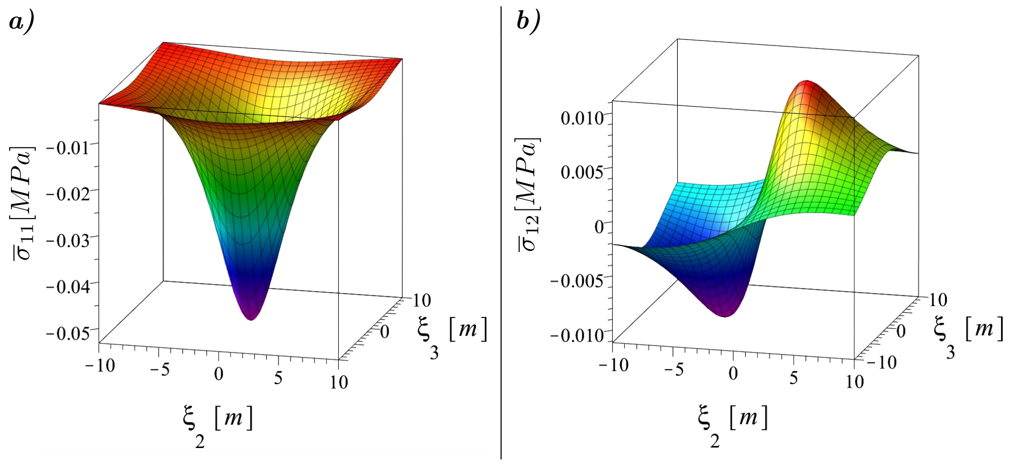

FIGURE 3. Graphs of normal and tangential thermal stresses $\overline{\sigma}_{11}(x,\xi)$, $\overline{\sigma}_{12}(x,\xi)$ in the half-space S ($\xi_1=0,5$; $-10\leq\xi_2,\xi_3\leq10$) caused by a unit heat source applied in the point $x_1=5m,x_2=0,x_3=0$.

Analyzing the Figure 3 graphs one can observe the following:

- the graph (Figure 3, a) has a local maximum in the point of application of the unit heat source $x_1=5m,x_2=0,x_3=0$, but the graph (Figure 3, b) has a discontinuity near this point;
- if $\xi_2,\xi_3\to\pm\infty$, then thermal stresses $\overline{\sigma}_{11}(x,\xi)$ and $\overline{\sigma}_{12}(x,\xi)\to0$.

Normal and tangential thermal stresses $\overline{\sigma}_{22}(x,\xi)$ $\overline{\sigma}_{23}(x,\xi)$ were constructed by using the formulas (20) and (21), which are presented in the Figure 4, a), b).

Analyzing the Figure 4 graphs one can observe the following:

- the graph (Figure 4, a) has a local maximum in the point of application of the unit heat source $x_1=5m,x_2=0,x_3=0$, but the graph (Figure 4, b) has a discontinuity near this point;
- if $\xi_2,\xi_3\to\pm\infty$, then thermal stresses $\overline{\sigma}_{22}(x,\xi)$ and $\overline{\sigma}_{23}(x,\xi)\to0$.

Normal and tangential thermal stresses $\overline{\sigma}_{33}(x,\xi)$, $\overline{\sigma}_{13}(x,\xi)$ were constructed by using the formulas (22) and (23), which are presented in the Figure 5, a), b).

Analyzing the Figure 5 graphs one can observe the following:

- the graph (Figure 5, a) has a local maximum in the point of application of the unit heat source $x_1=5m,x_2=0,x_3=0$, but the graph (Figure 5, b) has a discontinuity near this point;

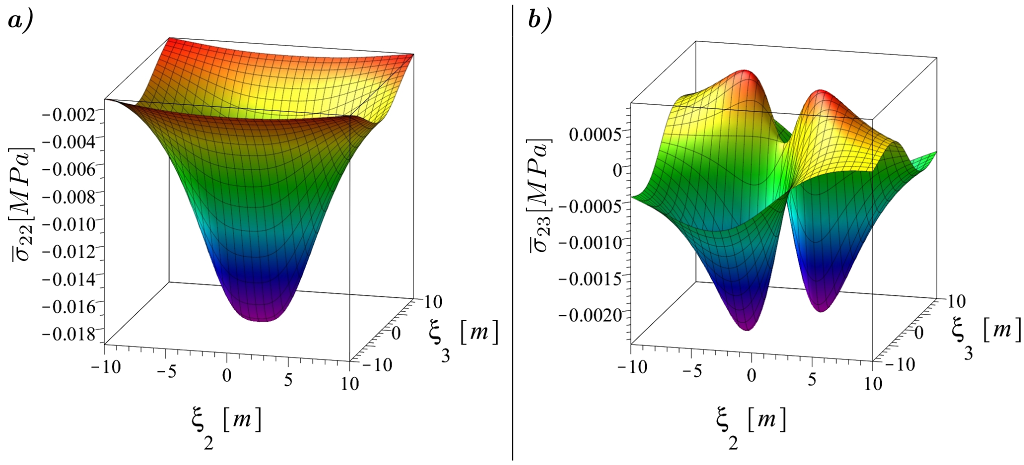

FIGURE 4. Graphs of normal and tangential thermal stresses $\overline{\sigma}_{22}(x,\xi)$, $\overline{\sigma}_{23}(x,\xi)$ in the half-space S ($\xi_1 = 0,5;\ -10 \leq \xi_2, \xi_3 \leq 10$) caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 0, x_3 = 0$.

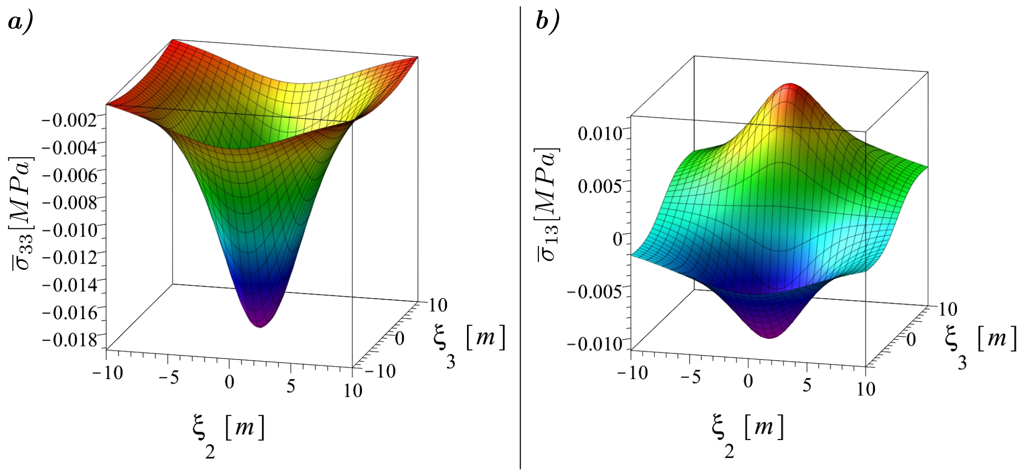

FIGURE 5. Graphs of normal and tangential thermal stresses $\overline{\sigma}_{33}(x,\xi)$, $\overline{\sigma}_{13}(x,\xi)$ in the half-space S ($\xi_1 = 0,5;\ -10 \leq \xi_2, \xi_3 \leq 10$) caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 0, x_3 = 0$.

- if $\xi_2, \xi_3 \to \pm\infty$, then thermal stresses $\overline{\sigma}_{33}(x,\xi)$ and $\overline{\sigma}_{13}(x,\xi) \to 0$.

3. CONCLUSIONS

The relations for thermoelastic displacements $U_i(x,\xi)$ (14) - (16) and thermal stresses $\overline{\sigma}_{ij}(x,\xi)$ (18) - (23) in the half-space S for the boundary conditions (7) were obtained for the first time. The expressions are presented in terms of elementary functions. Thermoelastic displacements $U_i(x,\xi)$ and thermal stresses $\overline{\sigma}_{ij}(x,\xi)$ are presented graphically using computer program Maple 18 with subsequent analysis of these graphs.

Using these expressions and soft Maple 18 can be obtained their graphical presentations caused by a unit heat source applied at any point of half-space S. Using the thermoelastic displacements $U_i(x,\xi)$ (14 - 16) and the thermal stresses $\overline{\sigma}_{ij}(x,\xi)$ (18 - 23) of a unit point heat source can be determined thermoelastic displacements (1) and thermal stresses (2) for a particular boundary value problems caused by the internal heat source applied at any point for half-space S and/or the temperature gradient applied of the marginal plan Γ_{10} for this half-space S.

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