NEW INFLUENCE FUNCTIONS FOR THERMAL DISPLACEMENTS AND STRESSES WITHIN HALF-SPACE

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Abstract. In this study new influence functions for the thermal displacements and stresses caused by a point heat source for three-dimensional boundary value problem of thermoelasticity within half-space were obtained. These results are presented in terms of elementary functions. Using the computer program Maple 18, the graphical presentations of thermal displacements and stresses caused by a unit point heat source were constructed.

1. Introduction

Determination of states of deformations and stresses caused by a internal heat source, temperature and other thermal actions on the body surface is a difficult mathematical problem that requires the creation of special theories. To obtain new integral solutions was required developed a new method, Harmonic Integral Representation Method (HIRM) proposed by V. Seremet. Using integral solutions can be calculated thermal displacements and stresses. By generalizing Maysel’s integral formula [1] and Green’s integral formulas, V. Seremet proposed a new form of these integral formulas [2, 3, 4, 5]:

$$u_i(\xi) = a^{-1} \int_V F(x)U_i(x,\xi)dV(x) - \int_{\Gamma_D} T(y) \frac{\partial U_i(y,\xi)}{\partial n_y}d\Gamma_D(y)$$

$$+ \int_{\Gamma_N} \frac{\partial T(y)}{\partial n_y} U_i(y,\xi)d\Gamma_N(y) + a^{-1} \int_{\Gamma_M} \left[ \alpha T(y) + a \frac{\partial T(y)}{\partial n_y} \right] U_i(y,\xi)d\Gamma_M(y); i = 1, 2, 3, (1)$$

where:

- $\Gamma_D$, $\Gamma_N$ and $\Gamma_M$ are the parts of the body surface $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_M$, which are given: the Dirichlet’s boundary conditions (temperature $T(y)$), the Neumann’s boundary conditions (heat flux $a \frac{\partial T(y)}{\partial n_y}$) and mixed boundary conditions (heat exchange between exterior medium and surface of the body represented by law $\left[ \alpha T(y) + a \frac{\partial T(y)}{\partial n_y} \right]$) are prescribed; $a$ is thermal conductivity; $F(x)$ is the internal heat source; $\alpha$ is the coefficient of convective heat conductivity; $\gamma = \alpha_t(2\mu + 3\lambda)$ is the thermoelastic constant; $\alpha_t$ is the coefficient of the linear thermal expansion, but $\lambda, \mu$ are Lame’s constants of elasticity.

The thermal stresses for three-dimensional canonical domains of Cartesian system of coordinates will be calculated by using the following type of Green’s integral formula [2]:

$$\sigma_{ij}(\xi) = a^{-1} \int_V F(x)\tau_{ij}(x,\xi)dV(x) - \int_{\Gamma_D} T(y) \frac{\partial \tau_{ij}(y,\xi)}{\partial n_y}d\Gamma_D(y) + \int_{\Gamma_N} \frac{\partial T(y)}{\partial n_y} \tau_{ij}(y,\xi)d\Gamma_N(y)$$

2010 Mathematics Subject Classification. 74G05, 35C15.

Key words and phrases. Green’s functions, integral solutions, thermoelasticity, thermal stresses, thermal displacements, volume dilatation, boundary conditions.
where:

\( \sigma_{ij} \) represents the influence functions for thermal stresses of a unit point heat source and \( \sigma_{ij} \) represent the thermal stress caused by internal heat source, temperature, heat flux or a heat exchange between exterior medium and surface of the body.

The thermal stresses \( \sigma_{ij} \) can be determined of Duhamel-Neumann law [6]:

\[
\sigma_{ij} = \mu (U_{i,j} + U_{j,i}) + \delta_{ij} (\lambda \Theta - \gamma G_T); \Theta = U_{k,k}(x,\xi); i,j,k = 1, 2, 3,
\]

where:

\( \delta_{ij} \) - Kronecker’s symbol, which is equal to 1, if \( i = j \) and 0, if \( i \neq j \).

2. THERMAL STRESSES \( \sigma_{ij} \) WITHIN HALF-SPACE CAUSED BY A UNIT POINT HEAT SOURCE

It is required to determine thermal stresses \( \sigma_{ij}(x,\xi); i, j = 1, 2, 3 \) of a boundary value problem in the half-space \( S(0 \leq x_1 < \infty, -\infty < x_2, x_3 < \infty) \) with thermal boundary condition of Dirichlet type for Green’s function:

\[
G_T = 0.
\]

The mechanical boundary conditions on the marginal plan \( \Gamma_{10} \) are:

\[
U_1 = U_2 = U_3 = 0.
\]

The mechanical and thermal boundary conditions of a boundary value problems within half-space are showed in the Figure 1.

![Figure 1. The scheme of the half-space \( S(0 \leq x_1 < \infty, -\infty < x_2, x_3 < \infty) \) with the mechanical boundary conditions \( U_1, U_2, U_3 \) and the thermal boundary conditions \( G_T \) on the marginal plan \( \Gamma_{10} \).](image)

To solve this problem it is necessary to determine thermoelastic displacements \( U_i(x,\xi) \) caused by a unit point heat source.

In the field literature [7], [8] boundary value problems within half-space are solved by using \( \Theta G \) - convolution method, but with other mechanical and thermal boundary conditions.

In this paper for the first time the thermoelastic displacements \( U_i(x,\xi); i = 1, 2, 3 \) and thermal stresses \( \sigma_{ij}(x,\xi) \) caused by a unit point heat source with boundary conditions [4]
- within half-space using the structural formulas have been proposed. These structural formulas were obtained by HIRM.

2.1. Determination of thermoelastic displacements \( U_i \). In the half-space \( S(0 \leq x_1 < \infty, -\infty < x_2, x_3 < \infty) \), thermoelastic displacements \( U_i(x, \xi); x \equiv (x_1, x_2, x_3); \xi \equiv (\xi_1, \xi_2, \xi_3) \) must be calculated. For this must be solved Lame equations:

\[
\mu \nabla^2 U_i(x, \xi) + (\lambda + \mu)\Theta_{\xi_i}(x, \xi) - \gamma G_{T,\xi_i}(x, \xi) = 0; \quad i = 1, 2, 3, \tag{6}
\]

where:

\[
\Theta_{\xi}(x, \xi) \equiv \frac{\partial G_{T}(x, \xi)}{\partial y_1} - \text{Green's function with reverse boundary condition for marginal plan } \Gamma_{10} \text{ on } y_1 = 0; -\infty < y_2, y_3 < \infty.
\]

\[
U_1(x, y) = U_2(x, y) = U_3(x, y) = 0; \quad x \in S; \quad G_T(y, \xi) = 0; \quad y \equiv (0, y_2, y_3) \in \Gamma_{10}. \tag{7}
\]

To determine the thermal displacements using the structural formulas \( U_i(x, \xi) \) and \( \Theta(x, \xi) \) which have been demonstrated in theorem 13 of the monograph [2]. In this case thermoelastic displacements take the following form:

\[
U_i(x, \xi) = \frac{\gamma}{2(\lambda + 2\mu)} \left[ \xi_i G_T(x, \xi) - x_i G_i(x, \xi) - 2x_1 \xi_1 B^{-1} \frac{\partial}{\partial y_1} W_T(x, \xi) \right], \tag{8}
\]

where:

\[
W_T(x, \xi) \text{ is regular part of the Green's function } \overline{G}_T(x, \xi) - \text{ Green's function with reverse boundary condition for marginal plan } \Gamma_{10}; \frac{\partial \overline{G}_T(x, \xi)}{\partial y_1} = -\frac{\partial \overline{G}_T(x, \xi)}{\partial y_1} = 0;
\]

\[
B = \frac{\lambda + 3\mu}{\lambda + \mu},
\]

and thermoelastic volume dilatation:

\[
\Theta(x, \xi) = \frac{\gamma}{\lambda + 2\mu} \left( G_T(x, \xi) + \frac{2\mu}{\lambda + \mu} B^{-1} x_1 \frac{\partial}{\partial x_1} W_T(x, \xi) \right). \tag{9}
\]

Green’s functions \( G_T; G_\theta \) and \( G_i; i = 1, 2, 3 \) are connected with the boundary conditions [10] as follows: if on the marginal plan \( \Gamma_{10} \) the thermoelastic displacements are known, then Green’s functions are equal to zero:

\[
U_1 = U_2 = U_3 = 0; \quad G_T = 0 \Rightarrow G_1 = G_2 = G_3 = G_\theta = 0. \tag{10}
\]

Green’s functions \( G_T; G_\theta; G_i; G_2 \) and \( G_3 \) for the half-space \( S \) are extracted from handbook [3] or encyclopedia [9] and these functions will be calculated by the following expressions:

\[
G_1 = G_2 = G_3 = G_T = G_\theta = G^{(1)} = \frac{1}{4\pi} (R^{-1} - R_1^{-1}), \tag{11}
\]

where:

\[
R = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2};
\]

\[
R_1 = \sqrt{(x_1 + \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}.
\]

Green’s function \( \overline{G}_T(x, \xi) \) with reverse boundary condition \( \overline{G}_{T,1} = 0 \) for marginal plan \( \Gamma_{10} \) is calculated from [3, 9]:

\[
\overline{G}_T(x, \xi) = \frac{1}{4\pi} (R^{-1} + R_1^{-1}). \tag{12}
\]

The regular part of the Green’s function \( \overline{G}_T(x, \xi) \) [12] is that part which contains inferior index 1, that part of the \( \overline{G}_T(x, \xi) \) which are reflected via marginal plan \( \Gamma_{10} \). So, \( W_T(x, \xi) \) of the formulas [8] and [9] is calculated with the following relation:

\[
W_T(x, \xi) = \frac{1}{4\pi} (R_1^{-1}). \tag{13}
\]
Substituting expressions (11) and (13) in the formula (8) are obtained the final expressions for thermoelastic displacements $U_i(x,\xi)$ within half-space $S$, which are presented in the following forms:

\[
U_1(x,\xi) = \frac{\gamma}{8\pi(\lambda + 2\mu)} \left( (\xi_1 - x_1)(R^{-1} - R_1^{-1}) - 2x_1 \xi_1 B^{-1} \frac{\partial}{\partial \xi_1} R_1^{-1} \right); \quad (14)
\]

\[
U_2(x,\xi) = \frac{\gamma}{8\pi(\lambda + 2\mu)} \left( (\xi_2 - x_2)(R^{-1} - R_1^{-1}) - 2x_1 \xi_1 B^{-1} \frac{\partial}{\partial \xi_2} R_1^{-1} \right); \quad (15)
\]

\[
U_3(x,\xi) = \frac{\gamma}{8\pi(\lambda + 2\mu)} \left( (\xi_3 - x_3)(R^{-1} - R_1^{-1}) - 2x_1 \xi_1 B^{-1} \frac{\partial}{\partial \xi_3} R_1^{-1} \right). \quad (16)
\]

Graphs of the thermoelastic displacements $U_i(x,\xi)$; $i = 1, 2, 3$ within half-space $S$ for $0 \leq \xi_1 \leq 10, \xi_2 = 1, -10 \leq \xi_3 \leq 10$ caused by a unit point heat source applied in the point $x_1 = 5m, x_2 = 2m, x_3 = 0$ were constructed using computer program Maple 18. The value of elastic and thermal constants are: the Poisson ration $\nu = 0, 3$; modulus of elasticity $E = 2, 1 \cdot 10^5 MPa$, and coefficient of linear thermal expansion $\alpha_t = 1, 2 \cdot 10^{-5}(K^{-1})$.

Thermoelastic displacements $U_1(x,\xi)$, $U_2(x,\xi)$ and $U_3(x,\xi)$ were constructed using formulas (14)-(16) are presented in the Figure 2 (a), (b) and (c).

Analyzing the Figure 2 one can observe the following:

- the boundary conditions (7) are respected: on the marginal plan $\Gamma_{10}$. $U_1(x,\xi) = 0$ (Figure 2 a); $U_2(x,\xi) = 0$ (Figure 2 b); $U_3(x,\xi) = 0$ (Figure 2 c); $\xi \equiv (\xi_1 = 0$; $-\infty < \xi_2, \xi_3 < \infty)$;

- graph of the thermoelastic displacement $U_2(x,\xi)$ has a local maximum: in the point of application of the unit heat source (Figure 2 b). The others graphs $U_1(x,\xi)$ and $U_3(x,\xi)$ have a discontinuity near this point.

- if $\xi_1 \to \infty, \xi_3 \to \pm \infty$, then thermoelastic displacements $U_1(x,\xi)$; $U_2(x,\xi)$ and $U_3(x,\xi)$ $\to 0$.

2.2. Determination of thermal stresses $\sigma_{ij}$. The thermal stresses $\sigma_{ij}(x,\xi)$ are calculated using Duhamel-Neumann law (3), but $\Theta(x,\xi)$ - thermoelastic volume dilatation which is determine by using the formula (9) and has the form:

\[
\Theta(x,\xi) = \frac{\gamma}{4\pi(\lambda + 2\mu)} \left( R^{-1} - R_1^{-1} + \frac{2\mu}{\lambda + \mu} B^{-1} x_1 \frac{\partial}{\partial x_1} R_1^{-1} \right). \quad (17)
\]

Substituting Green’s function $G_T(x,\xi)$ (11), thermoelastic volume dilatation $\Theta(x,\xi)$ (17) and expressions for thermoelastic displacements $U_i(x,\xi); i = 1, 2, 3$ (14)-(16) in the Duhamel-Neumann law (3) we obtain the expressions for thermoelastic influence functions for thermal stresses $\sigma_{ij}(x,\xi)$:

\[
\sigma_{11}(x,\xi) = \frac{\gamma \mu}{4\pi(\lambda + 2\mu)} \left\{ \left[ (\xi_1 - x_1) \frac{\partial}{\partial \xi_1} - 1 \right] R^{-1} - \left( \xi_1 \frac{\partial}{\partial \xi_1} - 1 \right) R_1^{-1} 
+ 2x_1 B^{-1} \frac{\partial}{\partial \xi_1} R^{-1} \left( 1 - 2\xi_1 \frac{\partial}{\partial \xi_1} R_1^{-1} \right) \right\}; \quad (18)
\]

\[
\sigma_{12}(x,\xi) = \frac{\gamma \mu}{8\pi(\lambda + 2\mu)} \left\{ \left[ (\xi_1 - x_1) \frac{\partial}{\partial \xi_2} + (\xi_2 - x_2) \frac{\partial}{\partial \xi_1} \right] (R^{-1} - R_1^{-1}) 
- 4x_1 \xi_1 B^{-1} \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} R_1^{-1} \right\}; \quad (19)
\]
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\[ a) \quad U_1 [K^{-1}] \]

\[ b) \quad U_2 [K^{-1}] \]

\[ c) \quad U_3 [K^{-1}] \]

**Figure 2.** Graphs of thermoelastic displacements \( U_i(x, \xi) \) in the half-space \( S \) for \( 0 \leq \xi_1 \leq 10; \xi_2 = 1; -10 \leq \xi_3 \leq 10 \) caused by a unit heat source applied in the point \( x_1 = 5m, x_2 = 2m, x_3 = 0.\)

\[
\sigma_{22}(x, \xi) = \frac{\gamma\mu}{4\pi(\lambda + 2\mu)} \left\{ \left( \xi_2 - x_2 \right) \frac{\partial}{\partial \xi_2} - 1 \right\} \left( R^{-1} - R^{-1}_1 \right) + 2x_1B^{-1} \left( \frac{\lambda}{\lambda + \mu} \frac{\partial}{\partial \xi_1} R_1^{-1} - \xi_1 \frac{\partial^2}{\partial \xi_2^2} R_1^{-1} \right) \right\}; \quad (20)
\]

\[
\sigma_{23}(x, \xi) = \frac{\gamma\mu}{8\pi(\lambda + 2\mu)} \left\{ \left( \xi_2 - x_2 \right) \frac{\partial}{\partial \xi_3} + \left( \xi_3 - x_3 \right) \frac{\partial}{\partial \xi_2} \right\} \left( R^{-1} - R^{-1}_1 \right) - 4x_1\xi_1B^{-1} \frac{\partial}{\partial \xi_2} \frac{\partial}{\partial \xi_3} R_1^{-1} \right\}; \quad (21)
\]

\[
\sigma_{33}(x, \xi) = \frac{\gamma\mu}{4\pi(\lambda + 2\mu)} \left\{ \left( \xi_3 - x_3 \right) \frac{\partial}{\partial \xi_3} - 1 \right\} \left( R^{-1} - R^{-1}_1 \right) + 2x_1B^{-1} \left( \frac{\lambda}{\lambda + \mu} \frac{\partial}{\partial \xi_1} R_1^{-1} - \xi_1 \frac{\partial^2}{\partial \xi_3^2} R_1^{-1} \right) \right\}; \quad (22)
\]
\[ \sigma_{13}(x, \xi) = \frac{\gamma \mu}{8\pi(\lambda + 2\mu)} \left\{ \left[ (\xi_1 - x_1) \frac{\partial}{\partial \xi_3} + (\xi_3 - x_3) \frac{\partial}{\partial \xi_1} \right] \left( R^{-1} - R_1^{-1} \right) - 4x_1\xi_1 B^{-1} \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_3} R_1^{-1} \right\} \].

Graphs of the thermal stresses \( \sigma_{ij}(x, \xi) \); \( i, j = 1, 2, 3 \) within half-space \( S \) for \( \xi_1 = 0, 5, -10 \leq \xi_2, \xi_3 \leq 10 \) caused by a unit point heat source applied in the point \( x_1 = 5m, x_2 = 0, x_3 = 0 \) were constructed using computer program Maple 18. The value of elastic and thermal constants were taken the same as for thermoelastic displacements \( U_i(x, \xi) \); \( i = 1, 2, 3 \).

Normal and tangential thermal stresses \( \sigma_{11}(x, \xi), \sigma_{12}(x, \xi) \) were constructed by using the formulas (18) and (19), which are presented in the Figure 3, a), b).

Analyzing the Figure 3 graphs one can observe the following:
- the graph (Figure 3, a) has a local maximum in the point of application of the unit heat source \( x_1 = 5m, x_2 = 0, x_3 = 0 \), but the graph (Figure 3, b) has a discontinuity near this point;
- if \( \xi_2, \xi_3 \to \pm \infty \), then thermal stresses \( \sigma_{11}(x, \xi) \) and \( \sigma_{12}(x, \xi) \to 0 \).

Normal and tangential thermal stresses \( \sigma_{22}(x, \xi), \sigma_{23}(x, \xi) \) were constructed by using the formulas (20) and (21), which are presented in the Figure 4, a), b).

Analyzing the Figure 4 graphs one can observe the following:
- the graph (Figure 4, a) has a local maximum in the point of application of the unit heat source \( x_1 = 5m, x_2 = 0, x_3 = 0 \), but the graph (Figure 4, b) has a discontinuity near this point;
- if \( \xi_2, \xi_3 \to \pm \infty \), then thermal stresses \( \sigma_{22}(x, \xi) \) and \( \sigma_{23}(x, \xi) \to 0 \).

Normal and tangential thermal stresses \( \sigma_{33}(x, \xi), \sigma_{13}(x, \xi) \) were constructed by using the formulas (22) and (23), which are presented in the Figure 5, a), b).

Analyzing the Figure 5 graphs one can observe the following:
- the graph (Figure 5, a) has a local maximum in the point of application of the unit heat source \( x_1 = 5m, x_2 = 0, x_3 = 0 \), but the graph (Figure 5, b) has a discontinuity near this point;
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3. Conclusions

The relations for thermoelastic displacements $U_i(x, \xi)$ (14) - (16) and thermal stresses $\sigma_{ij}(x, \xi)$ (18) - (23) in the half-space $S$ for the boundary conditions (7) were obtained for the first time. The expressions are presented in terms of elementary functions. Thermoelastic displacements $U_i(x, \xi)$ and thermal stresses $\sigma_{ij}(x, \xi)$ are presented graphically using computer program Maple 18 with subsequent analysis of these graphs.

Using these expressions and soft Maple 18 can be obtained their graphical presentations caused by a unit heat source applied at any point of half-space $S$. Using the thermoelastic...
displacements \( U_i(x, \xi) \) and thermal stresses \( \sigma_{ij}(x, \xi) \) of a unit point heat source can be determined for a particular boundary value problems caused by the internal heat source applied at any point for half-space \( S \) and/or the temperature gradient applied of the marginal plan \( \Gamma_{10} \) for this half-space \( S \).

References