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DETERMINING THERMOELASTIC STRAINS AND STRESSES CREATED BY THE TEMPERATURE FIELD AND CONSTRICTIONS EXERTED ON THE CONTOURS OF THE RECONDITIONED CYLINDRICAL PIECE

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ABSTRACT. The main purpose of this article is to determine the distribution of displacements and thermal stresses in composite coatings applied as a compensating layer on reconditioned car parts by means of the squeeze technique. To achieve the purpose means to meet some objectives like: to obtain axial symmetrical displacements and stress fields that are distributed on the thickness of the composite material layer; to validate the received results by solving the boundary value problem by means of the direct integration method and the Green's function method. As a result of mathematical modeling and theoretical analysis we propose a method to calculate displacements and stresses in different layers of the composite coating, taking into account their constraints too. Analytical expressions and graphs were built for both Green's functions and heat fields and thermoelastic displacements and stresses for axial symmetrical boundary problems of heat conduction, elastic and thermoelastic theories. The graphic representation has been made using the Maple 15 software. The obtained data can be used for engineering design calculations and choosing appropriate interference fits renovated with composite materials.

1. Introduction

It is extremely important to study the distribution of temperature and displacement fields as well as thermal stresses on the polymer coating thickness in order to correctly calculate the value of metalopolymeric joint fits. The article aims to determine how thermal fields are distributed in composite coatings applied as a compensating layer for reconditioned machine parts by means of the nominal size method. The importance and relevance of the study is justified by the role precise calculation of interference fits plays in insuring durability and availability of reconditioned tightened joints with composite materials. Having analyzed the data found in the specialty literature [1] - [4] and our own mathematical models, we formulated a new method to calculate temperature, displacements and stresses in different layers of polymer coatings. The method is based on the use of Green's functions that have a number of advantages as compared to traditional methods used in these calculations. Analytical expressions and graphs for the analyzed cases were developed with the help of the Maple 15 software. The data obtained will be used in calculations related to determining the character and values of displacements and internal tensions in different coating layers of the parts restored by means of polymeric composites. Ultimately, this will improve the calculation accuracy of tightened fits and, thus, increase the reliability of corresponding joints. Mathematical models for the temperature field, thermoelastic displacements and stresses formed inside polymer coatings

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were formulated using Green's functions. The Maple 15 software was used to present in a graphic way Green's functions and temperatures, thermoelastic displacements and stresses in different layers of the composite material coating, depending on the location of the polymer coating where values of these fields need to be estimated. We can develop graphs for different exploitation conditions with the help of this software. For instance, there are presented temperature, displacement and stress graphs, created by means of tightening and temperature for a real comprehensive interference piece which is reconditioned with the help of compensating wear composite materials based on polymers. To validate the obtained results, analytical expressions were established by means of two methods: the direct integration method and the Green's functions method.

To determine radial displacements $U_r(r)$ in the circular layer $V \equiv (r_1 \leq r \leq r_2, 0 \leq \varphi \leq 2\pi)$, created by tightening and temperature it is necessary to solve the differential equation:

$$\frac{d^2U_r(r)}{dr^2} + \frac{1}{r}\frac{dU_r(r)}{dr} - \frac{U_r(r)}{r^2} = (1+\mu)\alpha\frac{dT(r)}{dr} \tag{1}$$

with certain boundary conditions for $U_r(r)$ (we consider boundary conditions of the Dirichlet type in this paper). In equation (1): $T(r)$ stands for temperature, μ and α are the Poisson coefficient and the coefficient of linear thermal expansion respectively. Temperature field $T(r)$, created by constant temperatures T_1, T_2 given on the contours, $\Gamma_1 \equiv (r = r_1, 0 \leq \varphi \leq 2\pi), \Gamma_2 \equiv (r = r_2, 0 \leq \varphi \leq 2\pi)$ and by constant heat source S, is determined from the equation:

$$\frac{d^2T(r)}{dr^2} + \frac{1}{r}\frac{dT(r)}{dr} = -\frac{S}{a} \tag{2}$$

where a is thermal conductivity and the boundary conditions are

$$T(r = r_1) = T_1, T(r = r_2) = T_2 \tag{3}$$

The analytical expression for temperature $T(r)$, determined from boundary problem (2) and (3) can be written in the following way [5] (D. Șeremet, 2015):

$$T(r) = \left(\ln\frac{r_1}{r_2}\right)^{-1}\left[\frac{1}{4}a^{-1}S\left((r_1^2 - r_2^2)\ln r + (r_2^2\ln r_1 - r_1^2\ln r_2) - \left(\ln\frac{r_1}{r_2}\right)r^2\right) + T_1\ln\frac{r}{r_2} - T_2\ln\frac{r}{r_1}\right] \tag{4}$$

Further, we present two methods to solve equation (1):

a. Direct integration method
b. Green's functions method

2. Direct integration method

The general solution $U_r(r)$ of differential non-homogenous equation (1) is obtained as a sum of the general solution $U_{rgo}(r)$ of homogenous equation (1) ($S = 0$) and of the partial solution $U_{rp}(r)$ of non-homogeneous equation (1)

$$U_r(r) = U_{rgo}(r) + U_{rp}(r) \tag{5}$$

The general solution of homogeneous equation (1) is determined by the formula

$$U_{rgo}(r) = k_1 r + \frac{k_2}{r} \tag{6}$$

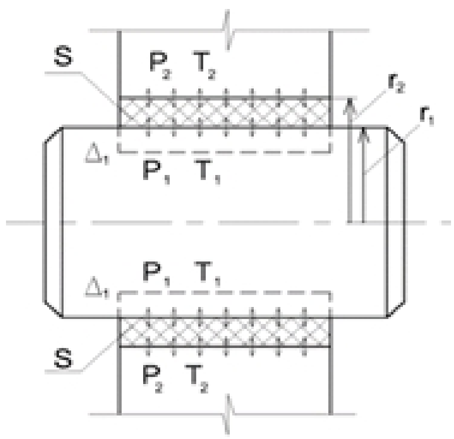

FIGURE 1. Clamping joint scheme reconditioned by the polymer composite layer applied on the comprehensive piece influenced by temperatures T_1, T_2, heat source S and tightening Δ_1

where r and $1/r$ are two linear independent particular solutions, and k_1, k_2 are arbitrary constants of integration. In case of temperature (4) and, therefore, the derivative

$$\frac{dT(r)}{dr} = \left[\frac{1}{4}a^{-1}S\left(\frac{(r_1^2 - r_2^2)}{r} - 2r\ln\frac{r_1}{r_2}\right) + \frac{(T_1 - T_2)}{r}\right]\left(\ln\frac{r_1}{r_2}\right)^{-1} \tag{7}$$

the partial solution is written in the following form:

$$U_{rp}(r) = \frac{(1+\mu)\alpha}{2}\left[\left(\frac{1}{4}a^{-1}S(r_1^2 - r_2^2) + (T_1 - T_2)\right)\left(\ln\frac{r_1}{r_2}\right)^{-1} r\ln r - \frac{S}{8a}r^3\right] \tag{8}$$

We should note that this solution can be obtained based on solution (6) considering k_1, k_2 functions of radius r and using the method of integration constant variation, which is rather solid. This method must be applied repeatedly for each separate function dT/dr. Substituting (6) in (5), we obtain the following general expression for the displacement

$$U_r(r) = k_1 r + \frac{k_2}{r} + U_{rp}(r) \tag{9}$$

in which constants of integration k_1 and k_2 will be determined from the mechanical boundary conditions on the contours $\Gamma_1 \equiv (r = r_1, 0 \le \varphi \le 2\pi)$ and $\Gamma_2 \equiv (r = r_2, 0 \le \varphi \le 2\pi)$ (Figure 1).

In case of the non-homogeneous boundary value problem of the Dirichlet type, these conditions will be written in the form:

$$U_r(r = r_1) = \Delta_1; U_r(r = r_2) = -\Delta_2 \tag{10}$$

where Δ_1 and Δ_2 are the tights given on contours Γ_1 and Γ_2 (Figure 1 reflects only tight Δ_1). Arbitrary coefficients of integration k_1, k_2 from solution (9) will be determined from boundary conditions (10). Introducing (9) in (10) we obtain the following system of equations:

$$\begin{cases} k_1 r_1 + k_2 r_1^{-1} = -U_{rp}(r_1) + \Delta_1 \\ k_1 r_2 + k_2 r_2^{-1} = -U_{rp}(r_2) - \Delta_2 \end{cases} \tag{11}$$

noting that:

$$c = \left[(4a)^{-1} S (r_1^2 - r_2^2) + (T_1 - T_2)\right] \left[\ln(r_1 r_2^{-1})\right]^{-1} \tag{12}$$

and respectively

$$P_1 = c \ln r_1 - (8a)^{-1} S r_1^2, P_2 = c \ln r_2 - (8a)^{-1} S r_2^2 \tag{13}$$

System (11) will be written in the following way:

$$\begin{cases} k_1 r_1 + k_2 r_1^{-1} = -2^{-1}(1+\mu)\alpha r_1 P_1 + \Delta_1 \\ k_1 r_2 + k_2 r_2^{-1} = -2^{-1}(1+\mu)\alpha r_2 P_2 - \Delta_2 \end{cases} \tag{14}$$

The solution of equations system (12) - (14) is obtained by applying the Kramer method:

So, we calculate:

a. The primary determinant

$$D = \begin{vmatrix} r_1 & r_1^{-1} \\ r_2 & r_2^{-1} \end{vmatrix} = \frac{(r_1^2 - r_2^2)}{r_1 r_2} \tag{15}$$

b. Secondary determinants

$$D_1 = \begin{vmatrix} -2^{-1}(1+\mu)\alpha r_1 P_1 + \Delta_1 & r_1^{-1} \\ -2^{-1}(1+\mu)\alpha r_2 P_2 - \Delta_2 & r_2^{-1} \end{vmatrix} = \frac{(1+\mu)\alpha}{2} \left(\frac{P_2 r_2^2 - P_1 r_1^2}{r_1 r_2} \right) + \frac{(\Delta_2 r_2 + \Delta_1 r_1)}{r_1 r_2} \tag{16}$$

$$D_2 = \begin{vmatrix} r_1 & -2^{-1}(1+\mu)\alpha r_1 P_1 + \Delta_1 \\ r_2 & -2^{-1}(1+\mu)\alpha r_2 P_2 - \Delta_2 \end{vmatrix} = -\frac{(1+\mu)\alpha}{2} r_1 r_2 (P_2 - P_1) - (\Delta_2 r_1 + \Delta_1 r_2) \tag{17}$$

c. The coefficients

$$k_1 = \frac{D_1}{D} = \frac{(1+\mu)\alpha}{2} \frac{P_2 r_2^2 - P_1 r_1^2}{r_1^2 - r_2^2} + \frac{(\Delta_2 r_2 + \Delta_1 r_1)}{r_1^2 - r_2^2} \tag{18}$$

$$k_2 = \frac{D_2}{D} = -\frac{(1+\mu)\alpha}{2} r_1^2 r_2^2 \frac{P_2 - P_1}{r_1^2 - r_2^2} - \frac{r_1 r_2 (\Delta_2 r_1 + \Delta_1 r_2)}{r_1^2 - r_2^2} \tag{19}$$

The final solution for thermoelastic displacements is written in the following form:

$$U_r(r) = U_{r\Delta}(r) + U_{rT}(r) \tag{20}$$

where

$$U_{r\Delta}(r) = \frac{(\Delta_2 r_2 + \Delta_1 r_1)}{r_1^2 - r_2^2} r - \frac{r_1 r_2 (\Delta_2 r_1 + \Delta_1 r_2)}{r_1^2 - r_2^2} \frac{1}{r} \tag{21}$$

$$\begin{aligned} U_{rT}(r) = \frac{b}{2} \Bigg\{ \Bigg[\frac{S}{8a} \left(d + 2 \left(\ln \frac{r_1}{r_2} \right)^{-1} (r_2^2 \ln r_2 - r_1^2 \ln r_1) \right) \\ + \frac{\Delta T}{l} \left(\ln \frac{r_1}{r_2} \right)^{-1} (r_2^2 \ln r_2 - r_1^2 \ln r_1) \Bigg] r \\ + (r_1 r_2)^2 \left(\frac{S}{8a} + \frac{\Delta T}{l} \right) \frac{1}{r} + \left(\ln \frac{r_1}{r_2} \right)^{-1} \left(\frac{S}{4a} l + \Delta T \right) r \ln r - \frac{S}{8a} r^3 \Bigg\} \end{aligned} \tag{22}$$

where

$$b = (1 + \mu)\alpha; d = r_1^2 + r_2^2; \Delta T = T_1 - T_2; l = r_1^2 - r_2^2 \tag{23}$$

The graphs of radial displacements: $U_{r\Delta}(r)$, created by tight $\Delta_1 = 0.3mm; U_{rT}(r)$, created by the temperature gradient $T_1 = 25°K, T_2 = 50°K$ and the constant heat source $S = 10(W/m^3)$ with the values of thermal conductivity - $a = 0.2(W/mK)$ and the value of the linear thermal dilatation coefficient $\alpha = 1.1 \cdot 10^{-2}K^{-1}$ and of the total displacements $U_r(r)$ for the radius values: $r_1 = 40mm, r_2 = 42mm$ are presented in figures 2a; 2b and 2c respectively.

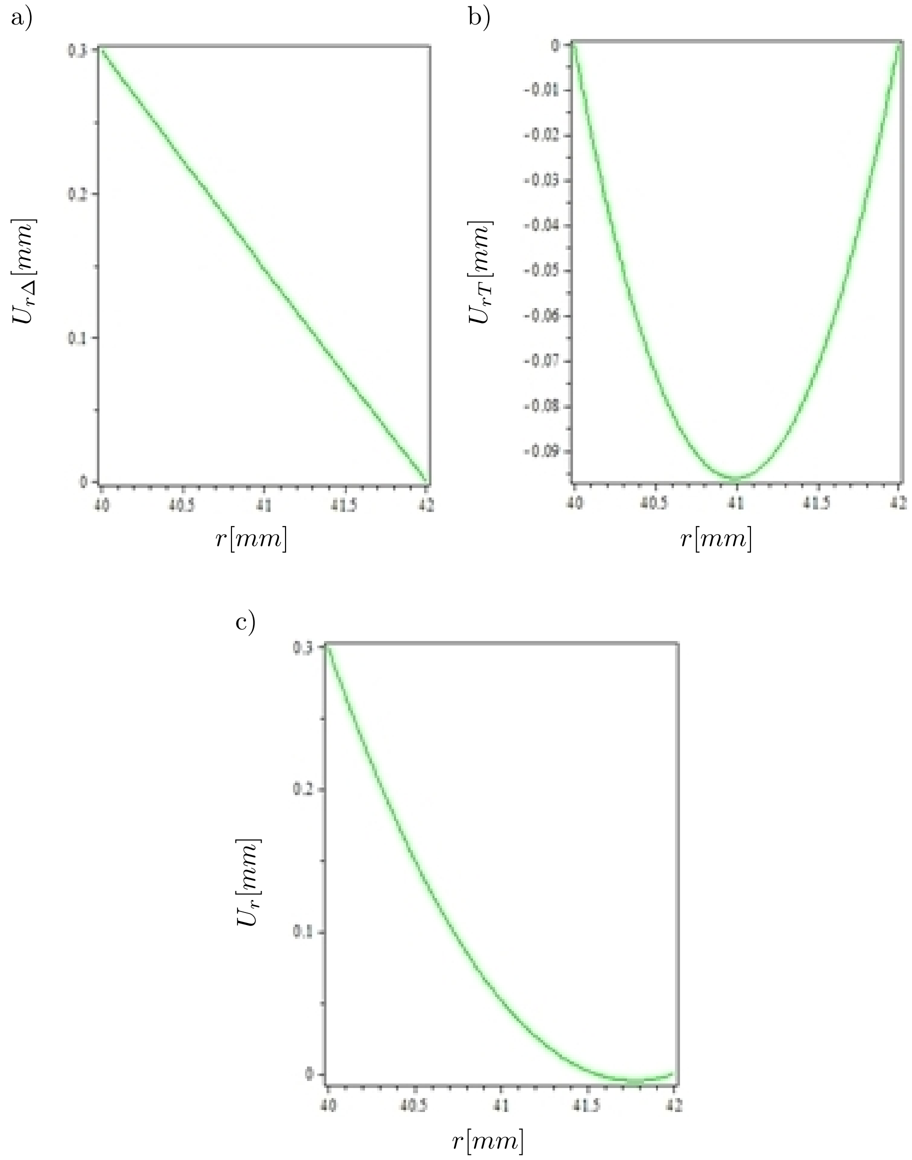

FIGURE 2. Graphs of radial displacements: $U_{r\Delta}(r)$ - Figure 2a, $U_{rT}(r)$ - Figure 2b and $U_r(r)$ - Figure 2c.

Having analyzed Figure 2, we observe that boundary conditions (10) are satisfied.

3. The Green's function method

3.1. Derivation of the Green's function for equation (1)**.** The differential equation for the derivation of the Green's function for equation (1) will be written in the following form:

$$2\pi r\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}-\frac{1}{r^2}\right)G(r,\rho)=-\delta(r-\rho) \tag{24}$$

where $\delta(r-\rho)$ is the Dirac's function. The Green's function may be written in two forms:

$$G(r,\rho)=\begin{cases}G_s(r,\rho)=c_1r+c_2r^{-1};r_1\le r\le\rho\\ G_d(r,\rho)=k_1r+k_2r^{-1};\rho\le r\le r_2,\end{cases} \tag{25}$$

with conjugation conditions of the functions $G_s(r,\rho)$ and $G_d(r,\rho)$ in the point $r=\rho$:

$$G_s(r=\rho-0,\rho)=G_d(r=\rho+0,\rho);\partial G_s(r=\rho-0,\rho)/\partial r-\partial G_d(r=\rho+0,\rho)/\partial r=(2\pi\rho)^{-1} \tag{26}$$

In equation (25) and further the functions $G_s(r,\rho)$ and $G_d(r,\rho)$ represent the expressions for the Green's function from the left and from the right respectively of the application point of the inner unitary pressure (point ρ); c_1,c_2,k_1 and k_2 represent some constants which will be determined from conjugation and boundary conditions. Next, we will introduce expressions (25) into conjugation conditions (26) and obtain the next system of equations in relation to the unknown quantities c_1-k_1 and c_2-k_2:

$$\begin{cases}(c_1-k_1)\rho+(c_2-k_2)\rho^{-1}=0\\ (c_1-k_1)-(c_2-k_2)\rho^{-2}=(2\pi\rho)^{-1}\end{cases} \tag{27}$$

System (27) is solved in the following way:

$$c_1-k_1=(4\pi\rho)^{-1};c_2-k_2=-\rho(4\pi)^{-1} \tag{28}$$

From the boundary conditions of the Dirichlet type we obtain

$$c_2=-c_1r_1^2;k_2=-k_1r_2^2 \tag{29}$$

Finally, from (28) and (29) we get:

$$c_1=-\left[4\pi\rho(r_2^2-r_1^2)\right]^{-1}(\rho^2-r_2^2);k_1=-\left[4\pi\rho(r_2^2-r_1^2)\right]^{-1}(\rho^2-r_1^2) \tag{30}$$

Introducing (29) and (30) in (25), we receive final expression for the Green's function:

$$G(r,\rho)=\begin{cases}G_s(r,\rho);r_1\le r\le\rho\\ G_d(r,\rho);\rho\le r\le r_2\end{cases}=-\left[4\pi(r_2^2-r_1^2)r\rho\right]^{-1}\begin{cases}(r^2-r_1^2)(\rho^2-r_2^2)\\ (\rho^2-r_1^2)(r^2-r_2^2)\end{cases} \tag{31}$$

3.2. Integral formula and analytical expression for thermoelastic displacements $U_r(r)$**.** Using the Green's function (31), we can receive a solution for the boundary value problem (1), (10) in form of integrals:

$$U_r(r)=-2\pi b\left[\int_{r_1}^{r_2}G_d(r,\rho)\rho d\rho+\int_{r_1}^{r}G_s(r,\rho)\rho d\rho\right]$$

$$-2\pi\Delta_2r_2\left.\frac{\partial G_s(r,\rho)}{\partial\rho}\right|_{\rho=r_2}+2\pi\Delta_1r_1\left.\frac{\partial G_d(r,\rho)}{\partial\rho}\right|_{\rho=r_1} \tag{32}$$

Calculating integrals (32) with the help of expression (31) for the Green's function and

$$\begin{aligned} \left[\partial G_s(r,\rho)/\partial\rho\right]\big|_{\rho=r_2} &= \left[2\pi(r_2^2-r_1^2)\right]^{-1}(r-r_1^2r^{-1}); \\ \left[\partial G_d(r,\rho)/\partial\rho\right]\big|_{\rho=r_1} &= -\left[2\pi(r_2^2-r_1^2)\right]^{-1}(r-r_2^2r^{-1}) \end{aligned} \tag{33}$$

we make sure that expressions for displacements $U_r(r)$ obtained by means of the Green's function method coincide with expressions (20)-(22) received by the direct integration method.

4. Thermal stresses in circular layer and pressures on its counters

Thermal stresses are determined using the Duhamel-Neumann formula, which in case of the axial symmetrical problem is written in the following way:

$$\sigma_{rr} = \frac{E}{1-\mu^2}\left(\frac{dU_r}{dr}+\mu\frac{U_r}{r}\right) - E\alpha T \tag{34}$$

Substituting expressions (20)-(22) in formula (34) we obtain the following analytical expressions for thermic radial stresses

$$\sigma_{rr}(r) = \sigma_{r\Delta}(r) + \sigma_{rT}(r) \tag{35}$$

where

$$\begin{aligned} \sigma_{r\Delta}(r) &= \frac{E}{1-\mu^2}\left(\frac{dU_{r\Delta}(r)}{dr}+\mu\frac{U_{r\Delta}(r)}{r}\right) \\ &= \frac{E}{1-\mu^2}\left[\frac{(\Delta_2 r_2+\Delta_1 r_1)}{r_1^2-r_2^2}(1+\mu) - \frac{r_1r_2(\Delta_1 r_2+\Delta_2 r_1)}{r_1^2-r_2^2}\frac{\mu-1}{r^2}\right] \end{aligned} \tag{36}$$

$$\begin{aligned} \sigma_{rT}(r) &= N\left(\frac{dU_{rT}(r)}{dr}+\mu\frac{U_{rT}(r)}{r}\right) - E\alpha T(r) \\ &= M\left\{\left[\frac{S}{8a}\left(d+2\left(\ln\frac{r_1}{r_2}\right)^{-1}(r_2^2\ln r_2 - r_1^2\ln r_1)\right)\right.\right. \\ &+\left.\frac{\Delta T}{l}\left(\ln\frac{r_1}{r_2}\right)^{-1}(r_2^2\ln r_2 - r_1^2\ln r_1)\right](1+\mu) + (r_1r_2)^2\left(\frac{S}{8a}+\frac{\Delta T}{l}\right)\left(\frac{\mu-1}{r^2}\right) \\ &+\left.\left(\ln\frac{r_1}{r_2}\right)^{-1}\left(\frac{S}{4a}l+\Delta T\right)((1+\mu)\ln r+1) - \frac{S}{8a}(3+\mu)r^2\right\} - E\alpha T(r) \end{aligned} \tag{37}$$

where

$$M = E\alpha[2(1-\mu)]^{-1}; N = E(1-\mu^2)^{-1} \tag{38}$$

Using the Maple 15 software, we have developed graphs for stresses from formulas (35) – (38) (Figure 3) for the following values of the constants: thermal conductibility - $a = 0.2(W/mK)$, heat source - $S = 10(W/m^3)$ and with the following sizes: $r_1 = 40mm, r_2 = 42mm$, $T_1 = 25K, T_2 = 25K$, $\alpha = 1.1\cdot 10^{-2}K^{-1}$, $E = 1.3\cdot 10^{-2}N/mm^2$, Poisson coefficient $\mu = 0.4$.

Finally, by means of (35) – (38) we can calculate radial pressures applied on the contours $r = r_1$ and $r = r_2$ of the compensating layer of polymeric composite material:

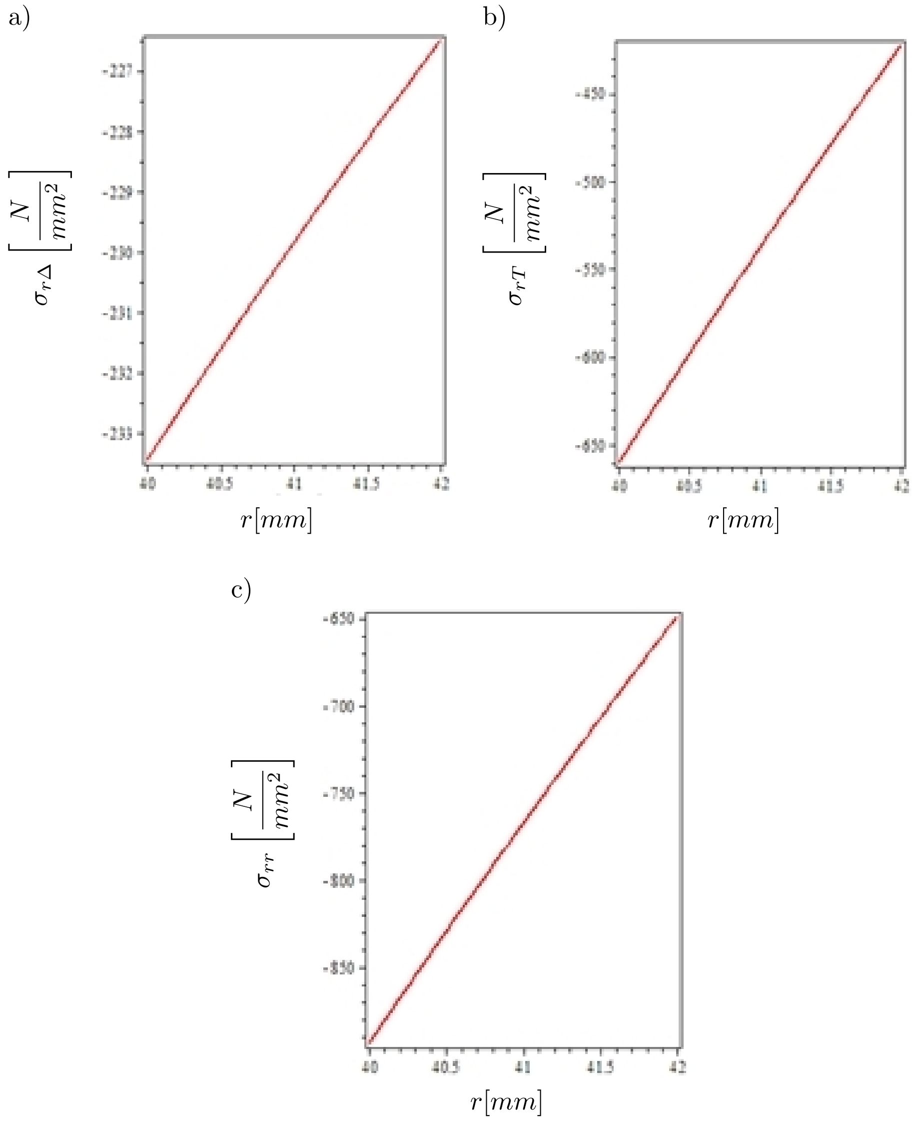

FIGURE 3. Graphs of stresses: $\sigma_{r\Delta}$ - Figure 3 a); σ_{rT} - Figure 3 b) and σ_{rr} - Figure 3c) depending on radius r.

$$p_1 = -\sigma_{rr}(r = r_1) = -(\sigma_{r\Delta}(r = r_1) + \sigma_{rT}(r = r_1)) \tag{39}$$

$$p_2 = \sigma_{rr}(r = r_2) = \sigma_{r\Delta}(r = r_2) + \sigma_{rT}(r = r_2) \tag{40}$$

$$\sigma_{r\Delta}(r = r_1) = N\left[\frac{\Delta_1}{r_1}\left(\mu + \frac{d}{l}\right) + \frac{2\Delta_2 r_2}{l}\right] \tag{41}$$

$$\sigma_{r\Delta}(r = r_2) = N\left[\frac{\Delta_2}{r_2}\left(\frac{d}{l} - \mu\right) + \frac{2\Delta_1 r_1}{l}\right] \tag{42}$$

$$\sigma_{rT}(r=r_1) = -M\left\{\frac{S}{4a}\left[d-\left(\ln\frac{r_1}{r_2}\right)^{-1}l\right]+\frac{\Delta T}{l}\left[2r_2^2-\left(\ln\frac{r_1}{r_2}\right)^{-1}l\right]\right\}-E\alpha T_1 \quad (43)$$

$$\sigma_{rT}(r=r_2) = M\left\{\frac{S}{4a}\left[d+\left(\ln\frac{r_1}{r_2}\right)^{-1}l\right]-\frac{\Delta T}{l}\left[2r_1^2-\left(\ln\frac{r_1}{r_2}\right)^{-1}l\right]\right\}-E\alpha T_2 \quad (44)$$

It is to mention that pressures p_1 and p_2 calculated with the help of formulas (39) - (44) and the Maple 15 software are similar to the results presented in Figure 3.

5. Conclusions

1. The results obtained by means of the direct integration method and the Green's function method for displacements $U_{r\Delta}(r)$, $U_{rT}(r)$, $U_r(r)$ and stresses $\sigma_{r\Delta}(r)$, $\sigma_{rT}(r)$, $\sigma_{rr}(r)$ are the same.
2. Despite the fact that final results for displacements and stresses, mentioned in the first conclusion and received by the presented methods, coincide, preference is given to the Green's function method, because it is more advantageous and needs a smaller amount of calculations. This is explained by the fact that it is necessary to calculate the corresponding integral for any function $dT(r)/dr$ in the Green's function method, but in order to obtain every particular solution of equation (1) for every $dT(r)/dr$ function by means of the direct integration method one needs to repeat the solid procedure provided by the method of integration constant variation.
3. In the future the obtained results for displacements and stresses will contribute to the solution of the boundary value problem of elasticity and thermoelasticity, and to the real calculation of resistance and rigidity of cylindrical pieces reconditioned by means of composite polymer materials.

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