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STABILITY IN NONLINEAR NEUTRAL VOLTERRA INTEGRAL EQUATIONS ON A TIME SCALE

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Abstract. Let $\mathbb{T}$ be a time scale that is unbounded above and below and such that $0 \in \mathbb{T}$. Let $id - \tau : \mathbb{T} \to \mathbb{T}$ be such that $(id - \tau)(\mathbb{T})$ is a time scale. We use the contraction mapping theorem to obtain stability results about the zero solution of the nonlinear neutral Volterra integral equation with variable delay

$$x^{\triangle}(t) = -a(t)\, x^{\sigma}(t) + c(t)\, x^{\widetilde{\triangle}}(t - \tau(t)) + \int_{t-\tau(t)}^{t} k(t,s)\, g(x(s))\, \triangle s, \; t \in \mathbb{T},$$

where $f^{\triangle}$ is the $\triangle$-derivative on $\mathbb{T}$ and $f^{\widetilde{\triangle}}$ is the $\triangle$-derivative on $(id - \tau)(\mathbb{T})$. The results obtained here extend some results due to Raffoul [11].

1. Introduction

In this paper, we are interested in the analysis of qualitative theory of stability in Volterra integral equations. Motivated by the papers [1], [2], [5]–[11] and the references therein, we consider the nonlinear neutral Volterra integral equation with variable delay

$$x^{\triangle}(t) = -a(t)\, x^{\sigma}(t) + c(t)\, x^{\widetilde{\triangle}}(t - \tau(t)) + \int_{t-\tau(t)}^{t} k(t,s)\, g(x(s))\, \triangle s, \; t \in \mathbb{T}, \tag{1}$$

where $\mathbb{T}$ is unbounded above and below. Throughout this paper we assume that $0 \in \mathbb{T}$ for convenience. We also assume that $a : \mathbb{T} \to \mathbb{R}$ is continuous, $k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is continuous and $c : \mathbb{T} \to \mathbb{R}$ is continuously delta-differentiable. In order for the function $x(t - \tau(t))$ to be well-defined and differentiable over $\mathbb{T}$, we assume that $\tau : \mathbb{T} \to \mathbb{R}$ is positive and twice continuously delta-differentiable, and that $id - \tau : \mathbb{T} \to \mathbb{T}$ is an increasing mapping such that $(id - \tau)(\mathbb{T})$ is closed where id is the identity function on $\mathbb{T}$. Our purpose here is to give, by using the contraction mapping principle (Smart [12]), asymptotic stability results of (1). In the special case $\mathbb{T} = \mathbb{R}$, in [11] Raffoul used the contraction mapping principle to show the asymptotic stability of the zero solution of (1).

We assume that reader is familiar with the notation and basic results for dynamic equations on time scale. For a review of this topic we direct the reader to the monographs [3], [4]. Throughout this paper, intervals subscripted with a $\mathbb{T}$ represent real intervals intersected with $\mathbb{T}$. For example, for $a, b \in \mathbb{T}$, $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \le t \le b\}$.

The organization of this paper is as follows. In Section 2, we present some preliminary material that we will need through the remainder of the paper. We present our main results on stability by using the contraction mapping principle in Section 3. The results obtained here extend some results due to Raffoul [11].

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2. Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Most of the following definitions, lemmas and theorems can be found in [3], [4].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \backslash \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t) [\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\triangle}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 1 ([3])**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*
(i) $(f + g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t)$.
(ii) $(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.
(ii) The product rules

$$(fg)^{\triangle}(t) = f^{\triangle}(t) g(t) + f^{\sigma}(t) g^{\triangle}(t),$$
$$(fg)^{\triangle}(t) = f(t) g^{\triangle}(t) + f^{\triangle}(t) g^{\sigma}(t).$$

(iv) If $g(t) g^{\sigma}(t) \neq 0$ then

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t) g(t) - f(t) g^{\triangle}(t)}{g(t) g^{\sigma}(t)}.$$

The next theorem is the chain rule on time scales [3].

Theorem 2 (Chain Rule)**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then $(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}$.*

In the sequel we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(\nu(t))$ where, $\nu(t) := t - \tau(t)$. Our next theorem is the substitution rule [3].

Theorem 3 (Substitution)**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\,\nu^{\triangle}(t)\,\triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)\,p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1+\mu(t)\,f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log\left(1 + \mu(z)\,p(z)\right) \triangle z\right). \tag{2}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^{\triangle} = p(t)\,y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 1 ([3])**.** *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)\,p(t))\,e_p(t,s)$;

(iii) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1+\mu(t)\,p(t)}$;

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)\,e_p(s,r) = e_p(t,r)$;

(vi) $e_p^{\triangle}(.,s) = p e_p(.,s)$ *and* $\left(\dfrac{1}{e_p(.,s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(.,s)}$.

3. Main results

We assume g is locally Lipschitz continuous. That is, there is a $L > 0$ so that if $|x|, |y| \leq L$, then

$$|g(x) - g(y)| \leq K\,\|x - y\|, \tag{3}$$

for a positive constant K.

Also, we assume

$$g(0) = 0. \tag{4}$$

In addition to the conditions on τ mentioned in Section 1, we need that

$$\tau^{\triangle}(t) \neq 1,\ \forall t \in \mathbb{T}. \tag{5}$$

Furthermore, the exponential function $e_{\ominus a}(t,0)$ must satisfy

$$e_{\ominus a}(t,0) \to 0 \text{ as } t \to \infty, \tag{6}$$

as well as the initial value problem $y^{\triangle}(t) = -a(t)\,y^{\sigma}(t)$, $y(0) = 1$. As such, we require that $a(t) \geq 0$ for all $t \in \mathbb{T}$. Since $a(t) \geq 0$ for all $t \in \mathbb{T}$, then $1 + \mu(t)\,a(t) \geq 1 > 0$ for all t and so $a \in \mathcal{R}^+$.

We have to invert equation (1). To do this, we use the variation of parameter formula to rewrite the equation as an integral mapping equation suitable for the contraction mapping theorem.

Lemma 2. *Suppose (5) holds. Then x is a solution of equation (1) if and only if*

$$x(t) = \left(x(0) - \frac{c(0)}{1-\tau^{\triangle}(0)} x(-\tau(0)) \right) e_{\ominus a}(t,0) + \frac{c(t)}{1-\tau^{\triangle}(t)} x(t-\tau(t)) + \int_0^t \left[-h(s) x^{\sigma}(s-\tau(s)) + \int_{s-\tau(s)}^{s} k(s,u) g(x(u)) \triangle u \right] e_{\ominus a}(t,s) \triangle s, \tag{7}$$

where

$$h(s) = \frac{\left(c^{\triangle}(s) + c^{\sigma}(s) a(s)\right)\left(1-\tau^{\triangle}(s)\right) + \tau^{\triangle\triangle}(s) c(s)}{\left(1-\tau^{\triangle}(s)\right)\left(1-\tau^{\triangle}(\sigma(s))\right)}. \tag{8}$$

Proof. We begin by rewriting (1) as

$$x^{\triangle}(t) + a(t) x^{\sigma}(t) = c(t) x^{\widetilde{\triangle}}(t-\tau(t)) + \int_{t-\tau(t)}^{t} k(t,s) g(x(s)) \triangle s.$$

Multiply both sides of the above equation by $e_a(t,0)$ and then we integrate from 0 to t to obtain

$$\int_0^t \left(e_a(s,0) x(s)\right)^{\triangle} \triangle s = \int_0^t \left[c(s) x^{\widetilde{\triangle}}(s-\tau(s)) + \int_{s-\tau(s)}^{s} k(s,u) g(x(u)) \triangle u \right] e_a(s,0) \triangle s.$$

As a consequence, we arrive at

$$e_a(t,0) x(t) - x(0) = \int_0^t \left[c(s) x^{\widetilde{\triangle}}(s-\tau(s)) + \int_{s-\tau(s)}^{s} k(s,u) g(x(u)) \triangle u \right] e_a(s,0) \triangle s.$$

Add $x(0)$ to both sides and multiply them by $e_{\ominus a}(t,0)$ to obtain

$$x(t) = x(0) e_{\ominus a}(t,0) + \int_0^t \left[c(s) x^{\widetilde{\triangle}}(s-\tau(s)) + \int_{s-\tau(s)}^{s} k(s,u) g(x(u)) \triangle u \right] e_{\ominus a}(t,s) \triangle s. \tag{9}$$

Here we have used Lemma 1 to simplify the exponential. We want to pull the factor $x^{\widetilde{\triangle}}(s-\tau(s))$ from under the integral in (9). Clearly

$$\int_0^t c(s) x^{\widetilde{\triangle}}(s-\tau(s)) e_{\ominus a}(t,s) \triangle s = \int_0^t x^{\widetilde{\triangle}}(s-\tau(s)) \left(1-\tau^{\triangle}(s)\right) \frac{c(s)}{\left(1-\tau^{\triangle}(s)\right)} e_{\ominus a}(t,s) \triangle s.$$

Using the integration by parts formula we get

$$\int_0^t f^{\triangle}(s) g(s) \triangle s = (fg)(t) - (fg)(0) - \int_0^t f^{\sigma}(s) g^{\triangle}(s) \triangle s,$$

and Theorems 2 and 3 implies

$$\begin{aligned}&\int_0^t c(s)\, x^{\widetilde{\triangle}}(s-\tau(s))\, e_{\ominus a}(t,s)\,\triangle s\\ &= \frac{c(t)}{1-r^{\triangle}(t)}x(t-\tau(t)) - \frac{c(0)}{1-\tau^{\triangle}(0)}x(-\tau(0))\, e_{\ominus a}(t,0)\\ &- \int_0^t h(s)\, x^{\sigma}(s-\tau(s))\, e_{\ominus a}(t,s)\,\triangle s, \end{aligned} \tag{10}$$

where h is given by (8). Finally, substituting the right hand side of (10) into (9) we obtain (7). Since each step is reversible, the converse follows easily. This completes the proof. □

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given bounded $\triangle$-differentiable initial function. We say $x := x(.,0,\psi)$ is a solution of (1) if $x(t) = \psi(t)$ for $t \leq 0$ and satisfies (1) for $t \geq 0$.

We say the zero solution of (1) is stable at t_0 if for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that $[\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ with $\|\psi\| < \delta]$ implies $|x(t, t_0, \psi)| < \epsilon$.

Let $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$ be the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ and define the set S_ψ by

$$S_\psi = \left\{\varphi \in C_{rd} : \|\varphi\| \leq L,\ \varphi(t) = \psi(t) \text{ if } t \leq 0 \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\right\}.$$

Then $(S_\psi, \|.\|)$ is a complete metric space where $\|.\|$ is the supremum norm.

For the next theorem we assume there is an $\alpha > 0$ such that

$$\left|\frac{c(t)}{1-\tau^{\triangle}(t)}\right| + \int_0^t \left(|h(s)| + K\int_{s-\tau(s)}^{s} |k(s,u)|\,\triangle u\right) e_{\ominus a}(t,s)\,\triangle s \leq \alpha < 1,\ t \geq 0, \tag{11}$$

and

$$t - \tau(t) \to \infty \text{ as } t \to \infty. \tag{12}$$

Theorem 4. *If (3)-(6), (11) and (12) hold, then every solution $x(.,0,\psi)$ of (1) with a small continuous initial function ψ, is bounded and tends to zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.*

Proof. For α and L, find an appropriate $\delta > 0$ such that

$$\left|1 - \frac{c(0)}{1-\tau^{\triangle}(0)}\right|\delta + \alpha L \leq L.$$

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given small bounded initial function with $\|\psi\| < \delta$. Define the mapping $P : S_\psi \to S_\psi$ by

$$(P\varphi)(t) = \psi(t) \text{ if } t \leq 0,$$

and

$$\begin{aligned}(P\varphi)(t) &= \left(\psi(0) - \frac{c(0)}{1-\tau^{\triangle}(0)}\psi(-\tau(0))\right) e_{\ominus a}(t,0) + \frac{c(t)}{1-\tau^{\triangle}(t)}\varphi(t-\tau(t))\\ &+ \int_0^t \left[-h(s)\,\varphi^{\sigma}(s-\tau(s)) + \int_{s-\tau(s)}^{s} k(s,u)\, g(\varphi(u))\,\triangle u\right] e_{\ominus a}(t,s)\,\triangle s,\ t \geq 0.\end{aligned}$$

Clearly, $P\varphi$ is continuous when φ is such. Let $\varphi \in S_\psi$, then using (11) in the definition of $P\varphi$ and applying (3) and (4), we obtain

$$\begin{aligned}
&\left|(P\varphi)(t)\right| \\
&\leq \left|1-\frac{c(0)}{1-\tau^{\triangle}(0)}\right|\delta+\left|\frac{c(t)}{1-\tau^{\triangle}(t)}\right|L \\
&+\int_0^t\left[\left|h(s)\right|\left|\varphi^{\sigma}(s-\tau(s))\right|+\int_{s-\tau(s)}^{s}\left|k(s,u)\right|\left|g(\varphi(u))\right|\triangle u\right]e_{\ominus a}(t,s)\triangle s \\
&\leq \left|1-\frac{c(0)}{1-\tau^{\triangle}(0)}\right|\delta \\
&+L\left\{\left|\frac{c(t)}{1-\tau^{\triangle}(t)}\right|+\int_0^t\left(\left|h(s)\right|+K\int_{s-\tau(s)}^{s}\left|k(s,u)\right|\triangle u\right)e_{\ominus a}(t,s)\triangle s\right\} \\
&\leq \left|1-\frac{c(0)}{1-\tau^{\triangle}(0)}\right|\delta+L\alpha,
\end{aligned}$$

which implies that $\left|(P\varphi)(t)\right| \leq L$ for the chosen δ. Thus we have $\|P\varphi\| \leq L$.

Next we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. By (6) and (12), the first term in the definition of $(P\varphi)(t)$ tends to zero. Also, the second term on the right-hand side tends to zero because of (12) and the fact that $\varphi \in S_\psi$. It remains to show that the integral term tends to zero as $t \to \infty$.

Let $\epsilon > 0$ be arbitrary and $\varphi \in S_\psi$. Then $\|\varphi\| \leq L$ and there exists $t_1 > 0$ such that $|\varphi(t)|$, $|\varphi(t-\tau(t))|$ and $|\varphi^{\sigma}(t-\tau(t))| < \epsilon$ for $t \geq t_1$. By condition (6), there exists $t_2 > t_1$ such that for $t > t_2$

$$e_{\ominus a}(t,t_1) < \frac{\epsilon}{\alpha L}.$$

For $t > t_2$, we have

$$\begin{aligned}
&\left|\int_0^t\left[-h(s)\varphi^{\sigma}(s-\tau(s))+\int_{s-\tau(s)}^{s}k(s,u)g(\varphi(u))\triangle u\right]e_{\ominus a}(t,s)\triangle s\right| \\
&\leq \int_0^t\left[\left|h(s)\right|\left|\varphi^{\sigma}(s-\tau(s))\right|+\int_{s-\tau(s)}^{s}\left|k(s,u)\right|\left|g(\varphi(u))\right|\triangle u\right]e_{\ominus a}(t,s)\triangle s \\
&\leq L\int_0^{t_1}\left(\left|h(s)\right|+K\int_{s-\tau(s)}^{s}\left|k(s,u)\right|\triangle u\right)e_{\ominus a}(t,s)\triangle s \\
&+\epsilon\int_{t_1}^{t_2}\left(\left|h(s)\right|+K\int_{s-\tau(s)}^{s}\left|k(s,u)\right|\triangle u\right)e_{\ominus a}(t,s)\triangle s \\
&\leq Le_{\ominus a}(t,t_1)\int_0^{t_1}\left(\left|h(s)\right|+K\int_{s-\tau(s)}^{s}\left|k(s,u)\right|\triangle u\right)e_{\ominus a}(t_1,s)\triangle s+\alpha\epsilon \\
&\leq \alpha Le_{\ominus a}(t,t_1)+\alpha\epsilon \leq \epsilon+\alpha\epsilon.
\end{aligned}$$

Hence $(P\varphi)(t) \to 0$ as $t \to \infty$.

It remains to show that P is a contraction under the supremum norm. For this, let $\varphi, \phi \in S_\psi$ then

$$
\begin{aligned}
& |(P\varphi)(t) - (P\phi)(t)| \\
& \leq \left| \frac{c(t)}{1-\tau^{\triangle}(t)} \right| \|\varphi - \phi\| + \int_0^t |h(s)| \left| \varphi^{\sigma}(s-\tau(s)) - \phi^{\sigma}(s-\tau(s)) \right| e_{\ominus a}(t,s) \triangle s \\
& + \int_0^t \left(\int_{s-\tau(s)}^s |k(s,u)| \left| g(\varphi(u)) - g(\phi(u)) \right| \triangle u \right) e_{\ominus a}(t,s) \triangle s \\
& \leq \left\{ \left| \frac{c(t)}{1-\tau^{\triangle}(t)} \right| + \int_0^t \left(|h(s)| + \int_{s-\tau(s)}^s |k(s,u)| \triangle u \right) e_{\ominus a}(t,s) \triangle s \right\} \|\varphi - \phi\| \\
& \leq \alpha \|\varphi - \phi\| .
\end{aligned}
$$

Thus, by the contraction mapping principle, P has a unique fixed point in S_ψ which solves (1), bounded and tends to zero as $t \to \infty$. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing L by ϵ. □

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